From Fierz-Pauli to Einstein-Hilbert

Gravity as a special relativistic field theory

Bert Janssen

Universidad de Granada & CAFPE

A pedagogical review

References: R. Feynman et al, *The Feynman Lectures on Gravitation*, 1995

T. Ortín, *Gravity and Strings*, 2004

Central Idea

Orionites with knowlegde of

- special relativity
- quantum field theory
- standard model & QCD
- NOT general relativity

Central Idea

Orionites with knowlegde of

- special relativity
- quantum field theory
- standard model & QCD
- NOT general relativity

discover gravity

$$F = G_N \frac{m_1 m_2}{r^2}$$

Central Idea

Orionites with knowlegde of

- special relativity
- quantum field theory
- standard model & QCD
- NOT general relativity

discover gravity

$$F = G_N \frac{m_1 m_2}{r^2}$$

They will try to describe gravity as a *special relativistic field theory*

$\longrightarrow$ try to find a bosonic field (graviton) that is responsable for gravity

Properties of the graviton:

Gravity

- is long range $\Longrightarrow$ Massless

Properties of the graviton:

Gravity

- is long range $\Longrightarrow$ Massless
- is static $\Longrightarrow$ integer spin (bosonic field)

Properties of the graviton:

Gravity

- is long range $\Longrightarrow$ Massless
- is static $\Longrightarrow$ integer spin (bosonic field)
- is universally attractive $\Longrightarrow$ even spin (spin 0, 2, 4, ...)

Properties of the graviton:

Gravity

- is long range $\Longrightarrow$ Massless
- is static $\Longrightarrow$ integer spin (bosonic field)
- is universally attractive $\Longrightarrow$ even spin (spin 0, 2, 4, ...)
- has deflection of light $\Longrightarrow$ not spin 0

  spin 0 propagator $\sim \; k^{-2}$

  only coupling $T^{\mu}{}_{\mu}\, k^{-2}\, T^{\nu}{}_{\nu}$

  electromagnetism (in $D = 4$): $T^{\mu}{}_{\mu} = 0$

Properties of the graviton:

Gravity

- is long range $\Longrightarrow$ Massless
- is static $\Longrightarrow$ integer spin (bosonic field)
- is universally attractive $\Longrightarrow$ even spin (spin 0, 2, 4, ...)
- has deflection of light $\Longrightarrow$ not spin 0

  spin 0 propagator $\sim \quad k^{-2}$

  only coupling $T^\mu{}_\mu \, k^{-2} \, T^\nu{}_\nu$

  electromagnetism (in $D = 4$): $T^\mu{}_\mu = 0$
- theoretical problems for spin $\geq 5/2 \Longrightarrow$ spin ≤ 2

Properties of the graviton:

Gravity

- is long range $\Longrightarrow$ Massless
- is static $\Longrightarrow$ integer spin (bosonic field)
- is universally attractive $\Longrightarrow$ even spin (spin 0, 2, 4, ...)
- has deflection of light $\Longrightarrow$ not spin 0

  spin 0 propagator $\sim \quad k^{-2}$

  only coupling $\; T^\mu{}_\mu \, k^{-2} \, T^\nu{}_\nu$

  electromagnetism (in $D = 4$): $\; T^\mu{}_\mu = 0$
- theoretical problems for spin $\geq 5/2$ $\Longrightarrow$ spin ≤ 2

$\Longrightarrow$ Spin 2 field $h_{\mu\nu}$: Representation theory: **16 = 9 + 6 + 1**

with **9** = symmetric & traceless

6 = antisymmetric = spin 1

1 = trace = spin 0

Properties of the graviton:

Gravity

- is long range $\Longrightarrow$ Massless
- is static $\Longrightarrow$ integer spin (bosonic field)
- is universally attractive $\Longrightarrow$ even spin (spin 0, 2, 4, ...)
- has deflection of light $\Longrightarrow$ not spin 0

  spin 0 propagator $\sim \; k^{-2}$

  only coupling $T^\mu{}_\mu \, k^{-2} \, T^\nu{}_\nu$

  electromagnetism (in $D = 4$): $T^\mu{}_\mu = 0$
- theoretical problems for spin $\geq 5/2$ $\Longrightarrow$ spin ≤ 2

$\Longrightarrow$ Spin 2 field $h_{\mu\nu}$: Representation theory: **16 = 9 + 6 + 1**

with **9** = symmetric & traceless

6 = antisymmetric = spin 1

1 = trace = spin 0

$\Longrightarrow$ Massless traceless symmetric spin 2 field $h_{\mu\nu}$:

Outlook

1. Introduction 1: Short review of General Relativity
2. Introduction 2: Electromagnetism as SRFT of spin 1

Outlook

1. Introduction 1: Short review of General Relativity
2. Introduction 2: Electromagnetism as SRFT of spin 1
3. Construction of Fierz-Pauli theory
4. Consistency problems with couplings
5. Deser's argument & General Relativity
6. (Philosophical) Conclusions

1 Short review of General Relativity

Einstein & Hilbert (1915):

Gravity is curvature of spacetime, caused by matter

$\longrightarrow$ Interaction between $g_{\mu\nu}$ and $T_{\mu\nu}$

$$S = \frac{1}{\kappa}\int d^4x\,\sqrt{|g|}\Big[R + \mathcal{L}_{\rm matter}\Big]$$

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \kappa\, T_{\mu\nu}$$

with

$$R_{\mu\nu} = \partial_\mu\Gamma^\lambda_{\nu\lambda} - \partial_\lambda\Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\lambda\nu} - \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma}$$

$$\Gamma^\rho_{\mu\nu} = \tfrac{1}{2}g^{\rho\sigma}\Big(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\Big)$$

$$T_{\mu\nu} = \frac{1}{\sqrt{|g|}}\frac{\delta\mathcal{L}_{\rm matter}}{\delta g^{\mu\nu}}$$

$\longrightarrow$ Very geometrical way to understand gravity

Linear perturbations of Minkowski space

Suppose $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$

$$h \; = \; h^{\mu}{}_{\mu} \; = \; \eta^{\mu\nu} h_{\mu\nu}$$

Linear perturbations of Minkowski space

Suppose $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$

$$h \;=\; h^{\mu}{}_{\mu} \;=\; \eta^{\mu\nu} h_{\mu\nu}$$

$$\Longrightarrow \qquad g^{\mu\nu} \approx \eta_{\mu\nu} - \varepsilon h_{\mu\nu}$$

$$R_{\mu\nu} \approx \tfrac{1}{2}\varepsilon\Big[\partial^2 h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial^\rho h_{\nu\rho} - \partial_\nu\partial^\rho h_{\mu\rho}\Big]$$

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R \;\approx\; \tfrac{1}{2}\varepsilon\Big[\partial^2 h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial^\rho h_{\nu\rho} - \partial_\nu\partial^\rho h_{\mu\rho} - \eta_{\mu\nu}\Big(\partial^2 h - \partial_\rho\partial_\lambda h^{\rho\lambda}\Big)\Big]$$

$$T_{\mu\nu} \approx 0 + \varepsilon\tau_{\mu\nu}$$

Linear perturbations of Minkowski space

Suppose $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$

$$h \;=\; h^{\mu}{}_{\mu} \;=\; \eta^{\mu\nu} h_{\mu\nu}$$

$$\Longrightarrow \qquad g^{\mu\nu} \approx \eta_{\mu\nu} - \varepsilon h_{\mu\nu}$$

$$R_{\mu\nu} \approx \tfrac{1}{2}\varepsilon \Big[\partial^2 h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\mu \partial^\rho h_{\nu\rho} - \partial_\nu \partial^\rho h_{\mu\rho} \Big]$$

