Lovelock Theories as extensions to General Relativity

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Abstract/resumen

In this thesis we will study Lovelock Theories, that is, some extensions to General Relativity with particularly good properties, for example, giving second-order differential equations and having Levi-Civita connection as a solution of first-order formalism. Despite their advantages, these theories had never been studied so deeply and in this thesis we will present several new results.

First of all, we explain basic concepts and set the mathematical base. In second chapter, we study the Einstein-Hilbert action. We will see that the solution to the metric-affine formalism is not only the Levi-Civita connection, but a set of connection that we will call Palatini connections. In third chapter, we talk about general properties of every Lovelock Theory, especially about projective invariance, which explains why Palatini connections are solutions of these theories. Finally, we study the Gauss-Bonnet action and we give a non-trivial solution of metric-affine formalism that is physically distinguishable of Levi-Civita, hence demonstrating the non-equivalence between metric and metric-affine formalisms.

En esta tesis se estudian las Teorías de Lovelock, unas extensiones a la Relatividad General con ciertas propiedades especialmente buenas, como por ejemplo, tener ecuaciones de movimiento de segundo orden y la conexión de Levi-Civita como solución al formalismo de primer orden. A pesar de sus ventajas, estas teorías nunca habían sido estudiadas tan a fondo y en esta tesis presentaremos varios resultados novedosos.

En primer lugar, se explican conceptos básicos y se sientan las bases matemáticas necesarias. En el segundo capítulo se estudia la acción de Einstein-Hilbert, donde veremos que la solución del formalismo métrico-afín no es únicamente Levi-Civita, sino otro conjunto de conexiones más general que llamaremos conexiones de Palatini. En el tercer capítulo se habla de propiedades globales de todas las Teorías de Lovelock, en especial de la invarianza proyectiva que da sentido a las conexiones de Palatini como solución de las Teorías de Lovelock. Por último, se estudia la acción de Gauss-Bonnet y se da una solución no trivial del formalismo métrico-afín que es físicamente distinguible de Levi-Civita, demostrando así la no equivalencia entre los formalismos métrico y métrico-afín.

Introduction

In 1915, ten years after publication of Special Relativity, Einstein proposed General Relativity. The purpose of this theory was to explain gravity while retaining compatibility with the new paradigm established by Special Relativity. This theory explained the effects of gravity in a totally different way as Newton's theory. While, in the latter, time and space were absolute and immutable, the former introduced the concept of space-time, modeled mathematically as a manifold.

This dynamics is governed by the so-called Einstein-Hilbert action, which gives the Einstein equations as equations of motion. These equations describe the detailed relation between curvature and energy-momentum distribution, and they constitute a system of partial differential equations, coupled to each other, non-linear and of second order. They are usually solved in situations with enough symmetry.

Despite the complexity of this theory, it led to lots of new research, taking into account new ideas, concepts and interpretations, perhaps black holes being the most famous ones, with the first black-hole solution of Einstein equations in 1916. After this solution proposed by Schwarzschild, several more appeared: Reissner-Nordström black hole (1916–1918), Einstein static universe (1917), de Sitter space (1917), Friedmann-Lemaître-Robinson-Walker solution (1922), Kerr black hole (1963), Kerr-Newman black hole (1965)... It also allowed to explain some post-Newtonian phenomena: Mercury's apsidal precession, gravitational lenses... other experiments like energy loss in binary pulsars and black holes, and the recent discovery of gravitational waves.

Furthermore, along the beginning of the 20$^{\text{th}}$ century, another theory appeared, partly based on Einstein research of photoelectric effect, indeed, and other observations that could not be explained with the classical theories. In the mid-1920s, Quantum Mechanics was formulated by Erwin Schrödinger, Werner Heisenberg, Max Born and others. As with General Relativity, this resulted in a new paradigm in physics.

Nowadays, these theories have greatly evolved with the aim, to a certain degree, of unifying all fundamental physics, that is, electromagnetism, gravitation, strong interaction and weak interaction. We would like to be able to explain all effects of physics using a single theory. In fact, unification has been almost achieved in the quantum paradigm with all of them except gravitation, thanks to the Standard Model.

When trying to unify gravitation with these theories, problems appear due to the different nature of gravitational theory with respect to the others. While gravitation is a geometrical theory, the others are gauge quantum theories. If we try to quantize gravitation, we get a non-renormalizable theory.

Gravitation also predicts its own limitations as irregularities—we know it is not the final answer. We work towards a formulation of quantum gravity that evaporates singularities and solve other problems with black holes, for instance. There are three popular approaches to this goal:

First one is directly looking for a theory that includes each one of the previously mentioned in a particular limit. If found, that would resolve everything, we would have the theory that includes every other, and all known physics (and potentially more) would be included in that one, until an experiment went out of that theory and led to new physics again. This is what String Theory and Loop Quantum Gravity try to do, for example. The problem with this strategy is that we expect those theories to become measurable at Plank length, fifteen orders of magnitude above what we can measure today in our best particle accelerator, so we do not have any evidence of how these theories should be.

This reason leads to the second approach: exploring extensions to Quantum Physics, which we can understand as low-energy corrections of that hypothetical quantum gravity theory, more accessible. If we explore extensions of that theory, we will not find the final answer, but we can get a larger set of physically acceptable theories, and so people looking for the big theory mentioned before have a greater set of physics to land in. For instance, physics beyond standard model experiments with this.

Finally, the third approach is as the second, but applied to General Relativity. There are a lot of possible theories for extending General Relativity, for example, $f(R)$-gravities, scalar-tensor theories, Ricci-based gravities or Lovelock Theories. Each of them has its advantages, but we will focus on the latter in this thesis. Lovelock Theories are a set of corrections of General Relativity with a lot of good properties that make them very good candidates for being physically acceptable. They appeared when trying to generalize the Einstein-Hilbert action, being the obvious way adding terms of quadratic order (or more) in the curvature. This, however, introduces a problem: equations of motion get derivatives of order greater than two, causing ghost solutions to appear. A particular combination that gave rise to second-order differential equations was already known by Lanczos in 1938, the Gauss-Bonnet term, and this was later generalized by Lovelock in ref. [31] for any order in the curvature: these are the Lovelock Theories—at zeroth and first order being cosmological constant and Einstein-Hilbert actions, respectively. It was shown in 1980, in refs. [52, 51], that although these corrections in higher orders of the curvature appear naturally in String Theory, they would engender ghosts if not in the exact combinations described by Lovelock.

Another property of Lovelock Theories is that equations of motion derived using first- and second-order formalisms lead to almost equivalent dynamics—completely equivalent for Einstein-Hilbert action, as we will see in this thesis—and this serves as a justification for using Levi-Civita connection (see refs. [5, 16, 13]), as it appears as a solution of the equations of motion.

When we talk about these two formalisms, we mean the method that we use to obtain the equations of motion from the action. They differ in which variables they take as fundamental. On the one hand, when we use the first-order formalism, connection and metric are both degrees of freedom of the theory and so one have to take the variation of the action with respect to each of them to get both equations of motion. On the other hand, when we use the second-order formalism, the connection is written in terms of the metric and only the metric is computed from its equations of motion. The latter is the traditional formalism, presented in General Relativity, where one assumes always that the connection is the Levi-Civita connection, while the former was first done by Einstein in 1925 in ref. [15] and it is gathering some research impetus in different areas. First-order formalism has mainly two advantages: the equations of motion are much easier to compute and, as I said before, Levi-Civita connection shows up as a solution of its equations of motion, not as an assumption.

Lovelock Theories have been studied as the natural extensions to GR for some years. They show up as corrections of String Theory to supergravity actions (see ref. [7]) and, in ADS/CFT, Gauss-Bonnet term is used for computing corrections for the viscosity in holographic hydrodinamics (see ref. [6]).

Moreover, in gravitation, they seem so natural that there is no reason for not adding Lovelock terms of higher order in any dynamic theory of gravity. In four dimensions, all terms are zero except zeroth (cosmological constant), first (Einstein-Hilbert) and second (Gauss-Bonnet), and this last one is topological, it does not contribute to dynamics, so we end up with the same action. In five dimensions or more, this term becomes dynamic, it contributes to the equations of motion, and we should add it to our action. Similarly, Lovelock term of order n is zero in a dimension $D < 2n$, topological in $D = 2n$ and dynamic in $D > 2n$. For this very reason, Lovelock Theories seem to be corrections in extra dimensions, and they actually appear as low-energy actions of String Theory, living in ten or eleven dimensions. We know, since the 1920s, how to get effective theories in a given number of dimensions from theories living in more, called dimensional reduction, so we could finish up with truly measurable effects.

This thesis is structured as follows. In Chapter 1 we will introduce first-and second-order formalisms and point out the differences between them. In Chapter 2 we will obtain the equations of motion corresponding to Einstein-Hilbert action using both formalisms and compare the solutions. In Chapter 3 we will discuss some aspects about Lovelock Theories in general and the projective symmetry in particular. Finally, in Chapter 4 we will find a solution of Gauss-Bonnet in the first-order formalism and discuss its implications.

Chapter 1

First- and second-order formalisms

The introduction of General Relativity ten years after Special Relativity was groundbreaking. The purpose of this new theory was to include gravitation in the new paradigm established by Special Relativity. To achieve that, Einstein changed from an immutable time and space to a dynamic space-time modelled by a manifold, where its curvature is related to the distribution of energy.

We can intuitively think of a manifold as a sheet floating in space. It is not normally flat, it becomes curved influenced by different circumstances, but if you look at it near enough, you can approximate it very well with its tangent plane at every point (see figure 1.1). We call this the *tangent space* at a point p, $T_p(\mathcal{M})$. This is a vector space where all vector quantities related with our manifold at the point p actually live, and its dimension coincides with what we call the dimension of the manifold. The set of all tangent spaces is called *tangent bundle*, $T(\mathcal{M})$.

The fact that tangent spaces are a local approximation of the manifold is closely related with the equivalence principle, the base of General Relativity. It says that *it is impossible to distinguish, only with local experiments, between an inertial observer in vacuum and another one in free fall in a gravitational field.* Essentially, that equivalence is related to the fact that we can take an orthonormal basis in any tangent space and approximate the manifold locally with it.

We also have coordinates in a manifold. They are given by the fact that we have a transformation from a neighbourhood of every point of the manifold to $\mathbb{R}^N$, so we can use any coordinates in $\mathbb{R}^N$ to build others in $\mathcal{M}$.

One more essential structure that we have in a manifold is a topology, which gives us a notion of neighbourhood or proximity. However, if we want to actually measure distances between points and angles between vectors (in the same

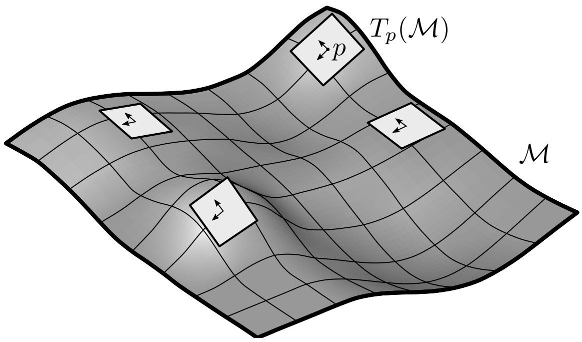

Figure 1.1: Manifold with some of its tangent spaces in several points.

tangent space), we need an additional mathematical structure. This structure is called *metric*, $g_{\mu\nu}$, and it may or may not be present in a manifold because it is not fundamentally required. In this thesis, nevertheless, we will always work with manifolds equipped with a metric.

1.1 Metric

As we are going to work with pseudo-Riemannian manifolds, we have to choose between two conventions for the signature of the metric: *mostly plus* and *mostly minus*. We will follow the mostly minus convention along all this thesis, giving to all the spatial lengths a negative character, and a positive one to the temporal ones. For example, in $D = 4$ Minkowski space, the metric would be

$$\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{1.1}$$

However, in general, we will call the metric g.

Once we have a pair $(\mathcal{M}, g)$, we can transform vectors to their dual space,

$$V^\sigma g_{\mu\sigma} = V_\mu, \tag{1.2}$$

or, analogously, with the inverse metric, change the type of any tensor, for example, from a $(0,2)$-type tensor to a $(1,1)$-type one,

$$T_{\mu\sigma} g^{\nu\sigma} = T_\mu{}^\nu. \tag{1.3}$$

A metric also allows us to compute the norm of a vector V,

$$|V| = \sqrt{V_\sigma V^\sigma} = \sqrt{g_{\sigma\eta} V^\sigma V^\eta} \tag{1.4}$$

and measure lengths of paths as

$$l = \int_\gamma \sqrt{g_{\sigma\eta} \dot{x}^\sigma \dot{x}^\eta} d\tau, \tag{1.5}$$

where x^μ is our set of coordinates and the dot means derivation with respect to the parameter of the curve, τ. We can also quantify angles between vectors in the same tangent space,

$$\alpha = \arccos \frac{g_{\sigma\eta} V^\sigma W^\eta}{\sqrt{g_{\theta\xi} V^\theta V^\xi g_{\varphi\chi} W^\varphi W^\chi}}, \tag{1.6}$$

and we could even calculate the shortest line between two points by taking the minimum of the distance between them as a functional of the trajectory γ,

$$s = \int_0^1 \sqrt{g_{\sigma\eta}(\gamma) \dot{\gamma}^\sigma \dot{\gamma}^\eta} d\tau, \tag{1.7}$$

where the parameter of the curve is again τ, $\gamma = \gamma(\tau)$, so that $\gamma(0) = p_1$ and $\gamma(1) = p_2$. This is what we call a *metric geodesic*, and it will play a very important role, as test particles follow geodesics in General Relativity. Metric geodesics are the generalization of straight lines for curved spaces in the way that they are the shortest curves between two given points. Let's compute them.

Taking the extremum of (1.7) using the Euler-Lagrange equation (we will use x^μ for the coordinates of the curve, for simplicity), we get

$$\frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0, \tag{1.8}$$

where $L = \sqrt{g_{\sigma\eta} \dot{x}^\sigma \dot{x}^\eta}$. Now we can multiply by $2L$ to get rid of the square root and make things easier,

$$\frac{d}{d\tau} \frac{\partial L^2}{\partial \dot{x}^\mu} - \frac{\partial L^2}{\partial x^\mu} = 2 \frac{dL}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu}. \tag{1.9}$$

If we focus on the first member of the equation,

$$\begin{aligned} \frac{d}{d\tau} \frac{\partial L^2}{\partial \dot{x}^\mu} - \frac{\partial L^2}{\partial x^\mu} &= \frac{d}{d\tau} \left(2 g_{\mu\sigma} \dot{x}^\sigma \right) - \dot{x}^\sigma \dot{x}^\eta \partial_\mu g_{\sigma\eta} \\ &= 2 \dot{x}^\sigma \dot{x}^\eta \partial_\eta g_{\mu\sigma} + 2 g_{\mu\sigma} \ddot{x}^\sigma - \dot{x}^\sigma \dot{x}^\eta \partial_\mu g_{\sigma\eta} \\ &= 2 g_{\mu\sigma} \ddot{x}^\sigma + \dot{x}^\sigma \dot{x}^\eta \left(\partial_\eta g_{\mu\sigma} + \partial_\sigma g_{\mu\eta} - \partial_\mu g_{\sigma\eta} \right) \\ &= 2 g_{\mu\sigma} \ddot{x}^\sigma + 2 g_{\mu\theta} \left\{ {\theta \atop \sigma\eta} \right\} \dot{x}^\sigma \dot{x}^\eta, \end{aligned} \tag{1.10}$$

where we have defined the *Christoffel symbols* as

$$\left\{ {\rho \atop \mu\nu} \right\} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right). \tag{1.11}$$

If we now operate on the second member of (1.9),

$$\begin{aligned} 2 \frac{\partial L}{\partial \dot{x}^\mu} \frac{dL}{d\tau} &= 2 \frac{g_{\mu\sigma} \dot{x}^\sigma}{L} \frac{d\dot{s}}{d\tau} \\ &= 2 g_{\mu\sigma} \dot{x}^\sigma \frac{\ddot{s}}{\dot{s}}, \end{aligned} \tag{1.12}$$

where we have used (1.7),

$$L = \frac{ds}{d\tau} = \dot{s}. \tag{1.13}$$

So, finally, rising the μ index, the equation of the metric geodesic is

$$\ddot{x}^\mu + \left\{ {\mu \atop \sigma\eta} \right\} \dot{x}^\sigma \dot{x}^\eta = \frac{\ddot{s}}{\dot{s}} \dot{x}^\mu. \tag{1.14}$$

We could choose the canonical parameterization, where the parameter is the curve length, $\tau = s$,

$$\ddot{x}^\mu + \left\{ {\mu \atop \sigma\eta} \right\} \dot{x}^\sigma \dot{x}^\eta = 0, \tag{1.15}$$

to get this simpler and, in some cases, more useful equation, getting rid of the last term.

As a quick introduction, that is the usefulness of a metric: measuring distances and norms of vectors. Still, there are other quantities that we cannot measure with a metric, for example, the curvature of the manifold. We will need another tool (not necessarily in combination with the metric) for computing them: the affine connection.

1.2 Affine connection

Apart from the metric, we can have other tools needed for other operations. For instance, even if we had a metric, we would not be able to compare the angle between vectors at different points of the manifold. For accomplishing that, we would need another instrument for translating vectors from one tangent space to another, that is, for translating vectors without changing their direction. We call this procedure *parallel transport*, and it is carried out by an *affine connection*. Once we have it in our manifold, $(\mathcal{M}, \Gamma)$, we can transport vectors along curves in the manifold, keeping them parallel to themselves by solving

$$\dot{\gamma}^\sigma \nabla_\sigma V^\mu = 0, \tag{1.16}$$

where $\dot{\gamma}$ is the vector tangent to the curve and the covariant derivative ∇ is defined as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\sigma}{}^\nu V^\sigma, \tag{1.17a}$$

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma_{\mu\nu}{}^\sigma V_\sigma, \tag{1.17b}$$

being Γ the affine connection. This derivative transforms as a tensor under general coordinate transformations thanks to the non-tensorial character of the affine connection, that counteracts the one coming from the partial derivative. This fact gives rise to (1.17) for vectors, but it is different when acting on scalars,

$$\nabla_\mu \phi = \partial_\mu \phi, \tag{1.18}$$

or, in general, on tensors,

$$\begin{aligned}\nabla_\mu S^{\nu_1\ldots\nu_n}{}_{\rho_1\ldots\rho_m} &= \partial_\mu S^{\nu_1\ldots\nu_n}{}_{\rho_1\ldots\rho_m} \\ &+ \Gamma_{\mu\sigma}{}^{\nu_1} S^{\sigma\nu_2\ldots\nu_n}{}_{\rho_1\ldots\rho_m} + \cdots + \Gamma_{\mu\sigma}{}^{\nu_n} S^{\nu_1\ldots\nu_{n-1}\sigma}{}_{\rho_1\ldots\rho_m} \\ &- \Gamma_{\mu\rho_1}{}^{\sigma} S^{\nu_1\ldots\nu_n}{}_{\sigma\rho_2\ldots\rho_m} - \cdots - \Gamma_{\mu\rho_m}{}^{\sigma} S^{\nu_1\ldots\nu_n}{}_{\rho_1\ldots\rho_{m-1}\sigma}.\end{aligned} \tag{1.19}$$

If we start transporting vectors, we can get some surprises. For example, let's take a two-dimensional sphere, $\mathbb{S}^2$. There, we pick a vector in the equator (vector V in figure 1.2) and transport it to the pole along two different curves: on the one hand, directly along a meridian, while on the other hand, we first move along the equator line and then to the pole. We will end up with two different vectors, even though they were parallel to the first. This is a consequence of the curvature of the manifold: parallelism depends on the trajectory followed.

