POLINOMIOS DE CHEBYSHEV Y APROXIMACIÓN UNIFORME

A) DEFINICIÓN

$$T_n(x) \equiv \cos(n \arccos(x)) \qquad n \geq 0 \quad , \quad x \in [-1,1]$$

Ejemplos: $T_0(x) = \cos(0) = 1$, $T_1(x) = \cos(1 \cdot \arccos(x)) = x$

$$T_2(x) = \cos(2\arccos(x)) = \,?$$

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B) \Rightarrow \cos(A)\cos(B) = \frac{\cos(A+B)+\cos(A-B)}{2}$$

Relación de recurrencia: $(\theta \equiv \arccos(x))$

$$\left.\begin{aligned} T_{n+1}(x) &= \cos((n+1)\theta) = \underbrace{\cos(n\theta)}_{T_n(x)}\underbrace{\cos(\theta)}_{T_1(x)} - \sin(n\theta)\sin(\theta) \\ T_{n-1}(x) &= \cos((n-1)\theta) = \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta) \end{aligned}\right\} +$$

$$T_{n+1}(x) + T_{n-1}(x) = 2\cos(n\theta)\cos(\theta) = 2T_n(x)\,x$$

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

$T_0(x) = 1$, $T_1(x) = x$

$$T_2(x) = 2x\,T_1(x) - T_0(x) = 2x \cdot x - 1 = 2x^2 - 1 = 2^{2-1}x^2 - 1$$

$$T_3(x) = 2x\,T_2(x) - T_1(x) = 2x(2x^2-1) - x = 4x^3 - 3x = 2^{3-1}x^3 - 3x$$

$$T_4(x) = 2x\,T_3(x) - T_2(x) = 2x(4x^3-3x) - (2x^2-1) = 8x^4 - 8x^2 + 1 = 2^{4-1}x^4 - 8x^2 + 1$$

$\vdots$

$\vdots$

$$T_n(x) = 2^{n-1} x^n + \cdots$$

Otra forma: ("Curiosidad") $\quad e^{\pm i\theta} = \cos\theta \pm i \sin\theta$ Formula de Euler

$$T_n(x) = \cos(n\theta) = \frac{e^{in\theta} + e^{-in\theta}}{2} = \frac{\left(\cos\theta + \sqrt{\cos^2\theta - 1}\right)^n + \left(\cos\theta + \sqrt{\cos^2\theta - 1}\right)^{-n}}{2}$$

$$= \left(\left(x + \sqrt{x^2-1}\right)^n + \left(x + \sqrt{x^2-1}\right)^{-n}\right)/2$$

B) ORTOGONALIDAD RESPECTO AL PRODUCTO ESCALAR:

$$\langle f | g \rangle = \int_{-1}^{1} \frac{f(x) \cdot g(x)}{\sqrt{1-x^2}}\, dx \quad , \quad \frac{1}{\sqrt{1-x^2}} = W(x) : \text{función peso}$$

Vamos a demostrar que $\langle T_n | T_m \rangle = \frac{\pi}{2} \delta_{n,m} = \begin{cases} \frac{\pi}{2} & \text{si } n = m \\ 0 & \text{si } n \neq m \end{cases}$

En efecto:

$$\langle T_n | T_m \rangle = \int_{-1}^{1} \frac{T_n(x)\, T_m(x)}{\sqrt{1-x^2}}\, dx \overset{\theta = \arccos(x)}{=} \int_{-1}^{1} \frac{\cos(n\theta)\cos(m\theta)}{\sqrt{1-x^2}}\, dx = \left| \begin{array}{l} \theta = \arccos x \\ d\theta = \frac{-dx}{\sqrt{1-x^2}} \end{array} \right|$$

$$= \int_{\pi}^{0} -\cos(n\theta)\cos(m\theta)\, d\theta = \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)\, d\theta =$$

$$= \int_0^{\pi} \frac{1}{2}\left[\cos((n+m)\theta) + \cos((n-m)\theta)\right] =$$

$$= \frac{1}{2}\left[\frac{\sin((n+m)\theta)}{n+m} + \frac{\sin((n-m)\theta)}{n-m}\right]_0^{\pi} = 0 \quad \text{si } n \neq m$$

$$\langle T_n | T_n \rangle = \int_0^{\pi} \frac{1}{2}\left(\cos(2n\theta) + 1\right) = \pi/2 \qquad \text{c.q.d.}$$