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \;\approx\; \tfrac{1}{2}\varepsilon \Big[\partial^2 h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\mu \partial^\rho h_{\nu\rho} - \partial_\nu \partial^\rho h_{\mu\rho} - \eta_{\mu\nu} \Big(\partial^2 h - \partial_\rho \partial_\lambda h^{\rho\lambda} \Big) \Big]$$

$$T_{\mu\nu} \approx 0 + \varepsilon \tau_{\mu\nu}$$

Einstein eqn $R_{\mu\nu} = -\kappa (T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T) \quad \Longrightarrow$

$$\tfrac{1}{2} \Big[\partial^2 h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\mu \partial^\rho h_{\nu\rho} - \partial_\nu \partial^\rho h_{\mu\rho} \Big] \approx -\kappa (\tau_{\mu\nu} - \tfrac{1}{2} \eta_{\mu\nu} \tau)$$

Intermezzo (

Linear perturbations of general solution

Suppose $g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon h_{\mu\nu}$

$$h \ = \ h^{\mu}{}_{\mu} \ = \ \bar{g}^{\mu\nu} h_{\mu\nu}$$

$$\Longrightarrow \qquad R_{\mu\nu} \approx \bar{R}_{\mu\nu} + \tfrac{1}{2}\varepsilon \Big[\nabla^2 h_{\mu\nu} + \nabla_\mu \partial_\nu h - \nabla_\mu \nabla^\rho h_{\nu\rho} - \nabla_\nu \nabla^\rho h_{\mu\rho} \Big]$$

$$T_{\mu\nu} \approx \bar{T}_{\mu\nu} + \varepsilon \tau_{\mu\nu}$$

Einstein eqn $R_{\mu\nu} = -\kappa (T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T) \quad \Longrightarrow$

$$[\varepsilon^0] : \qquad \bar{R}_{\mu\nu} = -\kappa(\bar{T}_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{T})$$

$$[\varepsilon^1] : \qquad \tfrac{1}{2}\Big[\nabla^2 h_{\mu\nu} + \nabla_\mu \partial_\nu h - \nabla_\mu \nabla^\rho h_{\nu\rho} - \nabla_\nu \nabla^\rho h_{\mu\rho} \Big]$$

$$\approx -\kappa \Big(\tau_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}\tau - \tfrac{1}{2} h_{\mu\nu} \bar{T} + \tfrac{1}{2}\bar{g}_{\mu\nu} h^{\rho\lambda} \bar{T}_{\rho\lambda} \Big)$$

)

Gauge invariance

Suppose $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$

$\delta x^\mu = \xi^\mu$, with $\xi^\mu = 0 + \varepsilon \chi^\mu$

Gauge invariance

Suppose $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$

$\delta x^\mu = \xi^\mu$, with $\xi^\mu = 0 + \varepsilon \chi^\mu$

$$\Longrightarrow \delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}$$

$$[\varepsilon^0]: \quad \delta \eta_{\mu\nu} = 0$$

$$[\varepsilon^1]: \quad \delta h_{\mu\nu} = \partial_\mu \chi_\nu + \partial_\nu \chi_\mu \qquad \longrightarrow \text{“Gauge invariance”}$$

Gauge invariance

Suppose $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$

$\delta x^\mu = \xi^\mu$, with $\xi^\mu = 0 + \varepsilon\chi^\mu$

$$\Longrightarrow \delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}$$

$$[\varepsilon^0]: \quad \delta\eta_{\mu\nu} = 0$$

$$[\varepsilon^1]: \quad \delta h_{\mu\nu} = \partial_\mu \chi_\nu + \partial_\nu \chi_\mu \qquad \longrightarrow \text{“Gauge invariance”}$$

$$\Longrightarrow \quad \delta\Gamma^\rho_{\mu\nu} \;=\; \varepsilon \partial_\mu \partial_\nu \chi^\rho$$

$$\delta R_{\mu\nu\rho\lambda} \;=\; \delta R_{\mu\nu} \;=\; 0$$

Gauge invariance

Suppose $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$

$\delta x^\mu = \xi^\mu$, with $\xi^\mu = 0 + \varepsilon \chi^\mu$

$$\Longrightarrow \delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}$$

$$[\varepsilon^0] : \quad \delta \eta_{\mu\nu} = 0$$

$$[\varepsilon^1] : \quad \delta h_{\mu\nu} = \partial_\mu \chi_\nu + \partial_\nu \chi_\mu \qquad \longrightarrow \text{“Gauge invariance”}$$

$$\Longrightarrow \quad \delta \Gamma^\rho_{\mu\nu} = \varepsilon \partial_\mu \partial_\nu \chi^\rho$$

$$\delta R_{\mu\nu\rho\lambda} = \delta R_{\mu\nu} = 0$$

NB: $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$ with $h_{\mu\nu} = \partial_\mu \chi_\nu + \partial_\nu \chi_\mu$

$\Longrightarrow$ $g_{\mu\nu} = \eta_{\mu\nu}$ since $h_{\mu\nu}$ is pure gauge

Let's forget

about all this

(for a moment)

2 Electromagnetism as SRFT of spin 1

Massless spin 1 field A_μ:

$$\partial_\mu \partial^\mu A^\nu = j^\nu$$

2 Electromagnetism as SRFT of spin 1

Massless spin 1 field A_μ:

$$\partial_\mu \partial^\mu A^\nu = j^\nu$$

$\longrightarrow$ not pos. def. energy density due to non-physical states

(A_μ: 4 components $\longleftrightarrow$ photon: 2 helicity states)

2 Electromagnetism as SRFT of spin 1

Massless spin 1 field A_μ:

$$\partial_\mu \partial^\mu A^\nu = j^\nu$$

$\longrightarrow$ not pos. def. energy density due to non-physical states

(A_μ: 4 components $\longleftrightarrow$ photon: 2 helicity states)

- We can impose Lorenz condition $\partial_\mu A^\mu = 0$ by hand

$\Longrightarrow$ $\partial_\mu j^\mu = 0$ (charge conservation)

2 Electromagnetism as SRFT of spin 1

Massless spin 1 field A_μ:

$$\partial_\mu \partial^\mu A^\nu = j^\nu$$

$\longrightarrow$ not pos. def. energy density due to non-physical states
(A_μ: 4 components $\longleftrightarrow$ photon: 2 helicity states)

- We can impose Lorenz condition $\partial_\mu A^\mu = 0$ by hand
  $\Longrightarrow \partial_\mu j^\mu = 0$ (charge conservation)
- We would like $\mathcal{L}$ such that eqns of motion yield $\mathcal{F}^\mu(A) = j^\mu$
  charge conservation $\Rightarrow \partial_\mu \mathcal{F}^\mu(A) = 0$
  gauge identity implying Lorenz cond.

Most general Lorentz-invar action, quadratic in ∂A

$$\mathcal{L} \;=\; \alpha\, \partial_\mu A_\nu \partial^\mu A^\nu \;\;+\;\; \beta\, \partial_\mu A_\nu \partial^\nu A^\mu$$

Eqns of motion: $\mathcal{F}^\nu(A) = \alpha\, \partial^2 A^\nu + \beta\, \partial_\mu \partial^\nu A^\mu = 0$

Gauge condition: $\partial_\nu \mathcal{F}^\nu(A) \equiv 0 \quad \Longrightarrow \quad \alpha = -\beta$

Most general Lorentz-invar action, quadratic in ∂A

$$\mathcal{L} \;=\; \alpha\,\partial_\mu A_\nu \partial^\mu A^\nu \;+\; \beta\,\partial_\mu A_\nu \partial^\nu A^\mu$$

Eqns of motion: $\mathcal{F}^\nu(A) = \alpha\,\partial^2 A^\nu + \beta\,\partial_\mu \partial^\nu A^\mu = 0$

Gauge condition: $\partial_\nu \mathcal{F}^\nu(A) \equiv 0 \quad \Longrightarrow \quad \alpha = -\beta$

The action is

$$\mathcal{L} \;=\; -\tfrac{1}{2}\Big[\partial_\mu A_\nu \partial^\mu A^\nu \;-\; \partial_\mu A_\nu \partial^\nu A^\mu\Big]$$

$$=\; -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$\mathcal{F}^\nu = \partial_\mu F^{\mu\nu}$