Thanks to this phenomena, we can measure the curvature of a space without having to embed it in a bigger one. The *curvature tensor* (also called *Riemann tensor*) is given by the connection as

$$R_{\mu\nu\rho}{}^\lambda = \partial_\mu \Gamma_{\nu\rho}{}^\lambda - \partial_\nu \Gamma_{\mu\rho}{}^\lambda + \Gamma_{\mu\sigma}{}^\lambda \Gamma_{\nu\rho}{}^\sigma - \Gamma_{\nu\sigma}{}^\lambda \Gamma_{\mu\rho}{}^\sigma. \tag{1.20}$$

Having a general connection, not the particular case of Levi-Civita that we will discuss later, the only symmetry this tensor has is an antisymmetry in the first two indices,

$$R_{\mu\nu\rho\lambda} = -R_{\nu\mu\rho\lambda}. \tag{1.21}$$

Due to that, it has three independent contractions,

$$R_{\mu\nu} = g^{\sigma\eta} R_{\mu\sigma\nu\eta}, \qquad \tilde{R}_{\mu\nu} = g^{\sigma\eta} R_{\mu\sigma\eta\nu}, \qquad \bar{R}_{\mu\nu} = g^{\sigma\eta} R_{\mu\nu\sigma\eta}, \tag{1.22}$$

that we will call the *Ricci tensor*, the *co-Ricci tensor* and the *a-Ricci tensor* respectively. The most useful one is the Ricci tensor, that can be computed directly as

$$R_{\mu\nu} = R_{\mu\sigma\nu}{}^\sigma = \partial_\mu \Gamma_{\sigma\nu}{}^\sigma - \partial_\sigma \Gamma_{\mu\nu}{}^\sigma + \Gamma_{\mu\eta}{}^\sigma \Gamma_{\sigma\nu}{}^\eta - \Gamma_{\mu\nu}{}^\sigma \Gamma_{\eta\sigma}{}^\eta. \tag{1.23}$$

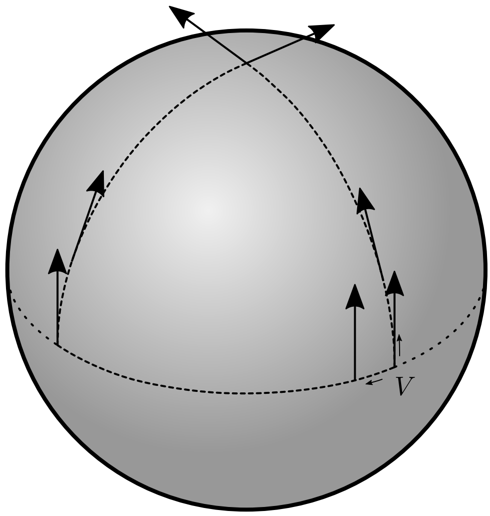

Figure 1.2: Transport of vector V along two paths (indicated with small arrows) to the pole of a sphere.

Contracting all these tensors (1.22), one can only find a single scalar, the *curvature scalar* (also called *Ricci scalar*),

$$R = g^{\sigma\eta} R_{\sigma\eta}. \tag{1.24}$$

These contractions of the Riemann tensor are so important because they are the quantities physically given by dynamics. We will see that in more detail later on.

There are other two important quantities related with the affine connection that get a special name: the *torsion* tensor,

$$T_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho}, \tag{1.25}$$

and the *non-metricity* tensor,

$$Q_{\mu\nu\rho} = -\nabla_{\mu} g_{\nu\rho}. \tag{1.26}$$

And there is a relation between the curvature tensor, the torsion and the commutator of two covariant derivatives acting on scalars,

$$[\nabla_{\mu}, \nabla_{\nu}]\, \phi = -T_{\mu\nu}{}^{\sigma} \nabla_{\sigma} \phi, \tag{1.27}$$

vectors,

$$[\nabla_{\mu}, \nabla_{\nu}]\, V^{\rho} = R_{\mu\nu\sigma}{}^{\rho} V^{\sigma} - T_{\mu\nu}{}^{\sigma} \nabla_{\sigma} V^{\rho}, \tag{1.28a}$$

$$[\nabla_{\mu}, \nabla_{\nu}]\, V_{\rho} = -R_{\mu\nu\rho}{}^{\sigma} V_{\sigma} - T_{\mu\nu}{}^{\sigma} \nabla_{\sigma} V_{\rho}, \tag{1.28b}$$

and, in general, over (n, m)-type tensors,

$$\begin{aligned}[\nabla_\mu, \nabla_\nu]\, S^{\rho_1\cdots\rho_n}{}_{\lambda_1\ldots\lambda_m} &= R_{\mu\nu\sigma}{}^{\rho_1} S^{\sigma\rho_2\cdots\rho_n}{}_{\lambda_1\ldots\lambda_m} + \cdots + R_{\mu\nu\sigma}{}^{\rho_n} S^{\rho_1\cdots\rho_{n-1}\sigma}{}_{\lambda_1\ldots\lambda_m} \\ &- R_{\mu\nu\lambda_1}{}^{\sigma} S^{\rho_1\cdots\rho_n}{}_{\sigma\lambda_2\ldots\lambda_m} - \cdots - R_{\mu\nu\lambda_m}{}^{\sigma} S^{\rho_1\cdots\rho_n}{}_{\lambda_1\ldots\lambda_{m-1}\sigma} \\ &- T_{\mu\nu}{}^{\sigma} \nabla_\sigma S^{\rho_1\cdots\rho_n}{}_{\lambda_1\ldots\lambda_m}, \end{aligned} \tag{1.29}$$

The affine connection also gives us one more thing: the other type of geodesics. The *affine geodesics* are characterized as the paths whose tangent vector are parallel. For calculating its expression, we have to impose that the tangent vector of the curve is transported parallel (1.16) along the own curve,

$$\begin{aligned} 0 &= \dot{\gamma}^\sigma \nabla_\sigma \dot{\gamma}^\mu \\ &= \ddot{\gamma}^\mu + \Gamma_{\sigma\eta}{}^{\mu} \dot{\gamma}^\eta \dot{\gamma}^\sigma, \end{aligned} \tag{1.30}$$

where we have used the chain rule,

$$\dot{\gamma}^\sigma \partial_\sigma \dot{\gamma}^\mu = \frac{d}{d\tau} \dot{\gamma}^\mu = \ddot{\gamma}^\mu. \tag{1.31}$$

It is very important to keep in mind that they do not necessarily coincide with metric geodesics. If we think of a straight line as we have always thought, we see that they have essentially two properties: they are the shortest paths between two given points, and they are the paths whose tangent vectors are all parallel. We can generalize these two properties, but, as we have different tools for measuring distances and parallelism, we will arrive at different results, in general. One remarkable exception to this is the Levi-Civita connection, for which both geodesics are the same. Actually, this connection has a lot of properties that make it very special, deserving to be studied in particular.

1.2.1 Levi-Civita connection

This specific choice of connection, which we will note with a ring $\mathring{\Gamma}$ but we will define later with precision, is not only very used by physicists but also by mathematicians because of several reasons. One of them is simplification, if we want to simplify the expressions involving torsion and non-metricity, we choose a connection with two properties: not having torsion,

$$0 = \mathring{T}_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} - \mathring{\Gamma}_{\nu\mu}{}^{\rho}, \tag{1.32}$$

and being metric compatible,

$$0 = \mathring{Q}_{\mu\nu\rho} = -\mathring{\nabla}_\mu g_{\nu\rho}. \tag{1.33}$$

This choice also simplifies the Riemann tensor (1.20), which gain more symmetries: antisymmetric in first two, last two, and symmetric by swapping first two and last two indices,

$$R_{\mu\nu\rho\lambda} = -R_{\nu\mu\rho\lambda}, \qquad R_{\mu\nu\rho\lambda} = -R_{\mu\nu\lambda\rho}, \qquad R_{\mu\nu\rho\lambda} = R_{\rho\lambda\mu\nu}. \tag{1.34}$$

Due to that, its only independent contraction (1.22) is the Ricci tensor, as the others are related with it or vanishes,

$$\tilde{R}_{\mu\nu} = -R_{\mu\nu}, \qquad \bar{R}_{\mu\nu} = 0. \tag{1.35}$$

It turns out that the only connection that fulfil those two conditions (torsionless and metric-compatible) is the Levi-Civita connection, and it is very easy to get its expression from them. Let's write (1.33) with cyclic permutations of the indices,

$$0 = \partial_\mu g_{\nu\rho} - \mathring{\Gamma}_{\mu\nu}{}^{\sigma} g_{\sigma\rho} - \mathring{\Gamma}_{\mu\rho}{}^{\sigma} g_{\nu\sigma}, \tag{1.36a}$$

$$0 = \partial_\nu g_{\rho\mu} - \mathring{\Gamma}_{\nu\rho}{}^{\sigma} g_{\sigma\mu} - \mathring{\Gamma}_{\nu\mu}{}^{\sigma} g_{\rho\sigma}, \tag{1.36b}$$

$$0 = \partial_\rho g_{\mu\nu} - \mathring{\Gamma}_{\rho\mu}{}^{\sigma} g_{\sigma\nu} - \mathring{\Gamma}_{\rho\nu}{}^{\sigma} g_{\mu\sigma}, \tag{1.36c}$$

and, if we add (1.36a) to (1.36b), and then subtract (1.36c), we end up with

$$0 = \partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} - \left(\mathring{\Gamma}_{\mu\nu}{}^{\sigma} + \mathring{\Gamma}_{\nu\mu}{}^{\sigma}\right) g_{\sigma\rho} + \mathring{T}_{\rho\mu}{}^{\sigma} g_{\sigma\nu} + \mathring{T}_{\rho\nu}{}^{\sigma} g_{\mu\sigma}, \tag{1.37}$$

where, if we use absence of torsion (1.32), we get

$$0 = \partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} - 2\mathring{\Gamma}_{\mu\nu}{}^{\sigma} g_{\sigma\rho}, \tag{1.38}$$

or, equivalently,

$$\mathring{\Gamma}_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right). \tag{1.39}$$

As we can see, this coincides with Christoffel symbols (1.11), which can be defined whether having a connection or not. As a consequence of this, metric (1.15) and affine (1.30) geodesics also coincide: this is another reason to choose the Levi-Civita connection.

Also, with the Levi-Civita connection we can easily fulfil the requirements of the equivalence principle because it can be set to zero at an arbitrary point, using the locally inertial coordinates.

The expression for the Levi-Civita connection (1.39) shows the fact that we can always construct the Levi-Civita connection corresponding to a metric. However, in principle, we are not forced to have a connection in our space, there exist metric manifolds, $(\mathcal{M}, g)$, without the notion of parallelism. However, in general, we will work with a metric-affine manifold, $(\mathcal{M}, g, \Gamma)$, where the connection either can be the Levi-Civita connection or it can be completely unrelated to the Levi-Civita connection of the chosen metric. We will have a look at the latter in the following section.

1.2.2 Other connections

If we choose a connection different from the Levi-Civita one, we have some strange phenomena—strange in the sense that we are not used to it. However, it is completely admissible from a mathematical point of view, so it deserves to be studied.

First of all, expressions involving torsion (1.25) and non-metricity (1.26) get harder, as they do not have to vanish, in general,

$$T_{\mu\nu}{}^{\rho} \neq 0, \qquad Q_{\mu\nu\rho} \neq 0. \tag{1.40}$$

It is important to notice that these two quantities are tensors, even though they involve the connection in their definitions, so they cannot be set to zero at a given point just by choosing appropriate coordinates, as we can do with the Levi-Civita connection. This leads to the first problem when dealing with general connections: the equivalence principle.

Basically, the equivalence principle tells us that we can choose some coordinates so that we can recover the dynamics of free particles locally. This is easy to carry out with metric geodesics (1.14), as we can always choose the correct parameterization (to cancel last term of the equation) and coordinates (to cancel Christoffel symbols) so that we end up with

$$\ddot{x} = 0, \tag{1.41}$$

recovering uniform linear motion.

However, with affine geodesics we can get in trouble. We will start by decomposing any connection in some useful parts to argue this,

$$\Gamma_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + \Xi_{\mu\nu}{}^{\rho}, \tag{1.42}$$

where we will call Ξ the *distortion* tensor. We can split the distortion in several different ways, for example, in its symmetric and antisymmetric parts,

$$\Xi_{\mu\nu}{}^{\rho} = \frac{1}{2}\left(S_{\mu\nu}{}^{\rho} + T_{\mu\nu}{}^{\rho}\right), \tag{1.43}$$

being T the torsion (1.25) and

$$S_{\mu\nu}{}^{\rho} = S_{\nu\mu}{}^{\rho} = \Xi_{\mu\nu}{}^{\rho} + \Xi_{\nu\mu}{}^{\rho}. \tag{1.44}$$

This decomposition can be put into the affine geodesic (1.30) to get

$$0 = \ddot{x}^{\mu} + \mathring{\Gamma}_{\sigma\eta}{}^{\mu}\dot{x}^{\sigma}\dot{x}^{\eta} + S_{\sigma\eta}{}^{\mu}\dot{x}^{\sigma}\dot{x}^{\eta} + T_{\sigma\eta}{}^{\mu}\dot{x}^{\sigma}\dot{x}^{\eta}, \tag{1.45}$$

where the last addend is zero because the torsion is antisymmetric and contracted with something symmetric. If we choose appropriate coordinates, we can again cancel out Levi-Civita connection, but we end up with

$$0 = \ddot{x}^\mu + S_{\sigma\eta}{}^\mu \dot{x}^\sigma \dot{x}^\eta, \tag{1.46}$$

where the last term is proportional to the tensor S, that cannot be set to zero at an arbitrary point with a change of coordinates due to its tensorial nature. Then, we seem to have a problem with the equivalence principle and arbitrary connections. Although questioning the equivalence principle has also been an object of research [14], in our cases, these problems disappear thanks to a reparameterization of the geodesics, as we will see in Chapter 2.

A new phenomenon is that the norm of a vector is not conserved when transporting along a curve, even if the vector itself does not change. This is straightforward as the non-metricity is not equal to zero and the norm depends on the metric,

$$\dot{\gamma}^\sigma \nabla_\sigma |V| = \dot{\gamma}^\sigma \nabla_\sigma \sqrt{g_{\eta\theta} V^\eta V^\theta} = \frac{1}{2\sqrt{g_{\eta\theta} V^\eta V^\theta}} V^\eta V^\theta \dot{\gamma}^\sigma \nabla_\sigma g_{\eta\theta}. \tag{1.47}$$

Other uncommon phenomena appear when we choose connection and metric independently. For instance, we can define a two-dimensional surface whose metric is the metric of a plane but whose connection is the one corresponding to a sphere in stereographic projection. Then, we would have that all vectors in figure 1.3 are parallel, so that the curvature of the manifold is the one of a sphere. However, the pole of the sphere would be at infinite distance, as we are using the metric of the plane for measuring distances.

As we mentioned before, due to the completely different nature of metric and affine geodesics, when choosing general connections they do not coincide. This can be a problem, as we would have to choose between one of them when postulating the movement of a test particle. Both of them derive from very well established physical principles: metric geodesics from the principle of least action, and affine geodesics as the generalization of Newton's second law. Rejecting any of them would be a problem, so it would be desirable that they coincide somehow. We will get more deeply into this in Chapter 2.

1.3 Second-order formalism

Now that we have a notion of what a connection is, let's start introducing what we call the second-order formalism.

In General Relativity, as in many theories, we apply the principle of least action to get the equations of motion of our system. We have a manifold equipped

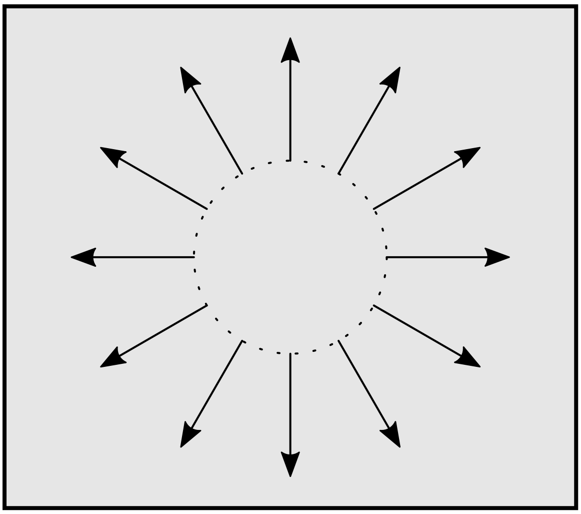

Figure 1.3: Plane with the affine connection of a sphere: all vectors are parallel transported along the dashed line.

with a metric and the Levi-Civita connection, being both dynamic, so the curvature is determined by the distribution of energy-momentum. The action, hence, will be of the form

$$S = \int d^4x \mathcal{L}(g, \Gamma(g)). \tag{1.48}$$

Let's get a closer look of how we obtain equations of motion. In the action, we substitute the curvature tensors in terms of the metric and the affine connection, using (1.24) and (1.23). After that, as we chose the Levi-Civita connection, we write it in terms of the metric (1.39). Thus, we have an action that only depends on the metric, and we can use Euler-Lagrange formula to get the conditions of extrema,

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad \Leftrightarrow \quad \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \partial_\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\sigma g^{\mu\nu})} + \partial_\eta \left(\partial_\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\eta \partial_\sigma g^{\mu\nu})} \right) = 0, \tag{1.49}$$

this is what we call the *second-order formalism* or *metric formalism.*

It is worth noticing that, at first glance at (1.49) and having terms in the action that are quadratic in second-order derivatives of the metric, one would expect to get equations of motion with derivatives of order greater than two. That, though, does not happen because, for this particular action, the sum of all problematic terms vanishes. This does not occur by chance, Einstein-Hilbert action is a particular case of Lovelock Theories and, as we will see in Chapter 3, they share this property.

The second-order formalism is very widely used, as it was the original procedure when General Relativity was developed and so it plays a role on its success. Despite that, mathematically speaking, metric and affine connection are not necessarily related, so it is more desirable to have a physical mechanism that naturally selects Levi-Civita connection among others. Here is where first-order formalism comes into play.

1.4 First-order formalism

In *first-order formalism*, also called *metric-affine formalism*, one does not assume any relation between the metric and the connection: they are both degrees of freedom of the theory. Accordingly, to get the extrema of the action, now it is necessary to calculate the variation of the action with respect to both the metric and the connection,

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0, \tag{1.50a}$$

$$\frac{\delta S}{\delta \Gamma_{\mu\nu}{}^{\rho}} = 0. \tag{1.50b}$$

This formalism was first done by Einstein in 1925 in [15], although it is wrongly called *Palatini formalism* sometimes [17], and it has been applied to several fields in and out of gravitation. For example, solid-state physics using torsion and non-metricity as a way to describe defects in crystals.