Teorema 1 $T_n(x_k) = 0$ para $x_k = \cos\left(\frac{2k-1}{2n}\pi\right)$, $k = 1, 2, \dots, n$

En efecto:

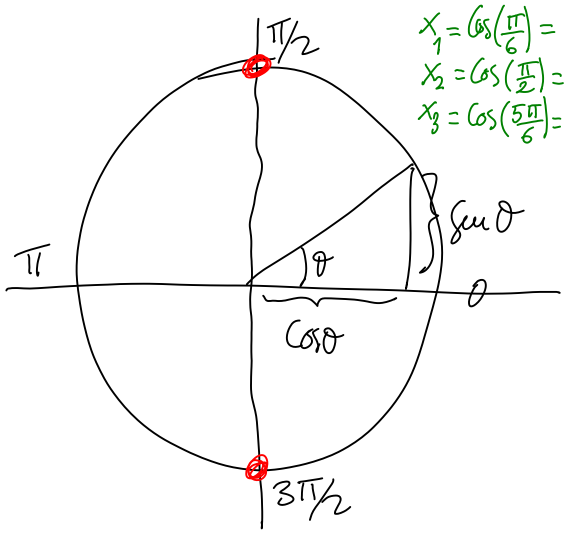
$$T_n(x_k) = \cos\left(n \arccos(x_k)\right) = \cos\left(\cancel{n}\, \frac{2k-1}{2\cancel{n}}\pi\right) = \cos\left(\frac{2k-1}{2}\pi\right) = 0$$

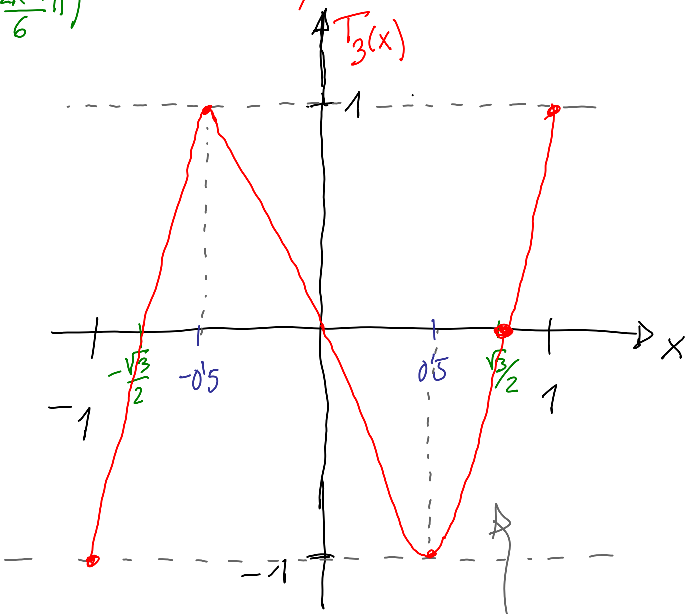
$n=3 \Rightarrow x_k = \cos\left(\frac{2k-1}{6}\pi\right)$

$x_1 = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$x_2 = \cos\left(\frac{\pi}{2}\right) = 0$

$x_3 = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$





Teorema 2 $T_n(\tilde{x}_k) = 0$ para $\tilde{x}_k = \cos\left(\frac{k\pi}{n}\right)$, con $T_n(\tilde{x}_k) = (-1)^k$

$k = 0, 1, \dots, n$

$n=3 \Rightarrow \tilde{x}_k = \cos\left(\frac{k\pi}{3}\right) \Rightarrow \tilde{x}_0 = 1$, $\tilde{x}_1 = \cos\left(\frac{\pi}{3}\right) = 0'5$

$\tilde{x}_2 = \cos\left(\frac{2\pi}{3}\right) = -0'5$, $\tilde{x}_4 = \cos(\pi) = -1$

Definición Polinomio mónico de grado n.

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \quad \text{con} \quad \boxed{a_n = 1}$$

Teorema 3 Sea $\tilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x) = x^n + \dots$ (Chebyshev mónico)

y sea $P_n(x)$ cualquier polinomio mónico de grado n

Entonces:

$$\underbrace{\max_{x\in[-1,1]} |P_n(x)|}_{\text{norma uniforme}} \geq \max_{x\in[-1,1]} |\tilde{T}_n(x)| = |\tilde{T}_n(\tilde{x}_k)| = \frac{|(-1)^k|}{2^{n-1}} = \frac{1}{2^{n-1}}$$