Most general Lorentz-invar action, quadratic in ∂A

$$\mathcal{L} \;=\; \alpha\, \partial_\mu A_\nu \partial^\mu A^\nu \;+\; \beta\, \partial_\mu A_\nu \partial^\nu A^\mu$$

Eqns of motion: $\mathcal{F}^\nu(A) = \alpha\, \partial^2 A^\nu + \beta\, \partial_\mu \partial^\nu A^\mu = 0$

Gauge condition: $\partial_\nu \mathcal{F}^\nu(A) \equiv 0 \quad \Longrightarrow \quad \alpha = -\beta$

The action is

$$\mathcal{L} \;=\; -\tfrac{1}{2}\Big[\partial_\mu A_\nu \partial^\mu A^\nu \;-\; \partial_\mu A_\nu \partial^\nu A^\mu\Big]$$

$$\;=\; -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$\mathcal{F}^\nu = \partial_\mu F^{\mu\nu}$

Gauge invariance:

$$\delta\mathcal{L} \;=\; \Big[\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} - \frac{\delta \mathcal{L}}{\delta A_\nu}\Big]\delta A^\nu \;=\; \partial_\mu F^{\mu\nu} \delta A_\nu \;\sim\; \partial_\mu \partial_\nu F^{\mu\nu} \Lambda$$

$$\Longrightarrow \delta A_\mu = \partial_\mu \Lambda$$

Coupling to matter

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Eqns of motion: $\partial_\mu\partial^\mu\phi = 0, \qquad \partial_\mu\partial^\mu\phi^* = 0, \qquad \partial_\mu F^{\mu\nu} = 0$

Invariant under: $\delta A_\mu = \partial_\mu\Lambda$ (local $U(1)$)

$\delta\phi = ie\lambda\phi$ (global $U(1)$)

Coupling to matter

$$\mathcal{L}_0 = \tfrac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Eqns of motion: $\partial_\mu\partial^\mu\phi = 0, \qquad \partial_\mu\partial^\mu\phi^* = 0, \qquad \partial_\mu F^{\mu\nu} = 0$

Invariant under: $\delta A_\mu = \partial_\mu\Lambda$ (local $U(1)$)

$\delta\phi = ie\lambda\phi$ (global $U(1)$)

Noether current for λ:

$$\delta_\lambda\mathcal{L}_0 \; = \; -\tfrac{ie}{2}\partial_\mu\lambda\Big(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\Big) \; = \; e\lambda\partial_\mu j^\mu_{(0)}$$

$j^\mu_{(0)}$ preserved on-shell: $\partial_\mu j^\mu_{(0)} \; = \; \tfrac{i}{2}\Big[\phi^*\partial_\mu\partial^\mu\phi - \phi\partial_\mu\partial^\mu\phi^*\Big]$

Coupling to matter

$$\mathcal{L}_0 = \tfrac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Eqns of motion: $\partial_\mu\partial^\mu\phi = 0, \qquad \partial_\mu\partial^\mu\phi^* = 0, \qquad \partial_\mu F^{\mu\nu} = 0$

Invariant under: $\delta A_\mu = \partial_\mu\Lambda$ (local $U(1)$)

$\delta\phi = ie\lambda\phi$ (global $U(1)$)

Noether current for λ:

$$\delta_\lambda\mathcal{L}_0 \;=\; -\tfrac{ie}{2}\partial_\mu\lambda\Big(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\Big) \;=\; e\lambda\partial_\mu j^\mu_{(0)}$$

$j^\mu_{(0)}$ preserved on-shell: $\partial_\mu j^\mu_{(0)} \;=\; \tfrac{i}{2}\Big[\phi^*\partial_\mu\partial^\mu\phi - \phi\partial_\mu\partial^\mu\phi^*\Big]$

$j^\mu_{(0)}$ is not source term for $F^{\mu\nu}$

$\longrightarrow$ add coupling term $\mathcal{L} \sim eA_\mu j^\mu_{(0)}$

Coupling term

$$\mathcal{L}_1 = \tfrac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + eA_\mu j^\mu_{(0)}$$

with $j^\mu_{(0)} = \frac{ie}{2}\Big(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\Big)$

Eqn of motion: $\partial_\mu F^{\mu\nu} = e j^\nu_{(0)}$

$$\partial_\mu\partial^\mu\phi - ie\partial_\mu A^\mu\phi - 2ieA_\mu\partial^\mu\phi = 0$$

$$\partial_\mu\partial^\mu\phi^* + ie\partial_\mu A^\mu\phi^* + 2ieA_\mu\partial^\mu\phi^* = 0$$

Problem: $j^\mu_{(0)}$ is not conserved on-shell!

$$0 \;=\; \partial_\mu\partial_\nu F^{\mu\nu} \;=\; \partial_\mu j^\mu_{(0)} \;\neq\; 0$$

$\longrightarrow$ INCONSISTENCY!!!

Coupling term

$$\mathcal{L}_1 = \tfrac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + eA_\mu j^\mu_{(0)}$$

with $j^\mu_{(0)} = \frac{ie}{2}\Big(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\Big)$

Eqn of motion: $\partial_\mu F^{\mu\nu} = e j^\nu_{(0)}$

$$\partial_\mu\partial^\mu\phi - ie\partial_\mu A^\mu\phi - 2ieA_\mu\partial^\mu\phi = 0$$

$$\partial_\mu\partial^\mu\phi^* + ie\partial_\mu A^\mu\phi^* + 2ieA_\mu\partial^\mu\phi^* = 0$$

Problem: $j^\mu_{(0)}$ is not conserved on-shell!

$$0 \;=\; \partial_\mu\partial_\nu F^{\mu\nu} \;=\; \partial_\mu j^\mu_{(0)} \;\neq\; 0$$

$\longrightarrow$ INCONSISTENCY!!!

Reason:

$j^\mu_{(0)}$ is Noether current of $\mathcal{L}_0$, not of $\mathcal{L}_1$

Real Noether current: $j^\mu_{(1)} = j^\mu_{(0)} + eA^\mu\phi^*\phi$

Noether procedure

Identify λ and Λ

$$\delta A_\mu = \partial_\mu \Lambda, \qquad \qquad \delta\phi = ie\Lambda\phi$$

Noether procedure

Identify λ and Λ

$$\delta A_\mu = \partial_\mu \Lambda, \qquad\qquad \delta\phi = ie\Lambda\phi$$

Then:

$$\delta\mathcal{L}_0 = -e\partial\Lambda j^\mu_{(0)} \neq 0$$

$$\longrightarrow \text{add } \mathcal{L} \sim eA_\mu j^\mu_{(0)}$$

Noether procedure

Identify λ and Λ

$$\delta A_\mu = \partial_\mu \Lambda, \qquad \delta\phi = ie\Lambda\phi$$

Then:

$$\delta\mathcal{L}_0 = -e\partial\Lambda j^\mu_{(0)} \neq 0$$

$$\longrightarrow \text{add } \mathcal{L} \sim eA_\mu j^\mu_{(0)}$$

$$\delta\mathcal{L}_1 = -e\partial\Lambda j^\mu_{(0)} + e\partial\Lambda j^\mu_{(0)} - e^2\partial_\mu\Lambda A^\mu\phi^*\phi \neq 0$$

$$\longrightarrow \text{add } \mathcal{L} \sim \tfrac{1}{2}e^2 A_\mu A^\mu \phi^*\phi$$

Noether procedure

Identify λ and Λ

$$\delta A_\mu = \partial_\mu \Lambda, \qquad \delta\phi = ie\Lambda\phi$$

Then:

$$\delta\mathcal{L}_0 = -e\partial\Lambda j^\mu_{(0)} \neq 0$$

$$\longrightarrow \text{add } \mathcal{L} \sim eA_\mu j^\mu_{(0)}$$

$$\delta\mathcal{L}_1 = -e\partial\Lambda j^\mu_{(0)} + e\partial\Lambda j^\mu_{(0)} - e^2\partial_\mu\Lambda A^\mu\phi^*\phi \neq 0$$

$$\longrightarrow \text{add } \mathcal{L} \sim \tfrac{1}{2}e^2 A_\mu A^\mu \phi^*\phi$$

$$\mathcal{L}_2 = \tfrac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + eA_\mu j^\mu_{(0)} + \tfrac{1}{2}e^2 A_\mu A^\mu\phi^*\phi$$

$$\longrightarrow \delta\mathcal{L}_2 \;=\; -e^2\partial_\mu\Lambda A^\mu\phi^*\phi \;+\; e^2\partial_\mu\Lambda A^\mu\phi^*\phi = 0$$