It has been gathering some research impetus for two decades, as we can see in the reviews [19, 43, 36], being proposed as a more simple solution to explore for solving dark matter and dark energy problems in cosmology [35] and different extensions of gravity as $f(R)$-gravity [34, 9, 10].

From a practical point of view, the first advantage of this formalism is that equations of motion are much easier to compute, as there are not any derivatives of the metric in the action anymore—they all came from Levi-Civita connection. Hence, Euler-Lagrange equation gets simplified to one single term and Einstein equation is easily obtained from (1.50a) as

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad \Leftrightarrow \quad \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} = 0. \tag{1.51}$$

On the other hand, from a theoretical point of view, the advantage of this formalism is that we can now justify the use of Levi-Civita connection if it is a solution of (1.50b), as done in [5, 16, 13]. In fact, we could go further: we can take this formalism as fundamental and see if Levi-Civita connection is the only solution of this, so its use is completely justified, or if actually there are other solutions that deserve the same status. We will go deeper on this topic in Chapter 2 and Chapter 4.

If first- and second-order formalism gave different results, we would have to choose between one of them, and that is not desirable. We will see in Chapter 2 that, for the Einstein-Hilbert action, they are completely equivalent.

Chapter 2

The Einstein-Hilbert action

At the beginning, when Einstein proposed General Relativity, he showed that the dynamics of the system was governed by his equations of motion,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa \mathcal{T}_{\mu\nu}, \tag{2.1}$$

without having an action from which to derive them. At about the same time, Hilbert presented the Einstein-Hilbert action as the base of the theory. Although it was in four dimensions, we will generalize it for any number of dimensions from now on,

$$S = \frac{1}{2\kappa} \int d^D x \sqrt{|g|} R, \tag{2.2}$$

where R is the curvature scalar introduced in (1.24) and $\kappa = 8\pi G$, with G the gravitational constant.

From this action, it is possible to derive the homogeneous Einstein equations. As we will see later, we can also add interaction terms and get the Einstein equations (2.1). This is traditionally done using the second-order formalism, the one introduced with this action, as we mentioned in Chapter 1. However, there are other ways to get the equations of motion, and we will introduce also the first order formalism to the reader, after an explanation in detail of the traditional one.

2.1 Second-order formalism

Conventionally, in General Relativity, the affine connection is already chosen and cannot change, that is, the Levi-Civita connection,

$$\mathring{\Gamma}_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right). \tag{2.3}$$

From that, one could write the curvature tensors in terms of the metric. For now, we will split the variation of the action (2.2) in the following manner,

$$\delta S = (\delta S)_1 + (\delta S)_2 + (\delta S)_3, \tag{2.4}$$

being

$$(\delta S)_1 = \frac{1}{2\kappa}\int d^D x\sqrt{|g|}g^{\sigma\eta}\delta\mathring{R}_{\sigma\eta}, \tag{2.5a}$$

$$(\delta S)_2 = \frac{1}{2\kappa}\int d^D x\sqrt{|g|}\mathring{R}_{\sigma\eta}\delta g^{\sigma\eta}, \tag{2.5b}$$

$$(\delta S)_3 = \frac{1}{2\kappa}\int d^D x\mathring{R}\delta\sqrt{|g|}. \tag{2.5c}$$

We would like to express all this in terms of the variation of the metric. However, it is more convenient to express it in terms of the variation of the inverse metric. As $g^{\mu\sigma}g_{\sigma\nu} = \delta^\mu_\nu$ and the variation of the Kronecker delta is zero, we can relate those two variations as

$$\delta g_{\mu\nu} = -g_{\mu\sigma}g_{\nu\eta}\delta g^{\sigma\eta}, \tag{2.6}$$

so they will share the extrema. Thus, we have (2.5b) ready.

For (2.5c), we will make use of the expression for the variation of the metric determinant (A.12) (see Appendix A for a full derivation), so we can write this part as

$$(\delta S)_3 = \frac{1}{2\kappa}\int d^D x\sqrt{|g|}\left(-\frac{1}{2}g_{\sigma\eta}\mathring{R}\right)\delta g^{\sigma\eta}. \tag{2.7}$$

Finally, for (2.5a), we have to go step by step. First of all, we will take into account that the Ricci tensor is given by

$$\mathring{R}_{\mu\nu} = \partial_\mu\mathring{\Gamma}_{\sigma\nu}{}^{\sigma} - \partial_\sigma\mathring{\Gamma}_{\mu\nu}{}^{\sigma} + \mathring{\Gamma}_{\mu\eta}{}^{\sigma}\mathring{\Gamma}_{\sigma\nu}{}^{\eta} - \mathring{\Gamma}_{\mu\nu}{}^{\sigma}\mathring{\Gamma}_{\eta\sigma}{}^{\eta}. \tag{2.8}$$

We want to express the variation of the Ricci tensor with respect to the metric, so we will start by expressing this variation in terms of the variation of the connection,

$$\mathring{\Gamma}_{\mu\nu}{}^{\rho} \to \mathring{\Gamma}_{\mu\nu}{}^{\rho} + \delta\mathring{\Gamma}_{\mu\nu}{}^{\rho}, \tag{2.9}$$

and, later, we will write that variation in terms of the variation of the metric. For the former, we will make use of the Palatini identity (B.4) (see Appendix B for a full derivation),

$$\delta\mathring{R}_{\mu\nu} = \mathring{\nabla}_\mu(\delta\mathring{\Gamma}_{\sigma\nu}{}^{\sigma}) - \mathring{\nabla}_\sigma(\delta\mathring{\Gamma}_{\mu\nu}{}^{\sigma}). \tag{2.10}$$

Thus, the contribution (2.5a) can be written as

$$
\begin{aligned}
(\delta S)_1 &= \frac{1}{2\kappa}\int d^Dx\sqrt{|g|}g^{\sigma\eta}\left(\mathring{\nabla}_\sigma(\delta\mathring{\Gamma}_{\theta\eta}{}^{\theta}) - \mathring{\nabla}_\theta(\delta\mathring{\Gamma}_{\sigma\eta}{}^{\theta})\right) \\
&= \frac{1}{2\kappa}\int d^Dx\sqrt{|g|}\mathring{\nabla}_\theta\left(g^{\sigma\theta}\cdot\delta\mathring{\Gamma}_{\eta\sigma}{}^{\eta} - g^{\sigma\eta}\cdot\delta\mathring{\Gamma}_{\sigma\eta}{}^{\theta}\right), \qquad (2.11)
\end{aligned}
$$

where we have used metric-compatibility. Nevertheless, this is an integral of the divergence of a vector. By Stokes' theorem, this is related with its boundary evaluation. If we call V the integration volume, the integral becomes

$$
(\delta S)_1 = \frac{1}{2\kappa}\int_{\partial V} d\Sigma_\theta\sqrt{|g|}\left(g^{\sigma\theta}\cdot\delta\mathring{\Gamma}_{\eta\sigma}{}^{\eta} - g^{\sigma\eta}\cdot\delta\mathring{\Gamma}_{\sigma\eta}{}^{\theta}\right), \qquad (2.12)
$$

where $d\Sigma$ is the differential of surface vector, whose direction is normal to the surface.

Now, we can compute the expression for the variation of the connection in terms of the variation of the metric and its derivative,

$$
\begin{aligned}
\delta\mathring{\Gamma}_{\mu\nu}{}^{\rho} &= \frac{1}{2}\delta g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \\
&\quad + \frac{1}{2}g^{\rho\sigma}\Big(\partial_\mu(\delta g_{\sigma\nu}) + \partial_\nu(\delta g_{\mu\sigma}) - \partial_\sigma(\delta g_{\mu\nu})\Big). \qquad (2.13)
\end{aligned}
$$

And, as a side note, we can write this variation in a manifestly covariant form, due to its tensorial nature,

$$
\delta\mathring{\Gamma}_{\mu\nu}{}^{\rho} = -\frac{1}{2}\left(g_{\sigma\mu}\mathring{\nabla}_\nu(\delta g^{\sigma\rho}) + g_{\sigma\nu}\mathring{\nabla}_\mu(\delta g^{\sigma\rho}) - g_{\mu\sigma}g_{\nu\eta}\mathring{\nabla}^\rho(\delta g^{\sigma\eta})\right), \qquad (2.14)
$$

but it is not very useful for our current goal.

Putting (2.13) into (2.12) leads to several terms in the integral. We can set a part of this contribution to zero by setting the variation of the metric to vanish at the boundary, but one subtle contribution stays: the one coming from the derivative of the variation of the metric. As a result, the final expression for (2.5a) is

$$
(\delta S)_1|_{\delta g=0} = \frac{1}{2\kappa}\int_{\partial V} d\Sigma_\theta\sqrt{|g|}g^{\sigma\theta}g^{\eta\xi}\Big(\partial_\sigma(\delta g_{\eta\xi}) - \partial_\xi(\delta g_{\eta\sigma})\Big). \qquad (2.15)
$$

Technically, in order to counteract this term and be able to obtain Einstein equations for any volume V, we should add one more term to the action, the Gibbons-Hawking-York term [50, 18],

$$
S_{\text{GHY}} = \frac{1}{2\kappa}\int_{\partial V} d^{D-1}x\sqrt{|\bar{h}|}n^\sigma\bar{h}^{\alpha\beta}\bar{\varepsilon}^{\eta}_{\alpha}\bar{\varepsilon}^{\theta}_{\beta}(\partial_\sigma g_{\eta\theta} - \partial_\eta g_{\sigma\theta}), \qquad (2.16)
$$

where y are the coordinates that describe the hypersurface ∂V, n is the orthogonal vector pointing to the outside of this surface and $\bar{h}$ is the induced metric. To calculate it, we can compute the length element in this surface,

$$\begin{aligned} d\bar{s}^2 &= g_{\sigma\eta}\, dx^\sigma dx^\eta|_{\partial V} \\ &= g_{\sigma\eta} \frac{\partial x^\sigma}{\partial y^\alpha} \frac{\partial x^\eta}{\partial y^\beta} dy^\alpha dy^\beta \\ &= \bar{h}_{\alpha\beta} dy^\alpha dy^\beta, \end{aligned} \tag{2.17}$$

where

$$\bar{h}_{\alpha\beta}(y) = \bar{\varepsilon}^\sigma_\alpha \bar{\varepsilon}^\eta_\beta g_{\sigma\eta}(x). \tag{2.18}$$

Considering this, we only get this boundary term from (2.5a). However, we get all the necessary terms from (2.5b) and (2.5c),

$$\delta S = \frac{1}{2\kappa} \int d^D x \sqrt{|g|} \left(\mathring{R}_{\sigma\eta} - \frac{1}{2} g_{\sigma\eta} \mathring{R} \right) \delta g^{\sigma\eta}. \tag{2.19}$$

Demanding the variation of the action to be zero for any symmetric variation of the metric, the symmetric part of the term between parenthesis has to be zero. As it is already symmetric,

$$\mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathring{R} = 0, \tag{2.20}$$

and we obtain the equations of motion of the metric (in vacuum): Einstein equations. These are second-order differential equations, as the curvature tensor and scalar have second-order derivatives of the metric. This happens thanks to the fact that the only term that could give higher-order derivatives, the term (2.5a), only has an effect on the boundary term, but not on the equations of motion.

If we want to include matter or energy, we can add other terms to the action,

$$S = \frac{1}{2\kappa} \int d^D x \sqrt{|g|} R + S_{\mathrm{m}}, \tag{2.21}$$

so when taking the variation of that, we get

$$\delta S = \delta S_{\mathrm{EH}} + \delta S_{\mathrm{m}}. \tag{2.22}$$

Then, for example, if the matter action depends on the metric and a scalar field, $S_{\mathrm{m}} = S_{\mathrm{m}}(g, \phi)$, we would get two terms from its variation,

$$\delta S_{\mathrm{m}} = \int d^D x \left(\frac{\delta S_{\mathrm{m}}}{\delta g^{\sigma\eta}} \delta g^{\sigma\eta} + \frac{\delta S_{\mathrm{m}}}{\delta \phi} \delta\phi \right). \tag{2.23}$$

Hence, the equation of motion of the metric will be

$$\mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathring{R} = -\kappa \mathcal{T}_{\mu\nu}, \tag{2.24}$$

getting the first member of the equation as we have shown along this section, and the energy-momentum tensor from the matter action as

$$\mathcal{T}_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}}, \tag{2.25}$$

depending on its explicit form as a functional of the metric.

On the other hand, the equation of motion corresponding to the scalar field would be computed as

$$0 = \frac{\delta S_{\mathrm{m}}}{\delta \phi}, \tag{2.26}$$

also depending on the functional dependency of the action.

2.2 First-order formalism

As we have studied in detail the second-order formalism, the one developed originally with the theory and that remains widely used, we now are ready to understand the change of paradigm that first-order formalism is.

As previously indicated, in first-order formalism, metric and connection are not related because we do not choose Levi-Civita. Instead, we let the connection vary freely, and we derive the equations of motion looking for an extremum of the action.

2.2.1 Equations of motion

Again, we start with the Einstein-Hilbert action (2.2). However, this time we consider the Riemann tensor to depend on the affine connection Γ as a variable of movement independent of the metric. Analogously to (2.5), we get

$$(\delta S)_1 = \frac{1}{2\kappa} \int d^D x \sqrt{|g|} g^{\sigma\eta} \delta R_{\sigma\eta}, \tag{2.27a}$$

$$(\delta S)_2 = \frac{1}{2\kappa} \int d^D x \sqrt{|g|} R_{\sigma\eta} \delta g^{\sigma\eta}, \tag{2.27b}$$

$$(\delta S)_3 = \frac{1}{2\kappa} \int d^D x R \delta \sqrt{|g|}. \tag{2.27c}$$

As these variations should be in terms of the variation of the (inverse) metric *and* the variation of the connection, we can follow the previous procedure for (2.27c), as in (2.7),

$$(\delta S)_3 = \frac{1}{2\kappa}\int d^D x\sqrt{|g|}\left(-\frac{1}{2}g_{\sigma\eta}R\right)\delta g^{\sigma\eta}. \tag{2.28}$$

That way, the total contribution from the variation of the metric is

$$\delta S|_{\delta\Gamma=0} = \int d^D x\sqrt{|g|}\frac{1}{2\kappa}\left(R_{\sigma\eta} - \frac{1}{2}g_{\sigma\eta}R\right)\delta g^{\sigma\eta}. \tag{2.29}$$

On the other hand, for (2.27a) we have to use again the Palatini identity (B.4) to get, analogously to (2.11),

$$(\delta S)_1 = \frac{1}{2\kappa}\int d^D x\sqrt{|g|}g^{\sigma\eta}\left(\nabla_\sigma(\delta\Gamma_{\theta\eta}{}^{\theta}) - \nabla_\theta(\delta\Gamma_{\sigma\eta}{}^{\theta}) + T_{\sigma\theta}{}^{\xi}\cdot\delta\Gamma_{\xi\eta}{}^{\theta}\right). \tag{2.30}$$

Note that, in contrast with (2.11), as the connection is not metric-compatible, this is not a divergence. We will do integration by parts, eliminating derivatives of any variation. Using (C.10) (in Appendix C there is a detailed explanation),

$$\begin{aligned}(\delta S)_1 = &-\frac{1}{2\kappa}\int d^D x\sqrt{|g|}\left(\nabla_\sigma g^{\sigma\eta} + \frac{1}{2}g^{\sigma\eta}g^{\xi\varphi}\nabla_\sigma g_{\xi\varphi} + g^{\sigma\eta}T_{\sigma\xi}{}^{\xi}\right)\delta\Gamma_{\theta\eta}{}^{\theta}\\ &+\frac{1}{2\kappa}\int d^D x\sqrt{|g|}\left(\nabla_\theta g^{\sigma\eta} + \frac{1}{2}g^{\sigma\eta}g^{\xi\varphi}\nabla_\theta g_{\xi\varphi} + g^{\sigma\eta}T_{\theta\xi}{}^{\xi}\right)\delta\Gamma_{\sigma\eta}{}^{\theta}\\ &+\frac{1}{2\kappa}\int d^D x\sqrt{|g|}g^{\sigma\eta}T_{\sigma\theta}{}^{\xi}\cdot\delta\Gamma_{\xi\eta}{}^{\theta}.\end{aligned} \tag{2.31}$$

Boundary terms do not appear as we can set the variation of the connection to be zero in the boundary. If we factor out the variation, we get the total variation of the action coming from the connection,

$$\begin{aligned}\delta S|_{\delta g=0} = -\frac{1}{2\kappa}\int d^D x\sqrt{|g|}\Big(&\nabla_\sigma g^{\sigma\beta}\delta^\alpha_\gamma + \frac{1}{2}g^{\sigma\beta}g^{\xi\varphi}\nabla_\sigma g_{\xi\varphi}\delta^\alpha_\gamma + g^{\sigma\beta}T_{\sigma\xi}{}^{\xi}\delta^\alpha_\gamma\\ &+\nabla_\gamma g^{\alpha\beta} + \frac{1}{2}g^{\alpha\beta}g^{\xi\varphi}\nabla_\gamma g_{\xi\varphi} + g^{\alpha\beta}T_{\gamma\xi}{}^{\xi}\\ &+g^{\sigma\beta}T_{\sigma\gamma}{}^{\alpha}\Big)\delta\Gamma_{\alpha\beta}{}^{\gamma}.\end{aligned} \tag{2.32}$$

With all this, we are ready to write the equations of motion. For the metric, from (2.29), we get

$$R_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{2.33}$$

where the parenthesis in subindices mean symmetrization (D.2) (see Appendix D for a full explanation of this notation), and for the connection, from (2.32),

$$\begin{aligned} 0 = \nabla_\sigma g^{\sigma\nu}\delta^\mu_\rho + \frac{1}{2}g^{\sigma\nu}g^{\xi\varphi}\nabla_\sigma g_{\xi\varphi}\delta^\mu_\rho + g^{\sigma\nu}T_{\sigma\xi}{}^{\xi}\delta^\mu_\rho \\ + \nabla_\rho g^{\mu\nu} + \frac{1}{2}g^{\mu\nu}g^{\xi\varphi}\nabla_\rho g_{\xi\varphi} + g^{\mu\nu}T_{\rho\xi}{}^{\xi} + g^{\sigma\nu}T_{\sigma\rho}{}^{\mu}. \end{aligned} \tag{2.34}$$

For adding minimally-coupled matter to the theory,

$$S = \frac{1}{2\kappa}\int d^Dx\sqrt{|g|}R + S_{\mathrm{m}}(g), \tag{2.35}$$

we proceed in the same way we did in the second-order formalism, as in (2.22), to get the equation of motion

$$R_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}R = -\kappa\mathcal{T}_{\mu\nu}, \tag{2.36}$$

or, if it also depends on a scalar field, we get (2.26), too.

Note that, as the variation of the metric is restricted to be symmetric, the equation of motion of the metric (2.33) involves only the symmetric part of the Ricci tensor, not the whole tensor. However, for the connection equation (2.34), we do not have any restrictions as the variation is a completely unconstrained tensor.

These two equations involve the metric and the connection simultaneously, so one should choose wisely how to proceed to obtain the general solution for them. First of all, we will simplify the connection equation.

2.2.2 Simplification

Following the steps in [3], we will simplify the equation of motion of the connection, as it is difficult to solve it in its current form.

This form gives us a hint of how to proceed, though. If we take a deeper look at (2.34), it seems that the first line is the trace of the second except for the last addend, so we think about subtracting that trace and check if we are getting an equivalent equation.

We will start by taking the δ^ρ_μ trace,

$$0 = (D-1)\cdot\nabla_\sigma g^{\sigma\nu} + (D-1)\cdot\frac{1}{2}g^{\sigma\nu}g^{\theta\xi}\nabla_\sigma g_{\theta\xi} + (D-2)\cdot g^{\sigma\nu}T_{\sigma\eta}{}^{\eta}. \tag{2.37}$$

From there, we can find the non-metricity trace,

$$\nabla_\sigma g^{\sigma\nu} = -\frac{1}{2}g^{\sigma\nu}g^{\theta\xi}\nabla_\sigma g_{\theta\xi} + \frac{D-2}{D-1}g^{\sigma\nu}T_{\sigma\eta}{}^{\eta}. \tag{2.38}$$

If we substitute that back in (2.34),

$$0 = -\frac{1}{D-1} g^{\sigma\nu} T_{\sigma\eta}{}^{\eta} \delta^{\mu}_{\rho} + \nabla_{\rho} g^{\mu\nu} + \frac{1}{2} g^{\mu\nu} g^{\sigma\eta} \nabla_{\rho} g_{\sigma\eta} + g^{\mu\nu} T_{\rho\sigma}{}^{\sigma} + g^{\sigma\nu} T_{\sigma\rho}{}^{\mu}, \quad (2.39)$$

we get a simpler equation. To see that this new equation is equivalent to (2.34), we can take the trace again. We will also obtain (2.37), so we can revert the steps, getting the equivalence.

Below, we will simplify it even more. Let's start by taking $g_{\mu\nu}$ trace,

$$0 = -\frac{1}{D-1} T_{\rho\sigma}{}^{\sigma} + g_{\sigma\eta} \nabla_{\rho} g^{\sigma\eta} + D \cdot \frac{1}{2} g^{\sigma\eta} \nabla_{\rho} g_{\sigma\eta} + D \cdot T_{\rho\sigma}{}^{\sigma} + T_{\sigma\rho}{}^{\sigma}. \quad (2.40)$$

Considering that

$$\begin{aligned} g_{\sigma\eta} \nabla_{\rho} g^{\sigma\eta} &= \nabla_{\rho}(g_{\sigma\eta} g^{\sigma\eta}) - g^{\sigma\eta} \nabla_{\rho} g_{\sigma\eta} \\ &= -g^{\sigma\eta} \nabla_{\rho} g_{\sigma\eta}, \end{aligned} \quad (2.41)$$

the trace results

$$\begin{aligned} 0 &= \left(D - 1 - \frac{1}{D-1} \right) T_{\rho\sigma}{}^{\sigma} + \left(\frac{D}{2} - 1 \right) g^{\sigma\eta} \nabla_{\rho} g_{\sigma\eta} \\ &= \frac{D(D-2)}{D-1} T_{\rho\sigma}{}^{\sigma} + \frac{D-2}{2} g^{\sigma\eta} \nabla_{\rho} g_{\sigma\eta}. \end{aligned} \quad (2.42)$$

Hence, we can now write the last addend in terms of the trace of the torsion,

$$\frac{1}{2} g^{\sigma\eta} \nabla_{\rho} g_{\sigma\eta} = -\frac{D}{D-1} T_{\rho\sigma}{}^{\sigma}. \quad (2.43)$$

Substituting back in (2.39),

$$0 = \nabla_{\rho} g^{\mu\nu} - \frac{1}{D-1} g^{\sigma\nu} T_{\sigma\eta}{}^{\eta} \delta^{\mu}_{\rho} - \frac{1}{D-1} g^{\mu\nu} T_{\rho\sigma}{}^{\sigma} + g^{\sigma\nu} T_{\sigma\rho}{}^{\mu}. \quad (2.44)$$

Here, again, we can retake the trace to recover (2.40), so the equivalence with the equation of motion is still guaranteed.

As the last simplification, we will lower all the indices. For the first addend, we should notice that

$$\begin{aligned} g_{\mu\sigma} g_{\nu\eta} \nabla_{\rho} g^{\sigma\eta} &= \nabla_{\rho}(g_{\mu\sigma} g_{\nu\eta} g^{\sigma\eta}) - g^{\sigma\eta} g_{\mu\sigma} \nabla_{\rho} g_{\nu\eta} - g^{\sigma\eta} g_{\nu\eta} \nabla_{\rho} g_{\mu\sigma} \\ &= \nabla_{\rho} g_{\mu\nu} - \delta^{\eta}_{\mu} \nabla_{\rho} g_{\nu\eta} - \delta^{\sigma}_{\mu} \nabla_{\rho} g_{\mu\sigma} \\ &= -\nabla_{\rho} g_{\mu\nu}. \end{aligned} \quad (2.45)$$

Then, changing the sign, lowering the indices, and rearranging them, the simplest equation that we can obtain is

$$0 = \nabla_\rho g_{\mu\nu} + \frac{1}{D-1} g_{\mu\rho} T_{\nu\sigma}{}^{\sigma} - \frac{1}{D-1} g_{\mu\nu} T_{\sigma\rho}{}^{\sigma} - g_{\mu\sigma} T_{\nu\rho}{}^{\sigma}, \tag{2.46}$$

which is much more easy to solve, even though it depends on the number of dimensions.

2.2.3 Particular solutions

In this subsection we will show some solutions and some properties of the solutions. First of all, it is straightforward to see that Levi-Civita connection is a solution, as every addend in (2.46) is proportional to the torsion or the non-metricity tensors.

Also, it is easy to see that absence of torsion implies metric compatibility from (2.46), as only the first term is not proportional to the torsion,

$$T_{\nu\rho}{}^{\mu} = 0 \Rightarrow 0 = \nabla_\rho g_{\mu\nu}. \tag{2.47}$$

Similarly, metric compatibility implies absence of torsion,

$$\nabla_\rho g_{\mu\nu} = 0 \Rightarrow 0 = \frac{1}{D-1} g_{\mu\rho} T_{\nu\sigma}{}^{\sigma} - \frac{1}{D-1} g_{\mu\nu} T_{\sigma\rho}{}^{\sigma} - g_{\mu\sigma} T_{\nu\rho}{}^{\sigma} \tag{2.48}$$

$$\Rightarrow 0 = T_{\nu\rho}{}^{\mu}. \tag{2.49}$$

These two results mean that if we impose that our connection is either metric-compatible or torsionless, we get the other property from the equation of motion and, then, the only solution is Levi-Civita connection.

2.2.4 General solution

After seeing some partial results or properties of the equation of motion of the connection, we will solve it generally.

As a naive thought, we would suspect Levi-Civita to be the only solution. We have already seen that this is a solution, but we will see that it is *not* the only solution.

We will follow a similar procedure than in (1.36) for Levi-Civita, that is, we will write the equation of motion (2.46), expanding the covariant derivative and the last torsion in terms of the connection, three times, permuting the indices,

$$0 = \partial_\rho g_{\mu\nu} - \Gamma_{\rho\mu}{}^{\sigma} g_{\sigma\nu} - \Gamma_{\nu\rho}{}^{\sigma} g_{\mu\sigma} + \frac{1}{D-1} g_{\mu\rho} T_{\nu\sigma}{}^{\sigma} - \frac{1}{D-1} g_{\mu\nu} T_{\sigma\rho}{}^{\sigma}, \quad (2.50a)$$

$$0 = \partial_\mu g_{\nu\rho} - \Gamma_{\mu\nu}{}^{\sigma} g_{\sigma\rho} - \Gamma_{\rho\mu}{}^{\sigma} g_{\nu\sigma} + \frac{1}{D-1} g_{\nu\mu} T_{\rho\sigma}{}^{\sigma} - \frac{1}{D-1} g_{\nu\rho} T_{\sigma\mu}{}^{\sigma}, \quad (2.50b)$$

$$0 = \partial_\nu g_{\rho\mu} - \Gamma_{\nu\rho}{}^{\sigma} g_{\sigma\mu} - \Gamma_{\mu\nu}{}^{\sigma} g_{\rho\sigma} + \frac{1}{D-1} g_{\rho\nu} T_{\mu\sigma}{}^{\sigma} - \frac{1}{D-1} g_{\rho\mu} T_{\sigma\nu}{}^{\sigma}, \quad (2.50c)$$

then, adding (2.50a) and (2.50b), and subtracting (2.50c),

$$0 = \partial_\rho g_{\mu\nu} + \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} - 2\Gamma_{\rho\mu}{}^{\sigma} g_{\sigma\nu} - \frac{2}{D-1} g_{\mu\nu} T_{\sigma\rho}{}^{\sigma}. \quad (2.51)$$

Raising the ν index and finding Γ,

$$\begin{aligned} \Gamma_{\rho\mu}{}^{\nu} &= \frac{1}{2} g^{\nu\sigma} \left(\partial_\rho g_{\mu\sigma} + \partial_\mu g_{\sigma\rho} - \partial_\sigma g_{\rho\mu} \right) - \frac{1}{D-1} g_{\mu\nu} T_{\sigma\rho}{}^{\sigma} \\ &= \mathring{\Gamma}_{\rho\mu}{}^{\nu} - \frac{1}{D-1} T_{\sigma\rho}{}^{\sigma} \delta^{\nu}_{\mu}. \end{aligned} \quad (2.52)$$

That way, we have written the connection in terms of the Levi-Civita connection and the irreducible elements of the torsion: in this case, only the trace of the torsion. Then, if we name this trace A,

$$\Gamma_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + A_\mu \delta^{\rho}_{\nu}, \quad (2.53)$$

we can see that, substituting it back in the equation of motion (2.46), it is automatically solved for any vector field A, so that the trace of the torsion is completely undetermined by the equations of motion. We have found the most general solution to them (2.53), that we will call Palatini connections following the nomenclature of [3], and we will denote the quantities related with this solution with a bar.

We have not shown yet what is the meaning of that vector field, but we will find it out shortly. Before that, let's check what happens with the equations of motion of the metric.

2.2.5 Curvature tensors

Below we will see some properties of the solution. We will start by calculating torsion and non-metricity of this solution, and the curvature tensors, as that will let us see what happens with the equation of motion of the metric (2.33).

It is a short calculation to get torsion and non-metricity of this solution,

$$\bar{T}_{\mu\nu}{}^{\rho} = A_\mu \delta^{\rho}_{\nu} - A_\nu \delta^{\rho}_{\mu}, \qquad \bar{Q}_{\mu\nu\rho} = -\bar{\nabla}_\mu g_{\nu\rho} = 2A_\mu g_{\nu\rho}, \quad (2.54)$$

which are both non-zero. Thus, the curvature tensors have less symmetries, as we will see in a moment.

If we put our solution (2.53) into the definition of the Riemann tensor (1.20), an easy calculation let us see that it yields

$$\bar{R}_{\mu\nu\rho}{}^{\lambda} = \mathring{R}_{\mu\nu\rho}{}^{\lambda} + F_{\mu\nu}\delta^{\lambda}_{\rho}, \tag{2.55}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.56}$$

This Riemann tensor does not have the same symmetries as the Levi-Civita one, it is just antisymmetric in the two first indices by definition.

From the expression for the Riemann tensor (2.55), contracting the second and fourth indices, as in (1.23), we get the Ricci tensor,

$$\bar{R}_{\mu\nu} = \mathring{R}_{\mu\nu} + F_{\mu\nu}, \tag{2.57}$$

which only differs with the Levi-Civita one in F, an antisymmetric tensor. That way, the symmetric part is the same as Levi-Civita, while the new antisymmetric part is F. It follows that the Ricci scalar is the same as in Levi-Civita,

$$\bar{R} = \mathring{R}. \tag{2.58}$$

If we put these results into the equation of motion of the metric (2.33), we obtain

$$\mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R} = -\kappa\mathcal{T}_{\mu\nu}\ , \tag{2.59}$$

as the matter field is minimally coupled, so the energy-momentum tensor does not change. Then, we get the same equation of motion of the metric, the Einstein equation and, as a consequence, the dynamics is the same. However, we will continue studying the dynamics to understand the meaning of A. The next thing that we will do is checking what happens with geodesics.

2.2.6 Geodesics

First of all, remember that there are two types of geodesics that in Levi-Civita match: metric and affine geodesics. As the metric is the one that satisfies the Einstein equation, metric geodesics remain the same in this formalism. Nevertheless, affine geodesics change. If we use the definition (1.30), we see

$$\begin{aligned} 0 &= \dot{x}^\sigma \bar{\nabla}_\sigma \dot{x}^\mu \\ &= \ddot{x}^\mu + \mathring{\Gamma}_{\sigma\eta}{}^{\mu}\dot{x}^\sigma\dot{x}^\eta + A_\sigma \dot{x}^\sigma \dot{x}^\mu. \end{aligned} \tag{2.60}$$

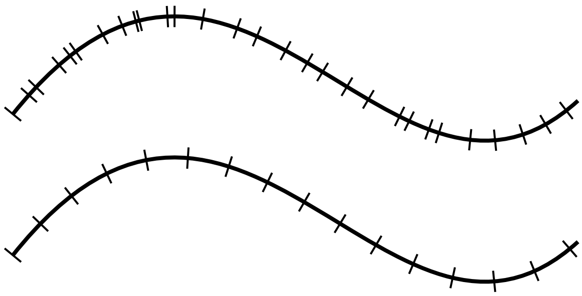

Figure 2.1: Same pregeodesics: geodesics with the same trajectory parameterized with different parameters, each marker representing an equal increment of it (top: parameterized with a general parameter; bottom: parameterized by arc length).

Now, we can compare this result with the metric geodesic (1.14), so we see that the difference can be understood as a change of parameterization,

$$-A_\sigma \dot{x}^\sigma \dot{x}^\mu = \frac{\ddot{s}}{\dot{s}} \dot{x}^\mu. \tag{2.61}$$

In this case,

$$s(\lambda) = \int_0^\lambda e^{-G(\lambda')} d\lambda', \qquad G(\lambda) = \int_0^\lambda \dot{x}^\rho A_\rho d\lambda', \tag{2.62}$$

and taking into account that the dot represents derivation with respect to the parameter of the curve, λ in this case.

We can now understand the degree of freedom introduced by A in this theory: it can be absorbed reparameterizing geodesics. Geodesics have the same trajectory, but they are not parameterized with the arc length (proper time), but with another parameter (see figure 2.1). This does not affect physics, as the parameterization does not play any role. Also, in Section 3.6, we will see that there is a symmetry of the Einstein-Hilbert action of which A is a parameter, and hence its absence of physical effect. We will also see other theories that do not have this Palatini connections as a solution, although there is always freedom to reparameterize.

As an interesting fact, without being a solution of the equations of motion, there are also other connections that give same pregeodesics as Levi-Civita. In fact, all the connections that have the same pregeodesics are called projectively related, and all the connections projectively related with Levi-Civita have the form [44]

$$\Gamma_{\mu\nu}{}^\rho = \mathring{\Gamma}_{\mu\nu}{}^\rho + A_\mu \delta^\rho_\nu + B_\nu \delta^\rho_\mu. \tag{2.63}$$

However, this kind of connections leads to much more complicated curvature tensors,

$$R_{\mu\nu\rho}{}^{\lambda} = \mathring{R}_{\mu\nu\rho}{}^{\lambda} + F_{\mu\nu}(A)\delta^{\lambda}_{\rho} + (\nabla_{\mu}B_{\rho} - B_{\mu}B_{\rho})\delta^{\lambda}_{\nu} - (\nabla_{\nu}B_{\rho} - B_{\nu}B_{\rho})\delta^{\lambda}_{\mu}, \quad (2.64)$$

except when

$$\nabla_{\mu}B_{\nu} = B_{\mu}B_{\nu}. \quad (2.65)$$

Apart from this highly exceptional case, the symmetric part of the Ricci tensor does not coincide con Levi-Civita any more, so we cannot recover the Einstein equation from it and then we cannot recover the dynamics.

2.2.7 Geodesic deviation

As geodesics are the same, we are going to take a look at the geodesic deviation. We look at this because in geodesic deviation we can see contributions of the Riemann tensor itself, not through its contraction, the Ricci tensor. As the Riemann tensor of the Palatini connections has a greater discrepancy of its Levi-Civita equivalent than the Ricci tensor, this is an interesting case.

First of all, we should derive the formula for a general connection, which was first done in [3]. Let $\gamma_{\eta}(\lambda)$ be a family of geodesics such that η identifies different geodesics of the family and λ parameterizes the points of the geodesic. This describes a surface $x(\lambda, \eta)$ where we can use λ and η as coordinates.

Hence, we can define two vectors tangent to the surface:

$$u^{\mu} = \frac{\partial x^{\mu}}{\partial \lambda}, \qquad s^{\mu} = \frac{\partial x^{\mu}}{\partial \eta}, \quad (2.66)$$

so that u gives a notion of the velocity along the geodesic and s of the distance between geodesics. As they form a basis of the surface $x(\lambda, \eta)$, it is followed that

$$[u, s]^{\mu} = 0 \Leftrightarrow u^{\sigma}\partial_{\sigma}s^{\mu} - s^{\sigma}\partial_{\sigma}u^{\mu} = 0 \quad (2.67)$$

$$\Leftrightarrow u^{\sigma}\nabla_{\sigma}s^{\mu} - s^{\sigma}\nabla_{\sigma}u^{\mu} = T_{\sigma\eta}{}^{\mu}u^{\sigma}u^{\eta}. \quad (2.68)$$

We can define two more vectors,

$$V^{\mu} = u^{\sigma}\nabla_{\sigma}s^{\mu}, \qquad A^{\mu} = u^{\sigma}\nabla_{\sigma}V^{\mu}, \quad (2.69)$$

which represents the change of separation between geodesics at first and second order. We can identify V with the recessional velocity and A with the relative acceleration between geodesics.