Nota bene:

$$\begin{aligned}
\mathcal{L}_2 &= \tfrac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{ie}{2}A_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + \tfrac{1}{2}e^2A_\mu A^\mu\phi^*\phi \\
&= \tfrac{1}{2}(\partial_\mu\phi^* + ieA_\mu\phi^*)(\partial^\mu\phi - ieA^\mu\phi) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&= D_\mu\phi^* D^\mu\phi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}
\end{aligned}$$

with: $D_\mu\phi = \partial_\mu\phi - ieA_\mu\phi$

$$\delta D_\mu\phi = D_\mu(\delta\phi)$$

Nota bene:

$$\begin{aligned}\mathcal{L}_2 &= \tfrac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{ie}{2}A_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + \tfrac{1}{2}e^2A_\mu A^\mu\phi^*\phi \\ &= \tfrac{1}{2}(\partial_\mu\phi^* + ieA_\mu\phi^*)(\partial^\mu\phi - ieA^\mu\phi) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= D_\mu\phi^* D^\mu\phi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\end{aligned}$$

with: $D_\mu\phi = \partial_\mu\phi - ieA_\mu\phi$

$$\delta D_\mu\phi = D_\mu(\delta\phi)$$

Lesson:

Lorentz symm & gauge invariance $\Longrightarrow$ free theory
Gauge invariance & consistent couplings
$\Longrightarrow$ fully consistent theory coupled to matter

3 Gravity as a SRFT of spin 2

Massless spin 2 field $h_{\mu\nu}$:

$$\partial_\rho \partial^\rho h^{\mu\nu} = \kappa T^{\mu\nu}$$

$\longrightarrow$ not pos. def. energy density due to non-physical states

($h_{\mu\nu}$: 4 components $\longleftrightarrow$ graviton: 2 helicity states)

3 Gravity as a SRFT of spin 2

Massless spin 2 field $h_{\mu\nu}$:

$$\partial_\rho \partial^\rho h^{\mu\nu} = \kappa T^{\mu\nu}$$

$\longrightarrow$ not pos. def. energy density due to non-physical states

($h_{\mu\nu}$: 4 components $\longleftrightarrow$ graviton: 2 helicity states)

Look for action $\mathcal{L}$ such that eqns of motion yield $\mathcal{H}_{\mu\nu}(h) = \kappa T^{\mu\nu}$

energy conservation $\partial_\mu T^{\mu\nu} = 0 \implies \partial_\mu \mathcal{H}^{\mu\nu}(h) = 0$

$\longrightarrow$ gauge invariant action

Most general Lorentz-invar action, quadratic in ∂h

$$\mathcal{L} = a\, \partial_\mu h_{\nu\rho} \partial^\mu h^{\nu\rho} \; + \; b\, \partial_\mu h_{\nu\rho} \partial^\nu h^{\mu\rho} \; + \; c\, \partial_\mu h \partial^\rho h^{\mu\rho} \; + \; d\, \partial_\mu h \partial^\mu h$$

with $h = h^\mu{}_\mu = \eta^{\mu\nu} h_{\mu\nu}$

Eqns of motion: $\mathcal{H}^{\nu\rho}(h) \equiv a\, \partial^2 h^{\nu\rho} + \; 2b\, \partial_\mu \partial^{(\nu} h^{\rho)\mu} \; + \; c\, \partial^{(\nu} \partial^{\rho)} h$

$$+\, c\, \eta^{\nu\rho} \partial_\mu \partial_\lambda h^{\mu\lambda} \; + \; 2d\, \eta^{\nu\rho} \partial^2 h \; = \; 0$$

Gauge condition: $\partial_\nu \mathcal{H}^{\nu\rho}(h) \equiv 0 \quad \Longrightarrow \quad a = -\frac{1}{2} b = \frac{1}{2} c = -d$

Most general Lorentz-invar action, quadratic in ∂h

$$\mathcal{L} = a\,\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} \;+\; b\,\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} \;+\; c\,\partial_\mu h \partial^\rho h^{\mu\rho} \;+\; d\,\partial_\mu h \partial^\mu h$$

with $h = h^\mu{}_\mu = \eta^{\mu\nu}h_{\mu\nu}$

Eqns of motion: $$\mathcal{H}^{\nu\rho}(h) \equiv a\,\partial^2 h^{\nu\rho} + \;2b\,\partial_\mu\partial^{(\nu}h^{\rho)\mu} \;+\; c\,\partial^{(\nu}\partial^{\rho)}h$$
$$+\,c\,\eta^{\nu\rho}\partial_\mu\partial_\lambda h^{\mu\lambda} \;+\; 2d\,\eta^{\nu\rho}\partial^2 h \;=\; 0$$

Gauge condition: $\partial_\nu\mathcal{H}^{\nu\rho}(h) \equiv 0 \quad \Longrightarrow \quad a = -\tfrac{1}{2}b = \tfrac{1}{2}c = -d$

The action then is

$$\mathcal{L} = \tfrac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} \;-\; \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} \;+\; \tfrac{1}{2}\partial_\mu h\partial^\rho h^{\mu\rho} \;-\; \tfrac{1}{4}\partial_\mu h\partial^\mu h$$

$$\mathcal{H}_{\mu\nu}(h) \equiv \partial^2 h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial^\rho\partial_\mu h_{\nu\rho} - \partial^\rho\partial_\nu h_{\mu\rho}$$
$$-\eta_{\mu\nu}\partial^2 h + \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda} = 0$$

(NB: $\mathcal{H}_{\mu\nu}(h)$ is first order of $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ for $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$)

Gauge invariance

$$\delta\mathcal{L} \;=\; \Big[\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu h_{\nu\rho})} - \frac{\delta\mathcal{L}}{\delta h_{\nu\rho}}\Big]\delta h^{\mu\nu} \;=\; \mathcal{H}^{\nu\rho}\delta h_{\nu\rho} \;\sim\; \partial_\nu\mathcal{H}^{\nu\rho}\chi$$

$$\Longrightarrow \delta h_{\mu\nu} = \partial_\mu\chi_\nu + \partial_\nu\chi_\mu$$

(NB: $\delta h_{\mu\nu}$ is first order of $\delta g_{\mu\nu}$ for $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$)

Gauge invariance

$$\delta\mathcal{L} \;=\; \Big[\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu h_{\nu\rho})} - \frac{\delta\mathcal{L}}{\delta h_{\nu\rho}}\Big]\delta h^{\mu\nu} \;=\; \mathcal{H}^{\nu\rho}\delta h_{\nu\rho} \;\sim\; \partial_\nu\mathcal{H}^{\nu\rho}\chi$$

$$\Longrightarrow \delta h_{\mu\nu} = \partial_\mu\chi_\nu + \partial_\nu\chi_\mu$$

(NB: $\delta h_{\mu\nu}$ is first order of $\delta g_{\mu\nu}$ for $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$)

- Transverse traceless gauge: $\partial_\mu h^{\mu\nu} = 0 = h$

$$\Longrightarrow \mathcal{H}_{\mu\nu}(h) = \partial_\rho\partial^\rho h_{\mu\nu} = 0$$

- De Donder gauge: $\partial_\mu \bar{h}^{\mu\nu} = 0$ with $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$

$$\Longrightarrow \mathcal{H}_{\mu\nu}(h) = \partial_\rho\partial^\rho \bar{h}_{\mu\nu} = 0$$