The geodesic deviation is the relation between this relative acceleration A and the curvature of the manifold, and can be written as

$$\begin{aligned} A^\mu &= u^\sigma \nabla_\sigma (u^\eta \nabla_\eta s^\mu) \\ &= u^\sigma \nabla_\sigma (s^\eta \nabla_\eta u^\mu + T_{\eta\theta}{}^\mu u^\eta s^\theta) \\ &= u^\sigma \nabla_\sigma (s^\eta \nabla_\eta u^\mu) + u^\sigma \nabla_\sigma (T_{\eta\theta}{}^\mu u^\eta s^\theta), \end{aligned} \tag{2.70}$$

where we have used (2.68). If we also use the Leibniz rule, we get

$$\begin{aligned} A^\mu &= (u^\sigma \nabla_\sigma s^\eta) \nabla_\eta u^\mu + u^\sigma s^\eta \nabla_\sigma \nabla_\eta u^\mu + u^\sigma \nabla_\sigma (T_{\eta\theta}{}^\mu u^\eta s^\theta) \\ &= (s^\sigma \nabla_\sigma u^\eta + T_{\sigma\theta}{}^\eta u^\sigma s^\theta) \nabla_\eta u^\mu + u^\sigma \nabla_\sigma (T_{\eta\theta}{}^\mu u^\eta s^\theta) \\ &\quad + u^\sigma s^\eta (\nabla_\eta \nabla_\sigma u^\mu + R_{\sigma\eta\theta}{}^\mu u^\theta - T_{\sigma\eta}{}^\theta \nabla_\theta u^\mu), \end{aligned} \tag{2.71}$$

and we have used again (2.68) in addition to (1.28a). If we expand the products, we realize that there are two terms than cancel each other,

$$\begin{aligned} A^\mu = (s^\sigma \nabla_\sigma u^\eta) \nabla_\eta u^\mu + u^\sigma \nabla_\sigma (T_{\eta\theta}{}^\mu u^\eta s^\theta) - R_{\sigma\eta\theta}{}^\mu s^\sigma u^\eta u^\theta \\ + u^\sigma s^\eta \nabla_\eta \nabla_\sigma u^\mu. \end{aligned} \tag{2.72}$$

If now we try to use the Leibniz rule again to write the last addend in a similar way to the first,

$$\begin{aligned} A^\mu = (s^\sigma \nabla_\sigma u^\eta) \nabla_\eta u^\mu + u^\sigma \nabla_\sigma (T_{\eta\theta}{}^\mu u^\eta s^\theta) - R_{\sigma\eta\theta}{}^\mu s^\sigma u^\eta u^\theta \\ + s^\eta \nabla_\eta (u^\sigma \nabla_\sigma u^\mu) - (s^\eta \nabla_\eta u^\sigma) \nabla_\sigma u^\mu, \end{aligned} \tag{2.73}$$

we can see that the first addend cancel with the last one, and the second last one is identically zero because of the geodesic equation. Hence, we end up with

$$A^\mu = u^\sigma \nabla_\sigma (T_{\eta\theta}{}^\mu u^\eta s^\theta) - R_{\sigma\eta\theta}{}^\mu s^\sigma u^\eta u^\theta. \tag{2.74}$$

We have done this calculation for a family of geodesic parameterized by just one parameter, but we can generalize it to a congruence of geodesics, where we call $\dot{x}$ the vector parallel to the geodesics and δx the displacement vector. Thus, the general formula is

$$\dot{x}^\sigma \nabla_\sigma (\dot{x}^\eta \nabla_\sigma \delta x^\mu) + R_{\sigma\eta\theta}{}^\mu \delta x^\sigma \dot{x}^\eta \dot{x}^\theta - \dot{x}^\sigma \nabla_\sigma (T_{\eta\theta}{}^\mu \dot{x}^\eta \delta x^\theta) = 0. \tag{2.75}$$

There are discrepancies with respect to the expression for the Levi-Civita connection: the last addend is new and the Riemann is different. As A cannot be physical, we should be able to get rid of these differences. To do that, we are going to reparameterize this surface analogously as we did for each geodesic.

In fact, writing explicitly the parameters, in our case, we can see that this expression for the geodesic deviation,

$$\frac{\partial x^\theta}{\partial\lambda}\bar{\nabla}_\theta\left(\frac{\partial x^\xi}{\partial\lambda}\bar{\nabla}_\xi\frac{\partial x^\mu}{\partial\eta}\right)+\bar{R}_{\theta\xi\varphi}{}^\mu\frac{\partial x^\theta}{\partial\eta}\frac{\partial x^\xi}{\partial\lambda}\frac{\partial x^\varphi}{\partial\lambda}-\frac{\partial x^\theta}{\partial\lambda}\bar{\nabla}_\theta\left(\bar{T}_{\xi\varphi}{}^\mu\frac{\partial x^\xi}{\partial\lambda}\frac{\partial x^\varphi}{\partial\eta}\right)=0, \tag{2.76}$$

can be mapped to the expression for Levi-Civita connections,

$$\frac{\partial x^\theta}{\partial\tau}\mathring{\nabla}_\theta\left(\frac{\partial x^\xi}{\partial\tau}\mathring{\nabla}_\xi\frac{\partial x^\mu}{\partial\sigma}\right)+\mathring{R}_{\theta\xi\varphi}{}^\mu\frac{\partial x^\theta}{\partial\sigma}\frac{\partial x^\xi}{\partial\tau}\frac{\partial x^\varphi}{\partial\tau}=0, \tag{2.77}$$

under the reparameterization

$$\frac{\partial x^\mu}{\partial\lambda}=\frac{\partial\tau}{\partial\lambda}\frac{\partial x^\mu}{\partial\tau}, \qquad \frac{\partial x^\mu}{\partial\eta}=\frac{\partial\tau}{\partial\eta}\frac{\partial x^\mu}{\partial\tau}+\frac{\partial x^\mu}{\partial\sigma}, \tag{2.78}$$

being

$$\tau=\tau(\lambda,\eta)=\int_0^\lambda e^{-G(\lambda',\eta)}d\lambda', \qquad G(\lambda,\eta)=\int_0^\lambda\frac{\partial x^\theta}{\partial\lambda'}A_\theta d\lambda'. \tag{2.79}$$

Thus, Palatini and Levi-Civita connections have the same geodesic deviation, as the expressions (2.76) and (2.77) are equivalent, and we have seen that this change of parameter is completely defined by A.

2.2.8 Homothety

Another remarkable property of Palatini solutions is that the parallel transport of a vector under any trajectory is homothetic to the Levi-Civita one. In fact, if we calculate the difference between the parallel transport under these two,

$$\dot{x}^\sigma\bar{\nabla}_\sigma V^\mu-\dot{x}^\sigma\mathring{\nabla}_\sigma V^\mu=\dot{x}^\sigma A_\sigma V^\mu, \tag{2.80}$$

we can see that it is proportional to the vector itself. This means that when parallel transporting a vector using Levi-Civita connection or any of the Palatini connections, the result is different, but the difference is only a proportionality coefficient. In particular, if we call $\mathring{V}$ the result of transporting along a curve $x^\mu(\lambda)$ with Levi-Civita connection, then the result along the same curve according to Palatini connections is given by

$$\bar{V}^\mu(\lambda)=e^{-G(\lambda)}\mathring{V}^\mu(\lambda), \tag{2.81}$$

where G was defined in (2.62). As we can see, the proportionality coefficient depends on the curve, but it does not depend on the vector itself. Then, we can say that Palatini transport is the same as Levi-Civita transport composed with a homothety. In other words, the vector changes, but only because its norm changes, the direction does not change.

Thus, the variation of the norm under parallel transport was expected as Palatini connections are not metric compatible. When a connection is not metric compatible, the scalar product is not conserved when transporting along a curve. In our case,

$$\dot{x}^\theta \bar{\nabla}_\theta \left(g_{\sigma\eta} V^\sigma W^\eta\right) = \dot{x}^\theta \bar{\nabla}_\theta g_{\sigma\eta} V^\sigma W^\eta = 2\dot{x}^\theta A_\theta g_{\sigma\eta} V^\sigma W^\eta = 2G'(\lambda) V_\sigma W^\sigma, \quad (2.82)$$

where we have taken into account that V and W are transported along x or, in other words,

$$\dot{x}^\sigma \nabla_\sigma V^\mu = \dot{x}^\sigma \nabla_\sigma W^\mu = 0. \quad (2.83)$$

As a curiosity, we can demonstrate that the Palatini connections are the only connections yielding this homothety property when transporting any vector along any curve. Let Γ be an arbitrary connection that we will decompose as in (1.42). Then, it yields homothetic parallel transport with respect to Levi-Civita if and only if

$$\dot{x}^\sigma \Xi_{\sigma\eta}{}^\mu V^\sigma = f(\lambda) V^\mu \quad (2.84)$$

for some function $f(\lambda)$ that may depend on the curve followed. If we want this to be true for all vectors, then it must be

$$\dot{x}^\sigma \Xi_{\sigma\nu}{}^\mu = f(\lambda)\delta^\mu_\nu. \quad (2.85)$$

This can also be written as

$$\dot{x}^\sigma \Xi_{\sigma\nu}{}^\mu = \dot{x}^\sigma A_\sigma \delta^\mu_\nu, \quad (2.86)$$

and if we want this to be true for every curve, it has to be

$$\Xi_{\mu\nu}{}^\rho = A_\mu \delta^\rho_\nu \quad (2.87)$$

or, in other words, the connection has to be a Palatini connection.

The meaning of this fact is that if we measure distances with the metric and directions with the affine connection, we do not need the Levi-Civita connection, but any of the Palatini connections to get the correct result.

2.2.9 Interpretation

We have seen that the most general connection allowed by the first-order formalism in the Einstein-Hilbert action, even allowing minimally coupled matter terms, is given by the non-symmetric and non-metric compatible connection

$$\bar{\Gamma}_{\mu\nu}{}^\rho = \mathring{\Gamma}_{\mu\nu}{}^\rho + A_\mu \delta^\rho_\nu, \quad (2.88)$$

with A an arbitrary but non-dynamical vector field. We will argue more about the nature of this vector A in Section 3.6.

We have shown some properties of this connection. First of all, it is the only connection, apart from the exceptional case of (2.65), that has the same pre-geodesics as Levi-Civita and at the same time conserves the form of the Einstein equations. On the other hand, its parallel transport is very similar to the Levi-Civita one, only differing in the resulting norm, due to non-metricity. Lastly, they preserve the dynamics of the second-order formalism with Levi-Civita connection.

Summing up, the answer to our original problem, whether the Levi-Civita connection was the only solution in the first-order formalism or not, is a bit subtler than expected: not only the Levi-Civita connection, but the entire family of Palatini connections are singled out by the variational principle and, from a mathematical point of view, there is no reason to assign a preferred status to Levi-Civita. However, since all Palatini connections lead to the same physics, the Levi-Civita connection has the virtue of being the simplest representative of a class of physically indistinguishable connections.

Chapter 3

Lovelock Theories

As we know that General Relativity cannot be the final answer, people are trying to find other theories. As we already said, there are several options, but the path followed by this thesis will be to find acceptable corrections to General Relativity.

However, Lovelock's theorem [32] says that, in four dimensions, the only possible second-order differential equations of motion coming from an action are the Einstein equations. That implies a kind of uniqueness in the Einstein-Hilbert plus the cosmological constant action, where only total derivatives can be added to modify it. An interesting case in four dimensions of a total derivative is the *Gauss-Bonnet term.*

This term is known since 1938, when Lanczos [30] tried to add more terms to the Einstein-Hilbert action. He discovered that the second-order curvature term given by

$$\alpha\mathring{R}^2 + \beta\mathring{R}_{\sigma\eta}\mathring{R}^{\sigma\eta} + \gamma\mathring{R}_{\sigma\eta\theta\xi}\mathring{R}^{\sigma\eta\theta\xi} \tag{3.1}$$

yields second-order differential equations if and only if $(\alpha, \beta, \gamma) = (1, -4, 1)$. With those coefficients, it is called nowadays the *Gauss-Bonnet term*,

$$\mathring{R}^2 - 4\mathring{R}_{\sigma\eta}\mathring{R}^{\sigma\eta} + \mathring{R}_{\sigma\eta\theta\xi}\mathring{R}^{\sigma\eta\theta\xi}. \tag{3.2}$$

This is expected because this term is a total derivative in four dimensions. We are going to demonstrate it in the following section.

3.1 Gauss-Bonnet as a total derivative

As can be seen in [49], it is not difficult to write Gauss-Bonnet Lagrangian as a total derivative in four dimensions. We will work with vielbein formalism as it is especially easy to demonstrate it. In this formalism, the metric degrees of freedom are represented by the vielbeins, e^a_μ. These are the components of a local orthonormal coframe. Thus, the metric become locally the Minkowski metric, $\eta_{\mu\nu}$, in any point of the manifold. The affine connection is then substituted by the components of the spin connection one-form, $\mathring{\omega}_{\mu a}{}^b$ using the appropriate basis transformation (called Vielbein Postulate).

We will also write the Gauss-Bonnet Lagrangian with delta notation, which is equal to (3.2) except for a proportionality coefficient, hence the actions are equivalent:

$$\int d^4x\sqrt{|g|}\delta^{\sigma\eta\theta\xi}_{\varphi\chi\psi\omega}\mathring{R}_{\sigma\eta}{}^{\varphi\chi}\mathring{R}_{\theta\xi}{}^{\psi\omega} \propto \int d^4x\sqrt{|g|}(\mathring{R}^2-4\mathring{R}_{\sigma\eta}\mathring{R}^{\sigma\eta}+\mathring{R}_{\sigma\eta\theta\xi}\mathring{R}^{\sigma\eta\theta\xi}), \quad (3.3)$$

where we have used the completely antisymmetrized delta (D.5).

With this conventions, in the tangent space, we would write the Gauss-Bonnet Lagrangian as

$$\int d^4x\sqrt{|g|}\delta^{\sigma\eta\theta\xi}_{\varphi\chi\psi\omega}\mathring{R}_{\sigma\eta}{}^{\varphi\chi}\mathring{R}_{\theta\xi}{}^{\psi\omega} = \int dx^\varphi dx^\chi dx^\psi dx^\omega \delta^{\sigma\eta\theta\xi}_{\varphi\chi\psi\omega}\sqrt{|g|}\varepsilon_{abcd}\mathring{R}_{\sigma\eta}{}^{ab}\mathring{R}_{\theta\xi}{}^{cd}, \quad (3.4)$$

where ε is the Levi-Civita symbol (there is more information about it in Appendix E). From there, we can expand the expression of the Ricci tensor in terms of the spin connection,

$$\begin{aligned}\sqrt{|g|}\varepsilon_{abcd}\mathring{R}_{\sigma\eta}{}^{ab}\mathring{R}_{\theta\xi}{}^{cd} &= 4\sqrt{|g|}\varepsilon_{abcd}(\partial_{[\sigma}\mathring{\omega}_{\eta]}{}^{ab} - \mathring{\omega}_{[\sigma}{}^{ae}\mathring{\omega}_{\eta]e}{}^{b})(\partial_{[\theta}\mathring{\omega}_{\xi]}{}^{cd} - \mathring{\omega}_{[\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d}) \\ &= 4\partial_{[\sigma}\left(\sqrt{|g|}\varepsilon_{|abcd|}\mathring{\omega}_{\eta}{}^{ab}\partial_\theta\mathring{\omega}_{\xi]}{}^{cd}\right) \\ &\quad - 8\sqrt{|g|}\varepsilon_{abcd}\partial_{[\sigma}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d} \\ &\quad + 4\sqrt{|g|}\varepsilon_{abcd}\mathring{\omega}_{[\sigma}{}^{ae}\mathring{\omega}_{\eta|e|}{}^{b}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d}. \end{aligned} \quad (3.5)$$

We can transform the second addend into a total derivative as follows. First, we try to transform it into a total derivative using the Leibniz rule and subtracting all the terms that appear,

$$\begin{aligned}\sqrt{|g|}\varepsilon_{abcd}\partial_{[\sigma}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d} &= \sqrt{|g|}\varepsilon_{abcd}\partial_{[\sigma}(\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d}) \\ &\quad - \sqrt{|g|}\varepsilon_{abcd}\mathring{\omega}_{[\eta}{}^{ab}\partial_\sigma\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d} \\ &\quad - \sqrt{|g|}\varepsilon_{abcd}\mathring{\omega}_{[\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\partial_\sigma\mathring{\omega}_{\xi]f}{}^{d} \end{aligned} \quad (3.6)$$

Thanks to the antisymmetrization of the Levi-Civita symbol, we can switch two spin connections to transform one term into the other and operate to get

$$\begin{aligned}\sqrt{|g|}\varepsilon_{abcd}\partial_{[\sigma}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d} &= \partial_{[\sigma}\left(\sqrt{|g|}\varepsilon_{|abcd|}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d}\right) \\ &\quad - 2\sqrt{|g|}\varepsilon_{abcd}\partial_{[\sigma}\mathring{\omega}_{\eta|f|}{}^{d}\mathring{\omega}_{\theta}{}^{ab}\mathring{\omega}_{\xi]}{}^{cf}. \end{aligned} \quad (3.7)$$

Then, since the spin connection is antisymmetric, we can be sure that the index f cannot be equal to c or d, so we can split that summation as follows (there is one summation over every repeated index, no matter how many times repeated),

$$\begin{aligned}\sqrt{|g|}\varepsilon_{abcd}\partial_{[\sigma}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d} = {}&\partial_{[\sigma}\left(\sqrt{|g|}\varepsilon_{|abcd|}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d}\right)\\&-2\sqrt{|g|}\varepsilon_{abcd}\partial_{[\sigma}\mathring{\omega}_{\eta|a|}{}^{d}\mathring{\omega}_{\theta}{}^{ab}\mathring{\omega}_{\xi]}{}^{ca}\\&-2\sqrt{|g|}\varepsilon_{abcd}\partial_{[\sigma}\mathring{\omega}_{\eta|b|}{}^{d}\mathring{\omega}_{\theta}{}^{ab}\mathring{\omega}_{\xi]}{}^{cb}.\end{aligned}\tag{3.8}$$

Now, we are going to operate to get

$$\begin{aligned}\sqrt{|g|}\varepsilon_{abcd}\partial_{[\sigma}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d} = {}&\partial_{[\sigma}\left(\sqrt{|g|}\varepsilon_{|abcd|}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d}\right)\\&+2\sqrt{|g|}\varepsilon_{abcd}\partial_{[\sigma}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{da}\mathring{\omega}_{\xi]a}{}^{c}\\&+2\sqrt{|g|}\varepsilon_{abcd}\partial_{[\sigma}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{db}\mathring{\omega}_{\xi]b}{}^{c}.\end{aligned}\tag{3.9}$$

Finally, we write the summation with the index f again, going back to standard Einstein summation convention and switch indices to get the same term that we started with,

$$\begin{aligned}\sqrt{|g|}\varepsilon_{abcd}\partial_{[\sigma}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d} = {}&\partial_{[\sigma}\left(\sqrt{|g|}\varepsilon_{|abcd|}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d}\right)\\&-2\sqrt{|g|}\varepsilon_{abcd}\partial_{[\sigma}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d}.\end{aligned}\tag{3.10}$$

Thus, we can conclude that

$$\sqrt{|g|}\varepsilon_{abcd}\partial_{[\sigma}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d} = \frac{1}{3}\partial_{[\sigma}\left(\sqrt{|g|}\varepsilon_{|abcd|}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d}\right),\tag{3.11}$$

so we have written it as a total derivative.

Regarding the last addend of (3.5), as ε is antisymmetric in all its indices and all of them are summation indices, we can calculate, for simplification,

$$\varepsilon_{1234}\mathring{\omega}_{[\sigma}{}^{1e}\mathring{\omega}_{\eta|e|}{}^{2}\mathring{\omega}_{\theta}{}^{3f}\mathring{\omega}_{\xi]f}{}^{4},\tag{3.12}$$

where you can see that, giving any value to indices e and f, you get a repeated spin connection ω in that product and, then, every term of the summation is zero.