Coupling to matter

$$\mathcal{L} \;=\; \mathcal{L}_{\rm FP} \;+\; \mathcal{L}_\phi \;+\; \kappa\, h_{\mu\nu} T^{\mu\nu}(\phi)$$

Eqns of motion: $\mathcal{H}_{\mu\nu} = \kappa T_{\mu\nu}(\phi) \qquad\qquad \partial^2\phi + ... = 0$

Gauge condition: $\partial_\mu \mathcal{H}^{\mu\nu} = 0 \;\;\Longrightarrow\;\; \partial_\mu T^{\mu\nu}(\phi) = 0$

Coupling to matter

$$\mathcal{L} \;=\; \mathcal{L}_{\rm FP} \;+\; \mathcal{L}_{\phi} \;+\; \kappa\, h_{\mu\nu} T^{\mu\nu}(\phi)$$

Eqns of motion: $\mathcal{H}_{\mu\nu} = \kappa T_{\mu\nu}(\phi) \qquad \partial^2\phi + ... = 0$

Gauge condition: $\partial_\mu \mathcal{H}^{\mu\nu} = 0 \;\Longrightarrow\; \partial_\mu T^{\mu\nu}(\phi) = 0$

Questions:

1. How good is this theory?

   $\longrightarrow$ Experimental verification

2. Is $\partial_\mu T^{\mu\nu}(\phi) = 0$ consistent with $\partial^2\phi + ... = 0$?

   $\longrightarrow$ Consistency problems

Newtonian limit

Point mass: $T^{\mu\nu} \;=\; -M\,\delta^\mu_0\,\delta^\nu_0\,\delta(x^\rho)$

$$\Longrightarrow \quad \mathcal{H}^{00} = -\kappa M \delta(x^\rho), \qquad \mathcal{H}^{ij} = 0$$

$$\Longrightarrow \quad h_{00} = \Phi, \qquad h_{ii} = \Phi\delta_{ij} \qquad \Phi = \frac{M}{8\pi|x|}$$

(Newtonian potential)

Action for mass m moving in gravitational potential Φ:

$$S \;=\; -m\int dt\Big\{\sqrt{1-v^2} + \frac{1}{\sqrt{1-v^2}}[1+v^2]\Phi\Big\}$$

$$\approx \; \int dt\Big\{\tfrac{1}{2}mv^2 - m\Phi - \tfrac{1}{4}mv^4 + \tfrac{3}{2}mv^2\Phi + \ldots\Big\}$$

$\Longrightarrow$ Good agreement for light deflection, time dilatation, ...

Perihelium Mercury is $0.75\times$ observed value

4 Consistency problems with couplings

$$\mathcal{L}_0 = \tfrac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} - \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} + \tfrac{1}{2}\partial_\mu h \partial^\rho h^{\mu\rho} - \tfrac{1}{4}\partial_\mu h \partial^\mu h + \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi$$

Eqns of motion: $\mathcal{H}_{\mu\nu} = 0 \qquad \partial_\mu \partial^\mu \phi = 0$

Invariant under: $\delta h_{\mu\nu} = \partial_\mu \chi_\nu + \partial_\nu \chi_\mu + \kappa\, \Sigma^\lambda \partial_\lambda h_{\mu\nu}$

$\delta \phi = \kappa\, \Sigma^\mu \partial_\mu \phi$

$\delta x^\mu = -\kappa\, \Sigma^\mu$ with Σ^μ global symm

4 Consistency problems with couplings

$$\mathcal{L}_0 = \tfrac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} - \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} + \tfrac{1}{2}\partial_\mu h \partial^\rho h^{\mu\rho} - \tfrac{1}{4}\partial_\mu h \partial^\mu h + \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi$$

Eqns of motion: $\mathcal{H}_{\mu\nu} = 0 \qquad \partial_\mu \partial^\mu \phi = 0$

Invariant under:

$$\delta h_{\mu\nu} = \partial_\mu \chi_\nu + \partial_\nu \chi_\mu + \kappa\, \Sigma^\lambda \partial_\lambda h_{\mu\nu}$$

$$\delta\phi = \kappa\, \Sigma^\mu \partial_\mu \phi$$

$$\delta x^\mu = -\kappa\, \Sigma^\mu$$

with Σ^μ global symm

Noether current for Σ^μ:

$$\delta_\Sigma \mathcal{L}_0 = \kappa\, \partial^\mu \Sigma^\nu \Big[T_{\mu\nu}(\phi) + T_{\mu\nu}(h) \Big]$$

with $T_{\mu\nu}(\phi)$ energy-momentum tensor of matter
$T_{\mu\nu}(h)$ gravitational energy-momentum tensor

$$
\begin{aligned}
T_{\mu\nu}(\phi) &= -\partial_\mu\phi\partial_\nu\phi + \tfrac{1}{2}\eta_{\mu\nu}(\partial\phi)^2 \\
T_{\mu\nu}(h) &= -\tfrac{1}{2}\partial_\mu h_{\rho\lambda}\partial_\nu h^{\rho\lambda} + \partial_\mu h_{\rho\lambda}\partial^\rho h_\nu{}^\lambda - \tfrac{1}{2}\partial_\mu h\partial^\rho h_{\nu\rho} \\
&\qquad -\tfrac{1}{2}\partial^\rho h\partial^\mu h_{\nu\rho} + \tfrac{1}{2}\partial_\mu h\partial^\mu h + \eta_{\mu\nu}\mathcal{L}_{\rm FP}
\end{aligned}
$$

$T_{\mu\nu}(\phi)$ and $T_{\mu\nu}(h)$ preserved on shell:

$$
\partial^\mu T_{\mu\nu}(\phi) = -\,\partial^2\phi\,\partial_\nu\phi \qquad \partial^\mu T_{\mu\nu}(h) = -\partial_\nu h_{\mu\rho}\mathcal{H}^{\mu\rho}
$$

$$
\begin{aligned}
T_{\mu\nu}(\phi) &= -\partial_\mu\phi\partial_\nu\phi + \tfrac{1}{2}\eta_{\mu\nu}(\partial\phi)^2 \\
T_{\mu\nu}(h) &= -\tfrac{1}{2}\partial_\mu h_{\rho\lambda}\partial_\nu h^{\rho\lambda} + \partial_\mu h_{\rho\lambda}\partial^\rho h_\nu{}^\lambda - \tfrac{1}{2}\partial_\mu h\partial^\rho h_{\nu\rho} \\
&\qquad -\tfrac{1}{2}\partial^\rho h\partial^\mu h_{\nu\rho} + \tfrac{1}{2}\partial_\mu h\partial^\mu h + \eta_{\mu\nu}\mathcal{L}_{\rm FP}
\end{aligned}
$$

$T_{\mu\nu}(\phi)$ and $T_{\mu\nu}(h)$ preserved on shell:

$$
\partial^\mu T_{\mu\nu}(\phi) = -\,\partial^2\phi\,\partial_\nu\phi \qquad \partial^\mu T_{\mu\nu}(h) = -\partial_\nu h_{\mu\rho}\mathcal{H}^{\mu\rho}
$$

$T_{\mu\nu}(\phi)$ is not source term for $h_{\mu\nu}$:

$\longrightarrow$ add coupling term $\mathcal{L} \sim \kappa h^{\mu\nu}T_{\mu\nu}(\phi)$

Coupling term

$$\mathcal{L}_1 = \tfrac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} - \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} + \tfrac{1}{2}\partial_\mu h \partial^\rho h^{\mu\rho} - \tfrac{1}{4}\partial_\mu h\partial^\mu h + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \tfrac{1}{2}\,\kappa h^{\mu\nu}T_{\mu\nu}(\phi)$$

Eqns of motion: $\mathcal{H}_{\mu\nu} = \kappa T_{\mu\nu}(\phi)$

$$\partial_\mu\partial^\mu\phi = \kappa\partial_\mu\Big[h^{\mu\nu}\partial_\nu\phi - \tfrac{1}{2}h\partial_\mu\phi\Big]$$

Problem: $T_{\mu\nu}(\phi)$ is no longer conserved on shell!

$$0 = \partial^\mu\mathcal{H}_{\mu\nu} = \kappa\,\partial^\mu T_{\mu\nu}(\phi) = \kappa\partial^2\phi\partial_\nu\phi \neq 0 \qquad \text{(-22)}$$

$\longrightarrow$ INCONSISTENCY!!!