Summing up, we have transformed this term into a total derivative,

$$\sqrt{|g|}\varepsilon_{abcd}\mathring{R}_{\sigma\eta}{}^{ab}\mathring{R}_{\theta\xi}{}^{cd} = 4\partial_{[\sigma}\left(\sqrt{|g|}\varepsilon_{|abcd|}\left(\mathring{\omega}_{\eta}{}^{ab}\partial_{\theta}\mathring{\omega}_{\xi]}{}^{cd} - \frac{2}{3}\mathring{\omega}_{\eta}{}^{ab}\mathring{\omega}_{\theta}{}^{cf}\mathring{\omega}_{\xi]f}{}^{d}\right)\right),\tag{3.13}$$

so that ends our demonstration.

Note that we have used that we are in four dimensions in two steps: writing the Levi-Civita symbol with four indices and saying that in (3.12) there are repeated indices. However, in more dimensions, this does not hold and Gauss-Bonnet is not a total derivative.

3.2 Lovelock's theorem

Now that we have demonstrated that Gauss-Bonnet is a total derivative in four dimensions, we can think about how can we extend our theory regarding the restrictions of the Lovelock's theorem. We have several options: focus on equations of motion that do not come from an action, let the equations be of order greater than two, or change the dimension we are working on.

We are not interested in emergence in this thesis, so we discard the idea of equations of motion that do not come from an action.

On the other hand, if we let the equations of motion to be of order greater than two, we will find problems eventually. Ostrogradsky's theorem [37] tells us that a non-degenerate Lagrangian dependent on time derivatives higher than the first, that is, a Lagrangian whose equations of motion are of order higher than two, corresponds to a linearly unstable Hamiltonian. The consequence is that there are some degrees of freedom that can reach arbitrarily negative energies. This, on its own, is not a problem, the instability appears when interacting with other degrees of freedom that are bounded from below. As there exist a vast phase space where the Hamiltonian is negative, by entropic argument, the modes will begin to populate them alone, while creating an equal amount of positive modes in the interacting degree of freedom by conservation of energy. Besides, while being this a classical instability, negative energy modes are particularly problematic in quantum physics, as they lead to negative norm states or negative energy states. As they are referred to as ghosts in quantum theory, higher-order derivative theories are often called ghost-like.

The only way to evade these problems is using degenerate Lagrangians [11, 48], that is, Lagrangians that give second-order derivative equations of motion although they depend on higher than first derivatives of the dynamical variable. Lovelock Theories are one example of these. Nevertheless, all Lovelock Theories vanish in four dimensions except cosmological constant, Einstein-Hilbert and Gauss-Bonnet, being this last one a total derivative, so we do not get anything new.

Finally, we could change the dimension we are working on. This seems reasonable as more general theories that are explored nowadays, like String Theory, seem to live in a greater number of dimensions. They connect with the four-dimensional world we live in through what is called dimensional reduction. In other words, they reduce to effective theories in four dimensions. The easiest method to achieve this is the Kaluza-Klein theory [27, 29, 28], which assumes that one coordinate is compact and very small, hence not observed directly. There are, however, more sophisticated methods, like Randrall-Sundrum model [41, 40], which describe the universe as a warped-geometry higher-dimensional universe where the interaction particles (except the graviton) are confined to a $(3+1)$-dimensional brane.

Summing up, we think that the most reasonable choice is working with theories in higher dimensions and study their properties. If we look for a term of arbitrary order in the curvature that also yields second-order differential equations, we will find Lovelock Theories.

It is particularly explanatory to demonstrate this fact for the Gauss-Bonnet term because we can see, as Lanczos did, that it is the only combination of second-order curvature terms that fulfils this condition. We are going to do it in the following section.

3.3 Equations of motion of Gauss-Bonnet

As this calculation is independent of the number of dimensions, we are going to do it in D dimensions. Thus, we are going to start with the following action,

$$S = \int d^D x \sqrt{|g|} (\alpha \mathring{R}^2 + \beta \mathring{R}_{\sigma\eta} \mathring{R}^{\sigma\eta} + \gamma \mathring{R}_{\sigma\eta\theta\xi} \mathring{R}^{\sigma\eta\theta\xi}) \tag{3.14}$$

and we will deal with every addend separately. Starting with the first as an example of how to proceed, we can follow the same decomposition than in equation (2.4),

$$\delta_{\mathring{R}^2} = \delta \left(\int d^D x \sqrt{|g|} g^{\sigma\eta} g^{\theta\xi} \mathring{R}_{\sigma\eta} \mathring{R}_{\theta\xi} \right) = (\delta S)_1 + (\delta S)_2 + (\delta S)_3, \tag{3.15}$$

where

$$(\delta S)_1 = 2 \int d^D x \sqrt{|g|} \mathring{R} g^{\sigma\eta} \delta \mathring{R}_{\sigma\eta}, \tag{3.16a}$$

$$(\delta S)_2 = 2 \int d^D x \sqrt{|g|} \mathring{R} \mathring{R}_{\sigma\eta} \delta g^{\sigma\eta}, \tag{3.16b}$$

$$(\delta S)_3 = \int d^D x \mathring{R}^2 \delta \sqrt{|g|}. \tag{3.16c}$$

Using (A.12), we can write the variation of the square root as a function of the determinant of the inverse metric (see Appendix A for a detailed derivation), then we have (3.16b) and (3.16c) ready. However, we need to compute some variations to write (3.16a) in terms of the variation of the inverse metric. First, we will use the Palatini identity (B.4),

$$(\delta S)_1 = 2\int d^D x\sqrt{|g|}\mathring{R}g^{\sigma\eta}\left(\mathring{\nabla}_\sigma(\delta\mathring{\Gamma}_{\theta\eta}{}^{\theta}) - \mathring{\nabla}_\theta(\delta\mathring{\Gamma}_{\sigma\eta}{}^{\theta})\right). \tag{3.17}$$

Next, we can make integration by parts (C.11) to get

$$(\delta S)_1 = -2\int d^D x\sqrt{|g|}\left(\mathring{\nabla}_\sigma\mathring{R}g^{\sigma\eta}\delta\mathring{\Gamma}_{\theta\eta}{}^{\theta} - \mathring{\nabla}_\theta\mathring{R}g^{\sigma\eta}\delta\mathring{\Gamma}_{\sigma\eta}{}^{\theta}\right). \tag{3.18}$$

Here, we can see some higher-order derivatives appearing, as we have derivatives of the Ricci tensor and the Ricci tensor has second derivatives of the metric. Afterwards, we can use the variation of the connection (2.14) to write it in terms of the variation of the inverse metric, getting

$$(\delta S)_1 = 2\int d^D x\sqrt{|g|}\left(\mathring{\nabla}_\sigma\mathring{R}g^{\sigma\eta}g_{\theta\xi}\mathring{\nabla}_\eta(\delta g^{\theta\xi}) - \mathring{\nabla}_\theta\mathring{R}\mathring{\nabla}_\xi(\delta g^{\theta\xi})\right). \tag{3.19}$$

We have to integrate by parts again to get the final expression

$$(\delta S)_1 = -2\int d^D x\sqrt{|g|}\left(\mathring{\nabla}^2\mathring{R}g_{\theta\xi} - \mathring{\nabla}_\xi\mathring{\nabla}_\theta\mathring{R}\right)\delta g^{\theta\xi}. \tag{3.20}$$

Adding the other variation terms in equations (3.16), we get the variation coming from the first of the three addends in Gauss-Bonnet action (3.14),

$$\delta_{\mathring{R}^2} = \int d^D x\sqrt{|g|}\left(2\mathring{R}_{\sigma\eta}\mathring{R} - \frac{1}{2}g_{\sigma\eta}\mathring{R}^2 - 2g_{\sigma\eta}\mathring{\nabla}^2\mathring{R} + 2\mathring{\nabla}_\eta\mathring{\nabla}_\sigma\mathring{R}\right)\delta g^{\sigma\eta}. \tag{3.21}$$

It is important to see that we get terms with fourth-order derivatives of the metric in the last two addends, as we have second derivatives of the curvature and the curvature has already second-order derivatives of the metric.

If we follow the same procedure for the other two addends, we will end up with the total variation as follows,

$$\begin{aligned}\delta S = \int d^D x\sqrt{|g|}\Bigg(&-\frac{1}{2}g_{\varphi\chi}(\alpha\mathring{R}^2 + \beta\mathring{R}_{\sigma\eta}\mathring{R}^{\sigma\eta} + \gamma\mathring{R}_{\sigma\eta\theta\xi}\mathring{R}^{\sigma\eta\theta\xi}) \\ &+ \alpha\Big(2\mathring{R}_{\varphi\chi}\mathring{R} - 2g_{\varphi\chi}\mathring{\nabla}^2\mathring{R} + 2\mathring{\nabla}_\varphi\mathring{\nabla}_\chi\mathring{R}\Big) \\ &+ \beta\Big(-2\mathring{R}_{\sigma\varphi\chi}{}^{\eta}\mathring{R}_\eta{}^{\sigma} - \frac{1}{2}g_{\varphi\chi}\mathring{\nabla}^2\mathring{R} + \mathring{\nabla}_\varphi\mathring{\nabla}_\chi\mathring{R} - \mathring{\nabla}^2\mathring{R}_{\varphi\chi}\Big) \\ &+ \gamma\Big(-4\mathring{R}_{\varphi\sigma}\mathring{R}_\chi{}^{\sigma} - 4\mathring{R}_{\sigma\varphi\chi}{}^{\eta}\mathring{R}_\eta{}^{\sigma} + 2\mathring{R}_{\varphi\sigma\eta\theta}\mathring{R}_\chi{}^{\sigma\eta\theta} \\ &\quad + 2\mathring{\nabla}_\varphi\mathring{\nabla}_\chi\mathring{R} - 4\mathring{\nabla}^2\mathring{R}_{\varphi\chi}\Big)\Bigg)\delta g^{\varphi\chi}. \end{aligned} \tag{3.22}$$

We can see there very clearly that choosing $(\alpha, \beta, \gamma) = (1, -4, 1)$ is the only way of cancelling the higher-order metric derivatives, getting the Gauss-Bonnet variation,

$$\delta S_{\text{GB}} = \int d^D x \sqrt{|g|} \bigg(-\frac{1}{2} g_{\varphi\chi} (\mathring{R}^2 - 4\mathring{R}_{\sigma\eta}\mathring{R}^{\sigma\eta} + \mathring{R}_{\sigma\eta\theta\xi}\mathring{R}^{\sigma\eta\theta\xi}) \\ + 2\mathring{R}_{\varphi\chi}\mathring{R} - 4\mathring{R}_{\varphi\sigma}\mathring{R}_{\chi}{}^{\sigma} + 4\mathring{R}_{\sigma\varphi\chi}{}^{\eta}\mathring{R}_{\eta}{}^{\sigma} + 2\mathring{R}_{\varphi\sigma\eta\theta}\mathring{R}_{\chi}{}^{\sigma\eta\theta} \bigg) \delta g^{\varphi\chi}. \tag{3.23}$$

So that ends our demonstration.

Note that we will have boundary terms appearing after the first integration by parts, as we had for Einstein-Hilbert. For the first addend, we have done the integration by parts in (3.18), but it happens for every addend. The final combination of boundary terms, with the coefficients of Gauss-Bonnet, should be cancelled adding the Myers term [33] (in general, it should be added to every Lovelock action), but as its computation is analogous to the Gibbons-Hawking-York term, we skipped the derivation.

3.4 Lovelock Theories

Once we have seen that the only possible combination of second-order terms in the curvature that gives second-order equations of motion is the Gauss-Bonnet term, we are ready to generalize this fact for any order in the curvature thanks to Lovelock Theories.

Lovelock Theories are a set of extensions to General Relativity proposed by Lovelock in 1971 [31]. They appeared as a very straightforward generalization of Einstein-Hilbert action, as they are a sum of terms increasing in curvature order,

$$\mathcal{L}_N = a_0 L_0 + a_1 L_1 + \cdots + a_N L_N, \tag{3.24}$$

whose zeroth-, first- and second-order terms are the cosmological constant, the Ricci scalar and Gauss-Bonnet, respectively,

$$L_0 = \sqrt{|g|}\Lambda, \quad L_1 = \sqrt{|g|}\mathring{R}, \quad L_2 = \sqrt{|g|} \left(\mathring{R}^2 - 4\mathring{R}_{\sigma\eta}\mathring{R}^{\sigma\eta} + \mathring{R}_{\sigma\eta\theta\xi}\mathring{R}^{\sigma\eta\theta\xi} \right). \tag{3.25}$$

This property makes these theories an ideal candidate for a generalization of General Relativity, because when we are in four dimensions, the equations of motion of these three terms are the ones of standard General Relativity. Moreover, the following terms are identically zero, so we end up with standard General Relativity, but only when $D = 4$. There is no reason for not adding them when we are in more than four dimensions.

The fact of vanishing after a critical term does not only happen in four dimensions. In fact, it happens for every dimension and depends on the dimension we are working on. If we call n the curvature order, terms with $n < D/2$ are dynamical, terms with $n = D/2$ (if D is even) are topological and terms with $n > D/2$ vanish. We can see this last statement trivially if we write the general formula for the n-th Lovelock term with delta notation,

$$L_n = \sqrt{|g|}\delta^{\sigma_1 \ldots \sigma_{2n}}_{\eta_1 \ldots \eta_{2n}} \mathring{R}_{\sigma_1 \sigma_2}{}^{\eta_1 \eta_2} \cdots \mathring{R}_{\sigma_{2n-1} \sigma_{2n}}{}^{\eta_{2n-1} \eta_{2n}}. \tag{3.26}$$

There, we can see that all terms with $n > D/2$ vanish because we have repeated antisymmetrized indices.

The fact of always having second order differential equations of motion and reducing to standard General Relativity in four dimensions make Lovelock Theories very good as a generalization of standard General Relativity. We do not have any reason not to include these terms in a theory with more than four dimensions. Nevertheless, they also seem to introduce some changes that could lead to some phenomenological problems.

On the one hand, it is known that in these theories gravity does not propagate at the speed of light. Instead, the speed depends on the curvature of the spacetime, possibly leading to causality inconsistencies [1, 12].

Besides, it is known that solutions of Gauss-Bonnet or higher do not have a Newtonian limit. This happens because there is no linear term in the action to which one should take the limit. One could take the limit to second-order terms, but the term obtained does not decay as a Newtonian potential. Thus, we think that a_1 should not be zero in order to be able to take the classical limit of our new theory.

On the other hand, as can be noticed in (3.24), there are some proportionality coefficients not determined by the theory. This happens because the properties of these theories are accomplished for any coefficients. They should be determined using other theoretical methods, as gauging (A)dS [45], checking about stability or consistency of solutions [42, 39], or, in the future, experimental measurements.

The important thing to remark here is that the advantages of Lovelock Theories are mathematical, formal, and the problems are purely phenomenological. They might be fixed or, at least, minimized with the appropriate choice of coefficients, probably decaying fast enough.

3.5 Levi-Civita in metric-affine formalism

As we have seen, there are a lot of properties that make Lovelock Theories a very good candidate for expanding General Relativity. However, we would also like to have a sort of equivalence between metric and metric-affine formalisms. As a requisite to that, Levi-Civita should be a solution for every Lovelock Gravity in metric-affine formalism. That is what we are going to demonstrate in this section.

Before we begin, we should clarify that the definition of Lovelock Theories in metric-affine formalism remains invariant if we use the delta notation,

$$L_n = \sqrt{|g|}\delta^{\sigma_1\ldots\sigma_{2n}}_{\eta_1\ldots\eta_{2n}} R_{\sigma_1\sigma_2}{}^{\eta_1\eta_2}\cdots R_{\sigma_{2n-1}\sigma_{2n}}{}^{\eta_{2n-1}\eta_{2n}}. \tag{3.27}$$

However, if we expand this summation, we will find differences with the expression that we introduced as Gauss-Bonnet (3.2) at the beginning of this chapter, as that expression was only valid for metric formalism. In particular, the expression reduces to

$$\begin{aligned} L_2 &= \sqrt{|g|}\delta^{\sigma_1\ldots\sigma_4}_{\eta_1\ldots\eta_4} R_{\sigma_1\sigma_2}{}^{\eta_1\eta_2} R_{\sigma_3\sigma_4}{}^{\eta_3\eta_4} \\ &= \frac{1}{6}\left(R^2 - R_{\sigma\eta}R^{\eta\sigma} + 2R_{\sigma\eta}\tilde{R}^{\eta\sigma} - \tilde{R}_{\sigma\eta}\tilde{R}^{\eta\sigma} + R_{\sigma\eta\theta\xi}R^{\sigma\eta\theta\xi}\right), \end{aligned} \tag{3.28}$$

where $\tilde{R}$ is the co-Ricci, one of the independent contractions of the Riemann tensor, defined in (1.22).

We should also introduce some notation, so we get some expressions that will be very useful later. We will deal with a general action that depends on the metric and on the connection through the Riemann tensor. We want to get properties about the connection equation, so we can skip the variation of the metric and write

$$\delta S|_{\delta g=0} = \int d^D x\sqrt{|g|}\Sigma^{\sigma\eta\theta}{}_{\xi}\delta R_{\sigma\eta\theta}{}^{\xi}, \tag{3.29}$$

where we have defined the tensor

$$\Sigma^{\mu\nu\rho}{}_{\lambda} = \frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta R_{\mu\nu\rho}{}^{\lambda}}. \tag{3.30}$$

With this definition, we can get the connection equation in terms of Σ following the same procedure as in Einstein-Hilbert, step by step, using the Palatini identity and integrating by parts, getting

$$0 = \nabla_\sigma \Sigma^{\sigma\mu\nu}{}_{\rho} + \frac{1}{2}g^{\sigma\eta}\nabla_\theta g_{\sigma\eta}\Sigma^{\theta\mu\nu}{}_{\rho} - \frac{1}{2}T_{\sigma\eta}{}^{\mu}\Sigma^{\sigma\eta\nu}{}_{\rho} + T_{\sigma\eta}{}^{\eta}\Sigma^{\sigma\mu\nu}{}_{\rho}. \tag{3.31}$$

As this is a completely generic calculation, we can see that this equation is valid for any action, even for Lovelock Theories, for which the tensor Σ is particularly simple.

From this expression we can already see some interesting facts. For example, all addends cancel trivially when choosing Levi-Civita, except the first. In order to see if this particular addend vanishes or not, we have to know the expression of Σ. Let's begin with an example. For Gauss-Bonnet,

$$\Sigma^{\mu\nu}{}_{\rho\lambda} = 2\delta^{\mu\nu\sigma\eta}_{\rho\lambda\theta\xi} R_{\sigma\eta}{}^{\theta\xi}. \tag{3.32}$$

If we want to see if Levi-Civita is a solution of (3.31), we can compute the divergence of these expressions to get the result of the first addend. In this case, we would get

$$\mathring{\nabla}_\sigma \mathring{\Sigma}^{\sigma\mu\nu}{}_\rho = 2\delta^{\sigma\mu\theta\xi}_{\eta\rho\varphi\chi} g^{\nu\eta} \mathring{\nabla}_\sigma \mathring{R}_{\theta\xi}{}^{\varphi\chi}, \tag{3.33}$$

where we can see that the summation over the indices σ, θ, ξ gives us something proportional to the Bianchi identity, so it vanishes. Same happens with the $(n+1)$-th Lovelock term. Using (3.27),

$$\Sigma^{\mu\nu}{}_{\rho\lambda} = (n+1)\delta^{\mu\nu\sigma_1\ldots\sigma_{2n}}_{\rho\lambda\eta_1\ldots\eta_{2n}} R_{\sigma_1\sigma_2}{}^{\eta_1\eta_2} \cdots R_{\sigma_{2n-1}\sigma_{2n}}{}^{\eta_{2n-1}\eta_{2n}}, \tag{3.34}$$

and from that expression we can also see the proportionality to the Bianchi identity,

$$\mathring{\nabla}_\theta \mathring{\Sigma}^{\theta\mu\nu}{}_\rho = n(n+1)\delta^{\theta\mu\sigma_1\ldots\sigma_{2n}}_{\xi\rho\eta_1\ldots\eta_{2n}} g^{\nu\xi} \mathring{\nabla}_\theta R_{\sigma_1\sigma_2}{}^{\eta_1\eta_2} R_{\sigma_3\sigma_4}{}^{\eta_3\eta_4} \cdots R_{\sigma_{2n-1}\sigma_{2n}}{}^{\eta_{2n-1}\eta_{2n}}. \tag{3.35}$$

Thus, we can now be sure that Levi-Civita is a solution of metric-affine formalism for every Lovelock theory, and this puts Lovelock Theories in a privileged status among other generalizations of General Relativity, as they are the only theories that have Levi-Civita as a solution in metric-affine formalism. We have mentioned its importance when we talked about metric and metric-affine formalisms and the justification of Levi-Civita connection: we have a reason for choosing Levi-Civita now. The solutions of metric formalism are included in metric-affine formalism, so when choosing Levi-Civita they are equivalent [16, 5, 13]. However, we would like them to be equivalent without the condition of choosing Levi-Civita in metric-affine formalism. We have discussed that the physics seem to remain the same for Einstein-Hilbert in Chapter 2. We will conclude that discussion in the next section and we will also answer that question for Gauss-Bonnet in Chapter 4.

3.6 Projective invariance

When we talked about Einstein-Hilbert in Chapter 2, we studied the effect of the vector A in the dynamics. In this section, we will present the final justification about why A is not physical at all.

It is related with transformations at the level of the action, so we will start reminding an example of action that is invariant under particular transformations.

If we take the action

$$S = \frac{1}{2}\int d^D x\sqrt{|g|}(\partial\phi)^2, \tag{3.36}$$

we can see that we could transform the matter field as

$$\phi \longrightarrow \phi' = \phi + c \tag{3.37}$$

and, given that c does not depend on any coordinate, the action remains invariant.

If we calculate the equations of motion of the matter field, we get

$$\nabla^2\phi = 0, \tag{3.38}$$

which is the scalar wave equation. If we find any solution of it, ϕ_0, we can see that applying the transformation (3.37) to that solution, we get a set of solutions, due to the symmetry that the action had. All the solutions are equivalent physically, we cannot determine the value of that constant using physical measurements, but they can be distinguished mathematically by that constant.

In our case, in Lovelock Theories, we also have a symmetry that plays an important role. Consider the following transformation to an arbitrary connection,

$$\Gamma_{\mu\nu}{}^{\rho} \longrightarrow \hat{\Gamma}_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho} + A_\mu\delta^\rho_\nu. \tag{3.39}$$

To see its effect on Lovelock Theories, we have to calculate the variation of the curvature tensor,

$$R_{\mu\nu\rho}{}^{\lambda} \longrightarrow \hat{R}_{\mu\nu\rho}{}^{\lambda} = R_{\mu\nu\rho}{}^{\lambda} + F_{\mu\nu}\delta^\lambda_\rho, \tag{3.40}$$

with F already defined in (2.56). We have already seen an example of this transformation, when taking Levi-Civita as the starting connection, in (2.55). When contracting it to get the Ricci tensor, we get

$$R_{\mu\nu} \longrightarrow \hat{R}_{\mu\nu} = R_{\mu\nu} + F_{\mu\nu}, \tag{3.41}$$

where we can see that this only affects the antisymmetric part of the Ricci tensor. The symmetric part remains invariant. Due to that, the Ricci scalar remains invariant,

$$R \longrightarrow \hat{R} = R, \tag{3.42}$$

hence, that is why in Einstein-Hilbert we had this symmetry in the action.

Once we have calculated this, we can see what happens in the general Lovelock action,

$$\begin{aligned} S &= \int d^D x\sqrt{|g|}\delta^{\sigma_1\ldots\sigma_{2n}}_{\eta_1\ldots\eta_{2n}}\hat{R}_{\sigma_1\sigma_2}{}^{\eta_1\eta_2}\cdots\hat{R}_{\sigma_{2n-1}\sigma_{2n}}{}^{\eta_{2n-1}\eta_{2n}} \\ &= \int d^D x\sqrt{|g|}\delta^{\sigma_1\ldots\sigma_{2n}}_{\eta_1\ldots\eta_{2n}}\left(R_{\sigma_1\sigma_2}{}^{\eta_1\eta_2} + F_{\sigma_1\sigma_2}g^{\eta_1\eta_2}\right)\cdots\hat{R}_{\sigma_{2n-1}\sigma_{2n}}{}^{\eta_{2n-1}\eta_{2n}} \\ &= \int d^D x\sqrt{|g|}\delta^{\sigma_1\ldots\sigma_{2n}}_{\eta_1\ldots\eta_{2n}}R_{\sigma_1\sigma_2}{}^{\eta_1\eta_2}\cdots R_{\sigma_{2n-1}\sigma_{2n}}{}^{\eta_{2n-1}\eta_{2n}}. \end{aligned} \tag{3.43}$$

They remain invariant due to the antisymmetrization of δ, which cancels with the metric in every factor.

Thus, we have seen that we have a symmetry in our action, the one presented in (3.39). This means that if we have a solution, we have a set. It is not difficult to see it right at the equation of motion of the action (3.31) as the tensor Σ,

$$\begin{aligned}\hat{\Sigma}^{\mu\nu}{}_{\rho\lambda} &= (n+1)\delta^{\mu\nu\sigma_1\ldots\sigma_{2n}}_{\rho\lambda\eta_1\ldots\eta_{2n}}\hat{R}_{\sigma_1\sigma_2}{}^{\eta_1\eta_2}\cdots\hat{R}_{\sigma_{2n-1}\sigma_{2n}}{}^{\eta_{2n-1}\eta_{2n}} \\ &= \Sigma^{\mu\nu}{}_{\rho\lambda},\end{aligned} \tag{3.44}$$

also remains invariant for the same reason.

With all this information, we can now completely understand why Palatini connections (2.53) are a solution of the equation of motion of the connection, as Levi-Civita was a particular solution and the transformation induces the complete set. Besides, this also happens in *every* Lovelock theory, for the same reasons.

Even though this transformation and the F tensor reminds us of the gauge transformations of the electromagnetic field, this has nothing to do with that. The difference is that, in this case, we can change the vector A in any way, being able to completely cancel the tensor F. The tensor F is not physical at all.

Thus, we have found the meaning of the vector A: it is the result of a symmetry in the Lagrangian. That is why we can now be absolutely sure that there are no physical meaning of A in Einstein-Hilbert or in any Lovelock theory. Moreover, from now on, we can ignore this part of the solution completely, as we know it does not contribute to the physics. We can always generate any solution of the set adding the arbitrary vector A again.

Also, in this chapter, we have also explained why Lovelock Theories are a very good proposal for extending General Relativity, explaining in detail the properties they have and demonstrating them. In the next chapter, we will try to find more solutions for Gauss-Bonnet and we will discuss its physical implications and the differences with respect to the Palatini solutions.

Chapter 4

Gauss-Bonnet

In previous chapters we showed what is metric-affine formalism and which properties should we expect from applying it to Lovelock Theories. We also talked about Levi-Civita as a solution and the projective invariance that all Lovelock Theories have.

In this chapter, we are going to apply all this knowledge to find a solution of Gauss-Bonnet, different from a projective transformation, and we are going to discuss its implications. This research is mainly published in [24].

4.1 The Weyl connection as a solution

We are going to consider the D-dimensional Gauss-Bonnet action in the metric-affine formalism,

$$S_{GB} = \int d^D x \sqrt{|g|} \delta^{\sigma_1 \ldots \sigma_4}_{\eta_1 \ldots \eta_4} R_{\sigma_1 \sigma_2}{}^{\eta_1 \eta_2} R_{\sigma_3 \sigma_4}{}^{\eta_3 \eta_4}, \tag{4.1}$$

and we are going to calculate the equations of motion for this action.

We will take benefit of the general connection equation that we calculated in (3.31), valid for any action,

$$\nabla_\sigma \Sigma^{\sigma\mu\nu}{}_\rho + \frac{1}{2} g^{\sigma\eta} \nabla_\theta g_{\sigma\eta} \Sigma^{\theta\mu\nu}{}_\rho - \frac{1}{2} T_{\sigma\eta}{}^\mu \Sigma^{\sigma\eta\nu}{}_\rho + T_{\sigma\eta}{}^\eta \Sigma^{\sigma\mu\nu}{}_\rho = 0, \tag{4.2}$$

where, for Gauss-Bonnet,

$$\Sigma^{\mu\nu}{}_{\rho\lambda} = g_{\rho\sigma} \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta R_{\mu\nu\sigma}{}^\lambda} = 2 \delta^{\mu\nu\sigma\eta}_{\rho\lambda\theta\xi} R_{\sigma\eta}{}^{\theta\xi}. \tag{4.3}$$

Besides, the connection can be decomposed into its Levi-Civita part and its distortion,

$$\Gamma_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + \Xi_{\mu\nu}{}^{\rho}. \tag{4.4}$$

Thus, if we substitute this decomposition into the connection equation, we get the equivalent equation

$$\mathring{\nabla}_\sigma \Sigma^{\sigma\mu\nu}{}_{\rho} + \Xi_{\sigma\eta}{}^{\nu}\Sigma^{\sigma\mu\eta}{}_{\rho} - \Xi_{\sigma\rho}{}^{\eta}\Sigma^{\sigma\mu\nu}{}_{\eta} = 0 \tag{4.5}$$

which is still valid for any action, not just for Gauss-Bonnet, as we have not used the expression for the Σ tensor.

Furthermore, using the antisymmetry of the Lovelock Σ tensor in the last two indices, it is easy to show that one can deduce the necessary (though not sufficient) condition for the connection,

$$(\Xi_{\sigma\eta\nu} + \Xi_{\sigma\nu\eta})\Sigma^{\sigma\mu\eta}{}_{\rho} + (\Xi_{\sigma\eta\rho} + \Xi_{\sigma\rho\eta})\Sigma^{\sigma\mu\eta}{}_{\nu} = 0, \tag{4.6}$$

valid for any Lovelock and, in particular, for Gauss-Bonnet.

It is very straightforward to get the equation of motion for the metric,

$$\delta^{\sigma\eta\alpha\beta}_{\mu\xi\gamma\delta} R_{\sigma\eta\nu}{}^{\xi} R_{\alpha\beta}{}^{\gamma\delta} + \delta^{\sigma\eta\alpha\beta}_{\nu\xi\gamma\delta} R_{\sigma\eta\mu}{}^{\xi} R_{\alpha\beta}{}^{\gamma\delta} - \frac{1}{2} g_{\mu\nu} \delta^{\sigma\eta\alpha\beta}_{\theta\xi\gamma\delta} R_{\sigma\eta}{}^{\theta\xi} R_{\alpha\beta}{}^{\gamma\delta} = 0. \tag{4.7}$$

It can be written in a more compact way using the value of the Σ tensor,

$$R_{\sigma\eta\mu}{}^{\xi}\Sigma^{\sigma\eta}{}_{\nu\xi} + R_{\sigma\eta\nu}{}^{\xi}\Sigma^{\sigma\eta}{}_{\mu\xi} - \frac{1}{2} g_{\mu\nu} R_{\sigma\eta}{}^{\theta\xi}\Sigma^{\sigma\eta}{}_{\theta\xi} = 0. \tag{4.8}$$

We will try to find a non-trivial connection, that is, not of the form of the Palatini connections (2.53), that solves the metric and connection equations. Our starting point will be the generalized Weyl connection,

$$\Gamma_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + A_\mu \delta^\rho_\nu + B_\nu \delta^\rho_\mu - C^\rho g_{\mu\nu}, \tag{4.9}$$

characterized by the three arbitrary vector fields A, B and C. Strictly speaking, A represents the projective symmetry of the action and can be gauged away completely. However for future reference, we prefer to maintain the calculation general for the moment. The Riemann for this connection is then given by

$$\begin{aligned} R_{\mu\nu\rho}{}^{\lambda} = \mathring{R}_{\mu\nu\rho}{}^{\lambda} &+ F_{\mu\nu}(A)\delta^\lambda_\rho + (\mathring{\nabla}_\mu B_\rho - B_\mu B_\rho)\delta^\lambda_\nu - (\mathring{\nabla}_\nu B_\rho - B_\nu B_\rho)\delta^\lambda_\mu \\ &- (\mathring{\nabla}_\mu C^\lambda - C_\mu C^\lambda) g_{\nu\rho} + (\mathring{\nabla}_\nu C^\lambda - C_\nu C^\lambda) g_{\mu\rho} - B_\sigma C^\sigma (\delta^\lambda_\mu g_{\nu\rho} - \delta^\lambda_\nu g_{\mu\rho}), \end{aligned} \tag{4.10}$$

where F was already defined in (2.56), and the Σ tensor is given by

$$\begin{aligned} \Sigma^{\mu\nu}{}_{\rho\lambda} = \mathring{\Sigma}^{\mu\nu}{}_{\rho\lambda} &+ \frac{1}{2}(D-3)\delta^{\mu\nu\sigma}_{\rho\lambda\eta}(\mathring{\nabla}_\sigma B^\eta - B_\sigma B^\eta + \mathring{\nabla}_\sigma C^\eta - C_\sigma C^\eta) \\ &+ \frac{1}{6}(D-2)(D-3)\delta^{\mu\nu}_{\nu\lambda} B_\sigma C^\sigma. \end{aligned} \tag{4.11}$$

Using the necessary condition (4.6), we find that

$$0 = 2(B_\sigma - C_\sigma)\Sigma_{(\mu}{}^{\nu\sigma}{}_{\rho)} + (B_{(\mu} - C_{(\mu})\Sigma^{\sigma\nu}{}_{\rho)\sigma}, \tag{4.12}$$

which is satisfied only when $B = C$. If we then gauge fix the projective symmetry by choosing also $A = B$, so that we can write the connection (4.9) as a non-integrable Weyl connection,

$$\Gamma_{\mu\nu}{}^\rho = \mathring{\Gamma}_{\mu\nu}{}^\rho + B_\mu \delta^\rho_\nu + B_\nu \delta^\rho_\mu - B^\rho g_{\mu\nu}, \tag{4.13}$$

with B for the moment an arbitrary vector field, whose precise form should be determined by the equations of motion. Filling in this expression into the connection equation (4.5) yields

$$\begin{aligned} 0 &= \mathring{\nabla}_\sigma \Sigma^{\sigma\mu}{}_{\nu\rho} + \Xi_{\sigma\eta\nu}\Sigma^{\sigma\mu}{}_{\theta\rho} g^{\eta\theta} - \Xi_{\sigma\rho\eta}\Sigma^{\sigma\mu}{}_{\nu\theta} g^{\eta\theta} \\ &= \frac{1}{12}(D-4)(2B_{[\rho}\mathring{R}\delta^\mu_{\nu]} + 4B^\sigma \mathring{R}_{\sigma[\nu}\delta^\mu_{\rho]} - 2B^\sigma \mathring{R}_{\nu\rho\sigma}{}^\mu) \\ &\quad + \frac{1}{6}(D-4)(D-3)(2B_\sigma \mathring{\nabla}_{[\nu} B^\sigma \delta^\mu_{\rho]} - 2B_{[\nu}\mathring{\nabla}_{|\sigma|} B^\sigma \delta^\mu_{\rho]} + 2B_{[\nu}\mathring{\nabla}_{\rho]} B^\mu) \\ &\quad - \frac{1}{6}(D-4)(D-3)(D-2) B_\sigma B^\sigma B_{[\nu}\delta^\mu_{\rho]}, \end{aligned} \tag{4.14}$$

which is satisfied for an arbitrary vector field B in $D = 4$. On the other hand, the metric equation (4.8) becomes

$$\begin{aligned} 0 &= \mathring{R}_{\sigma\eta\mu}{}^\xi \mathring{\Sigma}^{\sigma\eta}{}_{\nu\xi} + \mathring{R}_{\sigma\eta\nu}{}^\xi \mathring{\Sigma}^{\sigma\eta}{}_{\mu\xi} - \frac{1}{2} g_{\mu\nu} \mathring{R}_{\sigma\eta}{}^{\theta\xi} \mathring{\Sigma}^{\sigma\eta}{}_{\theta\xi} \\ &\quad + \frac{1}{3}(D-4)\Lambda^{(1)}_{\mu\nu} + \frac{1}{3}(D-4)(D-3)\Lambda^{(2)}_{\mu\nu} + \frac{1}{12}(D-4)(D-3)(D-2)\Lambda^{(3)}_{\mu\nu}, \end{aligned} \tag{4.15}$$

where the expression for $\Lambda^{(1)}$ is

$$\begin{aligned} \Lambda^{(1)}_{\mu\nu} &= \mathring{\nabla}_{(\mu} B_{\nu)} \mathring{R} + 2\mathring{\nabla}_\sigma B^\sigma \left(\mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathring{R} \right) + 2\mathring{\nabla}^\sigma B^\eta \mathring{R}_{\sigma(\mu\nu)\eta} - 2\mathring{\nabla}_{(\mu} B^\sigma \mathring{R}_{\nu)\sigma} \\ &\quad - 2\mathring{\nabla}^\sigma B_{(\mu}\mathring{R}_{\nu)\sigma} + 2\mathring{\nabla}_\sigma B_\eta \mathring{R}^{\sigma\eta} g_{\mu\nu} + (D-5) B_\sigma B^\sigma \left(\mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathring{R} \right) \\ &\quad - B_\mu B_\nu \mathring{R} - 2B^\sigma B^\eta \mathring{R}_{\sigma\eta} g_{\mu\nu} + 4(D-3) B^\sigma B_{(\mu}\mathring{R}_{\nu)\sigma} - 2B^\sigma B^\eta \mathring{R}_{\sigma(\mu\nu)\eta}, \end{aligned} \tag{4.16}$$

the expression for $\Lambda^{(2)}$ is given by

$$\begin{aligned} \Lambda^{(2)}_{\mu\nu} &= 2\mathring{\nabla}_{(\mu} B_{\nu)} \mathring{\nabla}_\sigma B^\sigma - 2\mathring{\nabla}_\sigma B_{(\mu}\mathring{\nabla}_{\nu)} B^\sigma - \mathring{\nabla}_\sigma B^\sigma \mathring{\nabla}_\eta B^\eta g_{\mu\nu} + \mathring{\nabla}_\sigma B^\eta \mathring{\nabla}_\eta B^\sigma g_{\mu\nu} \\ &\quad + (D-4)\mathring{\nabla}_{(\mu} B_{\nu)} B_\sigma B^\sigma - 2\mathring{\nabla}_\sigma B^\sigma B_\mu B_\nu + 2\mathring{\nabla}_\sigma B_{(\mu} B_{\nu)} B^\sigma \\ &\quad + 2B^\sigma B_{(\mu}\mathring{\nabla}_{\nu)} B_\sigma - 2B^\sigma B^\eta \mathring{\nabla}_\sigma B_\eta g_{\mu\nu} + (D-4) B_\sigma B^\sigma \mathring{\nabla}_\eta B^\eta g_{\mu\nu} \end{aligned} \tag{4.17}$$

and $\Lambda^{(3)}$ can be written as

$$\Lambda^{(3)}_{\mu\nu} = 4B_\sigma B^\sigma B_\mu B_\nu + (D-5)B_\sigma B^\sigma B_\eta B^\eta g_{\mu\nu}. \tag{4.18}$$

However, independently of the value of these Λ tensors, the equation of motion reduces to the equation of motion of the metric in the second-order formalism when $D = 4$,

$$\mathring{R}_{\sigma\eta\mu}{}^{\xi}\mathring{\Sigma}^{\sigma\eta}{}_{\nu\xi} + \mathring{R}_{\sigma\eta\nu}{}^{\xi}\mathring{\Sigma}^{\sigma\eta}{}_{\mu\xi} - \frac{1}{2}g_{\mu\nu}\mathring{R}_{\sigma\eta}{}^{\theta\xi}\mathring{\Sigma}^{\sigma\eta}{}_{\theta\xi} = 0. \tag{4.19}$$

In other words, the Weyl connection (4.13) is a solution of four-dimensional metric-affine Gauss-Bonnet gravity for any metric that satisfies the equations of the metric formalism, which are all metrics since Gauss-Bonnet is a topological term in four dimensions in metric formalism, as we demonstrated in Section 3.1.