Coupling term

$$\mathcal{L}_1 = \tfrac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} - \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho}$$
$$+ \tfrac{1}{2}\partial_\mu h\partial^\rho h^{\mu\rho} - \tfrac{1}{4}\partial_\mu h\partial^\mu h + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \tfrac{1}{2}\,\kappa h^{\mu\nu}T_{\mu\nu}(\phi)$$

Eqns of motion: $\mathcal{H}_{\mu\nu} = \kappa T_{\mu\nu}(\phi)$

$$\partial_\mu\partial^\mu\phi = \kappa\partial_\mu\Big[h^{\mu\nu}\partial_\nu\phi - \tfrac{1}{2}h\partial_\mu\phi\Big]$$

Problem: $T_{\mu\nu}(\phi)$ is no longer conserved on shell!

$$0 = \partial^\mu\mathcal{H}_{\mu\nu} = \kappa\,\partial^\mu T_{\mu\nu}(\phi) = \kappa\partial^2\phi\partial_\nu\phi \neq 0 \qquad \text{(-22)}$$

$\longrightarrow$ INCONSISTENCY!!!

Reason:

$T^{\mu\nu}(\phi)$ is Noether current of $\mathcal{L}_0$, not of $\mathcal{L}_1$

Total energy conserved, not only energy of ϕ

Gravity couples also to own energy $\longrightarrow$ non-linear theory

Gravity self-coupling

$$\mathcal{L}_2 = \tfrac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} - \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} + \tfrac{1}{2}\partial_\mu h\partial^\rho h^{\mu\rho} - \tfrac{1}{4}\partial_\mu h\partial^\mu h + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \tfrac{1}{2}\,\kappa h^{\mu\nu}T_{\mu\nu}(\phi) + \tfrac{1}{2}\,\kappa h^{\mu\nu}T_{\mu\nu}(h)$$

Gravity self-coupling

$$\begin{aligned}\mathcal{L}_2 \;=\; & \tfrac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} \;-\; \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} \;+\; \tfrac{1}{2}\partial_\mu h \partial^\rho h^{\mu\rho} \\ & - \tfrac{1}{4}\partial_\mu h \partial^\mu h \;+\; \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi \;+\; \tfrac{1}{2}\,\kappa h^{\mu\nu} T_{\mu\nu}(\phi) + \tfrac{1}{2}\,\kappa h^{\mu\nu} T_{\mu\nu}(h)\end{aligned}$$

$\longrightarrow$ TOO NAIVE!!!

Eqn of motion does not give

$$\mathcal{H}_{\mu\nu} = \kappa T_{\mu\nu}(\phi) + \kappa T_{\mu\nu}(h)$$

due to contribuition of $T_{\mu\nu}(h)$

Gravity self-coupling

$$\mathcal{L}_2 = \tfrac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} - \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} + \tfrac{1}{2}\partial_\mu h \partial^\rho h^{\mu\rho} - \tfrac{1}{4}\partial_\mu h \partial^\mu h + \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi + \tfrac{1}{2}\kappa h^{\mu\nu} T_{\mu\nu}(\phi) + \tfrac{1}{2}\kappa h^{\mu\nu} T_{\mu\nu}(h)$$

$\longrightarrow$ TOO NAIVE!!!

Eqn of motion does not give

$$\mathcal{H}_{\mu\nu} = \kappa T_{\mu\nu}(\phi) + \kappa T_{\mu\nu}(h)$$

due to contribuition of $T_{\mu\nu}(h)$

$\longrightarrow$ $T_{\mu\nu}(h)$ too simple for energy-monentum tensor for $h_{\mu\nu}$

We need $\mathcal{L}_{\rm corr} = \kappa\, h^{\mu\nu}\mathcal{L}_{\mu\nu} \sim \kappa\, h(\partial h)(\partial h)$ such that

$$\begin{aligned}\mathcal{L}_3 \;&=\; \tfrac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} \;-\; \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} \;+\; \tfrac{1}{2}\partial_\mu h\partial^\rho h^{\mu\rho} \\ &\qquad -\tfrac{1}{4}\partial_\mu h\partial^\mu h \;+\; \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi \;+\tfrac{1}{2}\,\kappa\, h^{\mu\nu}T_{\mu\nu}(\phi) \;+\; \kappa\, h^{\mu\nu}\mathcal{L}_{\mu\nu}\end{aligned}$$

yields

We need $\mathcal{L}_{\rm corr} = \kappa\, h^{\mu\nu}\mathcal{L}_{\mu\nu} \sim \kappa\, h(\partial h)(\partial h)$ such that

$$\mathcal{L}_3 = \tfrac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} - \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} + \tfrac{1}{2}\partial_\mu h \partial^\rho h^{\mu\rho} - \tfrac{1}{4}\partial_\mu h\partial^\mu h + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \tfrac{1}{2}\,\kappa\, h^{\mu\nu}T_{\mu\nu}(\phi) + \kappa\, h^{\mu\nu}\mathcal{L}_{\mu\nu}$$

yields

- $\mathcal{T}_{\mu\nu}(h) = \mathcal{L}_{\mu\nu} - \partial_\sigma\Big(h^{\rho\lambda}\frac{\delta\mathcal{L}_{\rho\lambda}}{\delta\partial_\sigma h_{\mu\nu}}\Big)$

We need $\mathcal{L}_{\rm corr} = \kappa\, h^{\mu\nu}\mathcal{L}_{\mu\nu} \sim \kappa\, h(\partial h)(\partial h)$ such that

$$\begin{aligned}\mathcal{L}_3 \;=\; & \tfrac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} \;-\; \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} \;+\; \tfrac{1}{2}\partial_\mu h\partial^\rho h^{\mu\rho} \\ & -\; \tfrac{1}{4}\partial_\mu h\partial^\mu h \;+\; \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi \;+ \tfrac{1}{2}\,\kappa\, h^{\mu\nu}T_{\mu\nu}(\phi) \;+\; \kappa\, h^{\mu\nu}\mathcal{L}_{\mu\nu}\end{aligned}$$

yields

- $\mathcal{T}_{\mu\nu}(h) = \mathcal{L}_{\mu\nu} - \partial_\sigma\Big(h^{\rho\lambda}\frac{\delta\mathcal{L}_{\rho\lambda}}{\delta\partial_\sigma h_{\mu\nu}}\Big)$
- $\mathcal{L}_3$ is invariant upto order κ^2 under

$$\delta h_{\mu\nu} = \partial_\mu\chi_\nu + \partial_\nu\chi_\mu + \kappa\,\chi^\rho\partial_\rho h_{\mu\nu} \qquad \delta\phi = \kappa\,\chi^\rho\partial_\rho\phi$$

We need $\mathcal{L}_{\rm corr} = \kappa\, h^{\mu\nu}\mathcal{L}_{\mu\nu} \sim \kappa\, h(\partial h)(\partial h)$ such that

$$\begin{aligned}\mathcal{L}_3 \;=\; & \tfrac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} \;-\; \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} \;+\; \tfrac{1}{2}\partial_\mu h \partial^\rho h^{\mu\rho} \\ & -\tfrac{1}{4}\partial_\mu h\partial^\mu h \;+\; \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi \;+\tfrac{1}{2}\,\kappa\, h^{\mu\nu}T_{\mu\nu}(\phi) \;+\; \kappa\, h^{\mu\nu}\mathcal{L}_{\mu\nu}\end{aligned}$$

yields

- $\mathcal{T}_{\mu\nu}(h) = \mathcal{L}_{\mu\nu} - \partial_\sigma\Big(h^{\rho\lambda}\frac{\delta\mathcal{L}_{\rho\lambda}}{\delta\partial_\sigma h_{\mu\nu}}\Big)$

- $\mathcal{L}_3$ is invariant upto order κ^2 under

$$\delta h_{\mu\nu} = \partial_\mu\chi_\nu + \partial_\nu\chi_\mu + \kappa\,\chi^\rho\partial_\rho h_{\mu\nu} \qquad \delta\phi = \kappa\,\chi^\rho\partial_\rho\phi$$

- $\mathcal{H}_{\mu\nu} = \kappa T_{\mu\nu}(\phi) + \kappa\mathcal{T}_{\mu\nu}(h) \qquad \partial^\mu\Big(T_{\mu\nu}(\phi) + \mathcal{T}_{\mu\nu}(h)\Big) = 0$

Most general term for $\mathcal{L}_{\rm corr}$:

$$\mathcal{L}_{\rm corr} = \tfrac{1}{2}\kappa\, h^{\mu\nu}\Big[\alpha\, \partial_\mu h_{\rho\lambda}\partial_\nu h^{\rho\lambda} \ + \ \beta\, \partial_\mu h_{\rho\lambda}\partial^\rho h^\lambda{}_\nu \ + \ ...\Big]$$

$\longrightarrow$ 20 terms with 16 coefficients

$\longrightarrow$ determine coefficients by demanding $\partial_\mu \mathcal{T}^{\mu\nu}(h) = \gamma^\nu_{\rho\lambda}\mathcal{H}^{\rho\lambda}$

Most general term for $\mathcal{L}_{\rm corr}$:

$$\mathcal{L}_{\rm corr} = \tfrac{1}{2}\kappa\, h^{\mu\nu}\Big[\alpha\, \partial_\mu h_{\rho\lambda}\partial_\nu h^{\rho\lambda} \ + \ \beta\, \partial_\mu h_{\rho\lambda}\partial^\rho h^\lambda{}_\nu \ + \ \ldots\Big]$$

$\longrightarrow$ 20 terms with 16 coefficients

$\longrightarrow$ determine coefficients by demanding $\partial_\mu \mathcal{T}^{\mu\nu}(h) = \gamma^\nu_{\rho\lambda}\mathcal{H}^{\rho\lambda}$

$$\Longrightarrow \mathcal{L}_{\rm corr} \ = \ \tfrac{1}{2}\kappa\, h^{\mu\nu}\Big[-\tfrac{1}{2}\partial_\mu h_{\rho\lambda}\partial_\nu h^{\rho\lambda} - \partial^\rho h_\mu{}^\lambda\partial_\rho h_{\lambda\nu} + \partial^\lambda h^\rho{}_{(\mu}\partial_\rho h_{\nu)\lambda}$$
$$+2\partial^\lambda h^\rho{}_{(\mu}\partial_{\nu)}h_{\rho\lambda} - \partial_{(\mu}h_{\nu)\lambda}\partial^\lambda h - \partial^\lambda h_{\mu\nu}\partial^\rho h_{\rho\lambda}$$
$$-\partial^\lambda h_{\lambda(\mu}\partial_{\nu)}h + \partial_\lambda h_{\mu\nu}\partial^\lambda h + \tfrac{1}{2}\partial_\mu h\partial_\nu h + \eta_{\mu\nu}\mathcal{L}_{\rm FP}\Big]$$

Eqns of motion: $\mathcal{H}_{\mu\nu} = \kappa\mathcal{T}_{\mu\nu}(h) + \kappa T_{\mu\nu}(\phi)$

$$\partial_\mu\partial^\mu\phi = \kappa\partial_\mu\Big[h^{\mu\nu}\partial_\nu\phi - \tfrac{1}{2}h\partial^m u\phi\Big]$$

Conservation of energy: $\partial^\mu\Big[\mathcal{T}_{\mu\nu}(h) + T_{\mu\nu}(\phi)\Big] = 0$

- Consistent result upto order κ
- Right values for perihelium of Mercury

  $\longrightarrow$ previous deficit due to neglecting self-interaction
- What about order $\kappa^2, \ \kappa^3, \ ... \ ?$

  $\longrightarrow$ Extremely tedious!!

- Consistent result upto order κ
- Right values for perihelium of Mercury

  $\longrightarrow$ previous deficit due to neglecting self-interaction
- What about order $\kappa^2, \ \kappa^3, \ ... \ ?$

  $\longrightarrow$ Extremely tedious!!

WE NEED A MORE GENERAL ARGUMENT!

5 Deser's argument & General Relativity

$$\mathcal{L}_0 = -\kappa\, \bar{h}_{\mu\nu}\Big[\partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu}\Big] \; + \eta^{\mu\nu}\Big[\Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\rho}\Big]$$

with $\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h$

$\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$

Invariant under: $\delta h_{\mu\nu} = \partial_\mu \chi_\nu + \partial_\nu \chi_\mu$

$\delta \Gamma^\rho_{\mu\nu} = \partial_\mu \partial_\nu \chi^\rho$

Eqns of motion: $[\bar{h}_{\mu\nu}] : \partial_\rho \Gamma^\rho_{\mu\nu} - \tfrac{1}{2}\partial_\mu \Gamma^\rho_{\nu\rho} - \tfrac{1}{2}\partial_\mu \Gamma^\rho_{\nu\rho} = 0$

$$[\Gamma^\rho_{\mu\nu}] : -\kappa\, \partial_\rho \bar{h}^{\mu\nu} + \kappa\, \partial_\lambda \bar{h}^{\lambda(\mu}\delta^{\nu)}_\rho + 2\eta^{\lambda(\mu}\Gamma^{\nu)}_{\rho\lambda} - \eta^{\lambda\sigma}\Gamma^{(\mu}_{\lambda\sigma}\delta^{\nu)}_\rho - \eta^{\mu\nu}\Gamma^\lambda_{\rho\lambda} = 0$$

$[\Gamma^\rho_{\mu\nu}]$ is algebraic eqn for $\Gamma^\rho_{\mu\nu}$ $\longrightarrow$ Constraint on $\Gamma^\rho_{\mu\nu}$

Constraint on $\Gamma_{\mu\nu}^{\rho}$:

$$\Gamma_{\mu\nu}^{\rho} = \tfrac{1}{2}\kappa\, \eta^{\rho\lambda}\Big[\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\Big]$$

$\longrightarrow$ $\Gamma_{\mu\nu}^{\rho}$ is not independent field

($\Gamma_{\mu\nu}^{\rho}$ is 1st order of Christoffel symbol for $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$)

Constraint on $\Gamma^\rho_{\mu\nu}$:

$$\Gamma^\rho_{\mu\nu} = \tfrac{1}{2}\kappa\, \eta^{\rho\lambda}\Big[\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\Big]$$

$\longrightarrow$ $\Gamma^\rho_{\mu\nu}$ is not independent field

($\Gamma^\rho_{\mu\nu}$ is 1st order of Christoffel symbol for $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$)

Substitute in $[\bar{h}_{\mu\nu}]$:

$$\partial^2 h_{\mu\nu} + \partial_\mu\partial_\nu - \partial^\rho\partial_\mu h_{\nu\rho} - \partial^\rho\partial_\nu h_{\mu\rho} = 0$$

$$\Longleftrightarrow \quad \mathcal{H}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\mathcal{H}_\rho{}^\rho = 0$$

(1st order of $R_{\mu\nu} = 0$ for $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$)

$\Longrightarrow$ $\mathcal{L}_0$ is equivalent to $\mathcal{L}_{\rm FP}$

We want to find correction to $\mathcal{L}_0$ which takes in account the self-interaction of $h_{\mu\nu}$, such that $\mathcal{H}_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu}$

$$\mathcal{L}_1 = -\kappa\, \bar{h}_{\mu\nu} \Big[\partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu} \Big] \; + (\eta^{\mu\nu} - \kappa h^{\mu\nu}) \Big[\Gamma^\lambda_{\mu\rho} \Gamma^\rho_{\nu\lambda} - \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\rho} \Big]$$

Eqns of motion:

$$[\bar{h}_{\mu\nu}]: \; R_{\mu\nu}(\Gamma) = 0$$

$$\begin{aligned}[\Gamma^\rho_{\mu\nu}]: \quad & -\kappa\, \partial_\rho \bar{h}^{\mu\nu} + \kappa\, \partial_\lambda \bar{h}^{\lambda(\mu} \delta^{\nu)}_\rho + 2(\eta^{\lambda(\mu} - \kappa \bar{h}^{\lambda(\mu}) \Gamma^{\nu)}_{\rho\lambda} \\ & -(\eta^{\lambda\sigma} - \kappa \bar{h}^{\lambda\sigma}) \Gamma^{(\mu}_{\lambda\sigma} \delta^{\nu)}_\rho - (\eta^{\mu\nu} - \kappa \bar{h}^{\mu\nu}) \Gamma^\lambda_{\rho\lambda} = 0\end{aligned}$$

We want to find correction to $\mathcal{L}_0$ which takes in account the self-interaction of $h_{\mu\nu}$, such that $\mathcal{H}_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu}$

$$\mathcal{L}_1 = -\kappa\, \bar{h}_{\mu\nu} \Big[\partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu} \Big] \; + (\eta^{\mu\nu} - \kappa h^{\mu\nu}) \Big[\Gamma^\lambda_{\mu\rho} \Gamma^\rho_{\nu\lambda} - \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\rho} \Big]$$

Eqns of motion:

$$[\bar{h}_{\mu\nu}]: \;\; R_{\mu\nu}(\Gamma) = 0$$

$$\begin{aligned} [\Gamma^\rho_{\mu\nu}]: \;\; & -\kappa\, \partial_\rho \bar{h}^{\mu\nu} + \kappa\, \partial_\lambda \bar{h}^{\lambda(\mu} \delta^{\nu)}_\rho + 2(\eta^{\lambda(\mu} - \kappa \bar{h}^{\lambda(\mu}) \Gamma^{\nu)}_{\rho\lambda} \\ & -(\eta^{\lambda\sigma} - \kappa \bar{h}^{\lambda\sigma}) \Gamma^{(\mu}_{\lambda\sigma} \delta^{\nu)}_\rho - (\eta^{\mu\nu} - \kappa \bar{h}^{\mu\nu}) \Gamma^\lambda_{\rho\lambda} = 0 \end{aligned}$$

- Correction term has no $\eta_{\mu\nu} \Longrightarrow$ No contribution to $\mathcal{T}_{\mu\nu}(h)$

  No further correction needed

We want to find correction to $\mathcal{L}_0$ which takes in account the self-interaction of $h_{\mu\nu}$, such that $\mathcal{H}_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu}$

$$\mathcal{L}_1 = -\kappa\, \bar{h}_{\mu\nu}\Big[\partial_\mu \Gamma_{\nu\rho}^\rho - \partial_\rho \Gamma_{\mu\nu}^\rho\Big] \;+ (\eta^{\mu\nu} - \kappa h^{\mu\nu})\Big[\Gamma_{\mu\rho}^\lambda \Gamma_{\nu\lambda}^\rho - \Gamma_{\mu\nu}^\lambda \Gamma_{\lambda\rho}^\rho\Big]$$

Eqns of motion:

$$[\bar{h}_{\mu\nu}]:\;\; R_{\mu\nu}(\Gamma) = 0$$

$$\begin{aligned}[\Gamma_{\mu\nu}^\rho]:\;\; & -\kappa\, \partial_\rho \bar{h}^{\mu\nu} + \kappa\, \partial_\lambda \bar{h}^{\lambda(\mu}\delta_\rho^{\nu)} + 2(\eta^{\lambda(\mu} - \kappa \bar{h}^{\lambda(\mu})\Gamma_{\rho\lambda}^{\nu)} \\ & -(\eta^{\lambda\sigma} - \kappa \bar{h}^{\lambda\sigma})\Gamma_{\lambda\sigma}^{(\mu}\delta_\rho^{\nu)} - (\eta^{\mu\nu} - \kappa \bar{h}^{\mu\nu})\Gamma_{\rho\lambda}^\lambda = 0\end{aligned}$$

- Correction term has no $\eta_{\mu\nu} \Longrightarrow$ No contribution to $\mathcal{T}_{\mu\nu}(h)$
  No further correction needed
- $R_{\mu\nu}(\Gamma) = \partial_\rho \Gamma_{\mu\nu}^\rho - \partial_\mu \Gamma_{\nu\rho}^\rho + \Gamma_{\mu\nu}^\lambda \Gamma_{\lambda\rho}^\rho - \Gamma_{\mu\rho}^\lambda \Gamma_{\nu\lambda}^\rho$ is Ricci tensor
  (Surprise!) What is the conecction $\Gamma_{\mu\nu}^\rho$?

Define

$$\begin{cases} \eta^{\mu\nu} - \kappa\, \bar{h}^{\mu\nu} = \hat{g}^{\mu\nu} \\ \hat{g}_{\mu\nu}\, \hat{g}^{\nu\rho} = \delta_\mu^\rho \\ g_{\mu\nu} = \sqrt{|\hat{g}|}\, \hat{g}_{\mu\nu} \end{cases}$$

$\longrightarrow$ $g_{\mu\nu}$ non-linear combination of $h^{\mu\nu}$ (infinite series)

$\longrightarrow$ $g_{\mu\nu}$ acts as metric

Define

$$\begin{cases} \eta^{\mu\nu} - \kappa\, \bar{h}^{\mu\nu} = \hat{g}^{\mu\nu} \\ \hat{g}_{\mu\nu}\, \hat{g}^{\nu\rho} = \delta_\mu^\rho \\ g_{\mu\nu} = \sqrt{|\hat{g}|}\, \hat{g}_{\mu\nu} \end{cases}$$

$\longrightarrow$ $g_{\mu\nu}$ non-linear combination of $h^{\mu\nu}$ (infinite series)

$\longrightarrow$ $g_{\mu\nu}$ acts as metric

Constraint on $\Gamma_{\mu\nu}^\rho$:

$$\Gamma_{\mu\nu}^\rho = \tfrac{1}{2}\, g^{\rho\lambda}\Big[\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\Big]$$

$\longrightarrow$ $\Gamma_{\mu\nu}^\rho$ is Levi-Civita conection

$\longrightarrow$ $R_{\mu\nu}(\Gamma) = R_{\mu\nu}(g)$

$\longrightarrow$ $[\bar{h}_{\mu\nu}]$ is Einstein eqn $R_{\mu\nu} = 0$

Eqn of motion $R_{\mu\nu} = 0$ invariant under general coord transf

However $\mathcal{L}_1$ is not!

$$\longrightarrow \text{add total derivative } \eta_{\mu\nu}\Big[\partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu}\Big]$$

Eqn of motion $R_{\mu\nu} = 0$ invariant under general coord transf

However $\mathcal{L}_1$ is not!

$\longrightarrow$ add total derivative $\eta_{\mu\nu}\Big[\partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu}\Big]$

$$\Longrightarrow \mathcal{L}_2 = (\eta^{\mu\nu} - \kappa\, \bar{h}_{\mu\nu})\Big[\partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu}\Big] + (\eta^{\mu\nu} - \kappa h^{\mu\nu})\Big[\Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\rho}\Big]$$

$$= g^{\mu\nu}\Big[\partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu}\Big] + g^{\mu\nu}\Big[\Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\rho}\Big]$$

$$= g^{\mu\nu} R_{\mu\nu}(g)$$

$\longrightarrow$ Einstein-Hilbert action!

6 Conclusions

- Fierz-Pauli $\mathcal{L}_{\mathrm{FP}}$ is consistent as free theory
- Couplings to matter $\Longrightarrow$ self-coupling of gravity
  $\Longrightarrow$ inconsistent
- Possible (but difficult) to construct consistent theory upto order κ^2
- General Relativity is consistent extension to all orders (unique?)
- Suprising geometrical interpretation: non-linear function of $h^{\mu\nu}$ is metric
  $\longrightarrow$ very elegant & much richer structure