4.2 A vector symmetry in four dimensions

In Section 3.6 we have seen that the existence of the nontrivial Palatini connections (2.53) as a solution in any metric-affine Lovelock theory is a consequence of the projective symmetry. In this section we will argue that our new solution (4.13) is also related to a symmetry, namely the conformal invariance of the four-dimensional Gauss-Bonnet action.

Conformal invariance and Weyl transformations have not been studied much in the context of metric-affine gravity. In [8] conformal rescalings of the metric are used to discuss the relations between the metric and Palatini formalism of $f(R)$ gravity in both the Einstein and the Jordan frame. More recently, in [20] a detailed classification was given of the metric-affine theories in terms of their scale invariance under rescalings of the metric, the coframe and/or the connection.

It is well known that the metric Gauss-Bonnet theory in $D = 4$ is invariant under conformal transformations of the metric,

$$g_{\mu\nu} \longrightarrow \hat{g}_{\mu\nu} = e^{2\phi}g_{\mu\nu}, \tag{4.20}$$

which on its turn change the Christoffel symbols as

$$\mathring{\Gamma}_{\mu\nu}{}^{\rho} \longrightarrow \hat{\Gamma}_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + \partial_\mu\phi\delta^\rho_\nu + \partial_\nu\phi\delta^\rho_\mu - \partial^\rho\phi g_{\mu\nu}. \tag{4.21}$$

On the other hand, as any metric-affine quadratic curvature term [4], the four-dimensional metric-affine Gauss-Bonnet theory is easily seen to have conformal weight zero, that is, to be invariant under the conformal transformations (4.20) of the metric, though in this context without a accompanying transformation in the affine connection, as the latter is independent of the metric.

The invariance of the $D = 4$ metric-affine Gauss-Bonnet term under the metric transformation (4.20) shows that in the metric formalism the transformation of the metric (4.20) and of the connection (4.21) are in fact quite independent of each other: the former acts effectively only on the explicit metrics in the contraction of the Riemann tensors and the effect of (4.21) remains constrained to the curvature tensors. One could therefore ask the question whether the metric-affine Gauss-Bonnet action is also invariant under (something similar to) the transformation (4.21), independently of a metric transformation.

In [2, 25, 26] it was already observed that actions with Gauss-Bonnet-like quadratic curvature invariants (that is, general combinations of quadratic contractions of the Riemann tensor, that reduce to the metric Gauss-Bonnet action when the Levi-Civita connection is imposed), when equipped with the (non-integrable) Weyl connection (4.13), can be written as the standard (Levi-Civita) Gauss-Bonnet action plus a series of non-minimal coupling terms for the Weyl field B, plus a kinetic term $F_{\sigma\eta}(B)F^{\sigma\eta}(B)$. Curiously enough, the non-minimal couplings vanish precisely in $D = 4$ and the kinetic term is multiplied by a coefficient that vanishes when the parameters of the extended Gauss-Bonnet term are chosen such that the action is the actual metric-affine Gauss-Bonnet term (3.28). In other words, the metric-affine Gauss-Bonnet action (3.28) does not see the difference between the substituting the Weyl or the Levi-Civita connection.

Inspired by this and by the fact that in the previous section we found that the integrable Weyl connection (4.13) is a solution to the metric and the connection equation, it seems logical to check the invariance of the Gauss-Bonnet action in metric-affine formalism (4.1) under the transformation

$$\Gamma_{\mu\nu}{}^{\rho} \longrightarrow \hat{\Gamma}_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho} + B_\mu \delta^\rho_\nu + B_\nu \delta^\rho_\mu - B^\rho g_{\mu\nu}, \tag{4.22}$$

not just as a deformation of the Levi-Civita connection (as in [2, 25, 26]), but as a transformation acting on general connections in the action (4.1), much in the same way as the projective transformations (3.39). Note that the $B_\mu \delta^\rho_\nu$ term can be undone by a projective transformation with parameter $-B_\mu$, so we can actually simplify the transformation (4.22) to

$$\Gamma_{\mu\nu}{}^{\rho} \longrightarrow \hat{\Gamma}_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho} + B_\nu \delta^\rho_\mu - B^\rho g_{\mu\nu}. \tag{4.23}$$

Up to boundary terms coming from integrating by parts, the four-dimensional action then transforms as

$$\begin{aligned} \mathcal{L}_{\rm GB}(g,\Gamma) \longrightarrow \hat{\mathcal{L}}_{\rm GB}(g,\hat{\Gamma}) = \mathcal{L}_{\rm GB}(g,\Gamma) &- 4B^\sigma B^\eta (R_{\sigma\eta} + \tilde{R}_{\sigma\eta}) \\ &- 2Q^{\sigma\eta\theta}\Lambda^{(1)}_{\sigma\eta\theta} - 2Q^{(1)\sigma}\Lambda^{(2)}_\sigma + 2Q^{(2)\sigma}\Lambda^{(3)}_\sigma, \end{aligned} \tag{4.24}$$

where $\tilde{R}$ is the co-Ricci tensor (1.22), $Q^{(1)}_\mu = Q_{\mu\sigma}{}^{\sigma}$ and $Q^{(2)}_\mu = Q^{\sigma}{}_{\sigma\mu}$ are the two traces of the non-metricity tensor (1.26), the expression for $\Lambda^{(1)}$ is

$$\begin{aligned}\Lambda^{(1)}_{\mu\nu\rho} = {} & B_\mu(R_{\nu\rho} + \tilde{R}_{\nu\rho}) + B^\sigma(R_{\sigma\nu\mu\rho} + R_{\sigma\nu\rho\mu}) \\ & - B^\sigma B_\nu(Q_{\sigma\mu\rho} - 2Q_{\rho\sigma\mu}) - B_\mu B_\nu(Q^{(1)}_\rho - Q^{(2)}_\rho) \\ & + 2B_\mu\nabla_\nu B_\nu + 4B_\nu\nabla_\rho B_\mu + 2B_\mu B^\sigma T_{\sigma\rho\mu} + 2B_\mu B_\nu T_{\rho\sigma}{}^\sigma, \end{aligned} \tag{4.25}$$

the expression for $\Lambda^{(2)}$ is given by

$$\begin{aligned}\Lambda^{(2)}_\mu = {} & B^\sigma(R_{\sigma\mu} - \tilde{R}_{\sigma\mu} - g_{\sigma\mu}R) - 2B_\mu B_\sigma B^\sigma \\ & - 2B_\mu\nabla_\sigma B^\sigma + 2B^\sigma\nabla_\sigma B_\mu + 3B_\mu B^\sigma Q^{(2)}_\sigma \end{aligned} \tag{4.26}$$

and $\Lambda^{(3)}$ can be written as

$$\Lambda^{(3)}_\mu = B^\sigma(R_{\sigma\mu} + R_{\sigma\mu}) + 2B_\mu\nabla_\sigma B^\sigma + 2B^\sigma\nabla_\sigma B_\mu + 2B_\mu B^\sigma T_{\sigma\eta}{}^\eta. \tag{4.27}$$

We can see then that in fact the four-dimensional metric-affine Gauss-Bonnet term (4.1) with a general connection is not invariant under this simplified Weyl transformation (4.23). However, taking into account that the Ricci and the co-Ricci tensor are in general related to each other as

$$\tilde{R}_{\mu\nu} = -R_{\mu\nu} + g^{\sigma\eta}\nabla_\mu Q_{\sigma\nu\eta} + g^{\sigma\eta}\nabla_\sigma Q_{\mu\nu\eta} + g^{\sigma\eta}T_{\mu\sigma}{}^\theta Q_{\theta\nu\eta}, \tag{4.28}$$

it is clear that the difference between $\mathcal{L}_{\text{GB}}(g,\Gamma)$ and $\hat{\mathcal{L}}_{\text{GB}}(g,\hat{\Gamma})$ is proportional to the non-metricity tensor, its derivatives and its traces. In other words, the simplified Weyl transformation (4.23) is indeed a symmetry, not of the full four-dimensional metric-affine Gauss-Bonnet action, but of the restriction of this theory to the subset of metric-compatible connections, which turns out to be a consistent truncation of the full theory [24]. The symmetry transformation (4.23) not only generalizes the results of [2, 25, 26], but also explains why the Weyl connection (4.13) appears as a solution to the Palatini formalism in the four-dimensional Gauss-Bonnet action: it arises by acting on the Levi-Civita solution first with the new vector symmetry (4.23) and then with a projective transformation (3.39) with the same parameter. Note that the order of these transformations is important, as the vector transformation is only a symmetry on the subset of metric-compatible connections. This subset itself is not invariant under projective transformations, since any projective transformation necessarily induces a non-trivial non-metricity:

$$Q_{\mu\nu\rho} \longrightarrow \hat{Q}_{\mu\nu\rho} = Q_{\mu\nu\rho} + 2A_\mu g_{\nu\rho}. \tag{4.29}$$

4.3 Solution space and interpretation

While looking for solutions of the connection equation of metric-affine Gauss-Bonnet theory (4.1), we have identified a number of transformations in the theory.

Besides the invariance under projective transformations (3.39), present in any dimension, we also found a vector transformation, the simplified Weyl transformation (4.23), which is a symmetry specifically in four-dimensions and only if we consider the theory to be restricted to metric-compatible connections ($\mathcal{L}_{\rm GB}|_{Q=0}$). However, this vector transformation will play an important role in the full (four-dimensional) theory $\mathcal{L}_{\rm GB}$.

To our knowledge, this vector symmetry (4.23) of the truncated theory is new, although a special case was already observed in [2, 25, 26]. Both the A and B vector transformations seem somehow to be related to the conformal invariance of the four-dimensional Gauss-Bonnet action in the metric formalism (4.20) and (4.21).

Note that the conformal weight of the four-dimensional Gauss-Bonnet term is zero, both in the metric as in the metric-affine formalism. Therefore, in the metric case, the $\partial\phi$ terms that come from the transformation of the Levi-Civita connection cancel out among each other, and hence the transformation rules for the metric (4.20) and the connection (4.21) do not interfere with each other in the variation of the Gauss-Bonnet action (4.1). Moreover, in the metric-affine formalism, where the metric and the affine connection are independent variables, one can separate both transformations completely, finding that the action is invariant under both of them separately, at least in the subset of metric-compatible connections. The remarkable thing is that the metric-compatible Gauss-Bonnet term allows not only for integrable Weyl vectors $B_\mu = \partial_\mu\phi$, but also for non-integrable ones, $B_\mu \neq \partial_\mu\phi$, as the transformation is no longer related to a conformal transformation of the metric.

To understand the mathematical structure of the space of solutions of the full four-dimensional Gauss-Bonnet action (4.1), it is necessary to see how projective transformations (3.39) and simplified Weyl transformations (4.23) act on the connections. It is straightforward to see that the projective transformation changes both the trace of the torsion and the non-metricity, but that the B vector transformation only acts on the trace of the torsion and leaves the non-metricity invariant:

$$T_{\mu\nu}{}^{\rho} \longrightarrow T_{\mu\nu}{}^{\rho} + 2(A_{[\mu} + B_{[\mu})\delta^{\rho}_{\nu]}, \qquad Q_{\mu\nu\rho} \longrightarrow Q_{\mu\nu\rho} + 2A_\mu g_{\nu\rho}. \tag{4.30}$$

There is a certain similarity, although also mayor differences, between our simplified Weyl transformation (4.23) and the torsion/non-metricity duality discussed in [21]. There it was shown that in $f(R)$ gravity the same physical situation can be described by different geometrical descriptions, either in terms of the torsion or in terms of the non-metricity, due to the fact that the projective symmetry of these theories interchanges the degrees of freedom of the torsion and the non-metricity (see also [3] for a similar observation in the context of the Einstein-Hilbert action). As can be seen from (4.30), this property is not limited to four-dimensional $f(R)$ gravity, but is present in any projectively invariant theory that allows the Weyl connection as a solution. However, an important difference between our case and [21] is that the B vector transformation in general is not a duality that relates physically equivalent situations, but, as we will show, a solution generating transformation, that maps certain connections onto other physically nonequivalent ones.

As we mentioned before, the B transformation is a symmetry when the theory is restricted to the subset of metric-compatible connections, but not of the full theory. This means that the connection space in the truncated theory $\mathcal{L}_{\rm GB}|_{Q=0}$ can be divided into equivalence classes, which are the orbits of the B transformations. Two connections in the same orbit differ by the trace of the torsion and are physically indistinguishable, as the B transformation is a symmetry in $\mathcal{L}_{\rm GB}|_{Q=0}$. Two connections in distinct orbits differ also in the traceless parts of the torsion.

However, from the point of view of the full theory $\mathcal{L}_{\rm GB}$, the B transformation is not a symmetry, but a solution-generating transformation, as different solutions of the (consistently) truncated theory $\mathcal{L}_{\rm GB}|_{Q=0}$ are guaranteed to be also solutions of the full theory. Within the $Q = 0$ subset of the full theory, the B transformation hence maps solutions of the connection equation in other, physically nonequivalent solutions. On the other hand, outside the $Q = 0$ subset, the flow of the B transformations also exists, but possibly map solutions of the theory into connections that do not satisfy the equations of motion.

Finally, the projective transformation (3.39) does not maintain solutions inside the $Q = 0$ subset, as it changes the trace of the non-metricity (as well as the trace of the torsion). The orbits of the A transformation that cross the $Q = 0$ subset have a pure-trace non-metricity,

$$Q_{\mu\nu\rho} = \frac{1}{4} Q_{\mu\sigma}{}^{\sigma} g_{\nu\rho}, \tag{4.31}$$

while the connections that have additional non-trivial parts of non-metricity lay on orbits of A that do not intersect the $Q = 0$ subset. Since the projective transformation is a symmetry of the full action, all connections on the same orbit of A are indistinguishable and hence physically equivalent. We can get a visual idea of this space of solutions in figure 4.1.

With this structure in mind, we can see that the two-vector family of solutions we have found for the metric-affine Gauss-Bonnet action is of the general form

$$\Gamma_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + A_\mu \delta_\nu^\rho + B_\nu \delta_\mu^\rho - B^\rho g_{\mu\nu} \tag{4.32}$$

These solutions span a subset that is generated on the one hand by the B orbit in the $Q = 0$ subset that contains the Levi-Civita connection and on the other hand by the A flow intersecting precisely this $\mathring{\Gamma}$ orbit. As far as we know, these

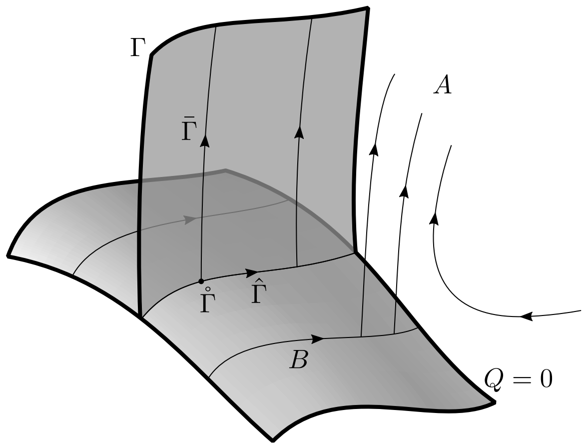

Figure 4.1: An outline of the structure of solution space.

are the only connections that are known to be solutions to Gauss-Bonnet in first-order formalism. But it should be clear that if a new solution $\breve{\Gamma}$ were to be found on another one of the B orbits in the $Q = 0$ subset, the flows of the A and B transformations would generate a new two-vector family of solutions,

$$\Gamma_{\mu\nu}{}^{\rho} = \breve{\Gamma}_{\mu\nu}{}^{\rho} + A_{\mu}\delta_{\nu}^{\rho} + B_{\nu}\delta_{\mu}^{\rho} - B^{\rho}g_{\mu\nu}. \tag{4.33}$$

It seems therefore reasonable to expect a (connected or disconnected) family of non-intersecting subsets of solutions, each one characterized by the orbits of the B transformation that form the intersection with the $Q = 0$ plane.

At the moment of writing [24], we believed this structure not to be unique for the four-dimensional Gauss-Bonnet action, but for any Lovelock theory in critical dimensions (that is, for the n-th order Lovelock term in $D = 2n$ dimensions). We also believed the existence of the non-trivial solutions was an indication of the non-topological character of Lovelock theories in critical dimensions, in the presence of non-metric-compatible connections. Later, it was demonstrated in [23] that the solution (4.32) was present in every Lovelock Theory in its critical dimension and that the cause of the non-topological character of Einstein-Hilbert in two dimensions was the traceless part of the non-metricity. This last statement was proven for every Lovelock theory in its critical dimension.

On the other hand, not much is known about the solutions of the four-dimensional Gauss-Bonnet action that are not generated through the flows of the A and B transformations from the $Q = 0$ subset, that is, that have at least one part of the non-metricity that is not pure trace, $Q_{\mu\nu\rho} \neq \frac{1}{4}Q_{\mu\sigma}{}^{\sigma}g_{\nu\rho}$ (besides the general property that they can be divided in the equivalence classes formed by the A orbits). Similarly, to our knowledge, there are no connections, other than Palatini connections (2.53), known to be a solution of the Gauss-Bonnet action in dimensions higher than four.

However, the fact that we have found non-trivial (that is, non-equivalent) solutions for the specific four-dimensional case, disproves the commonly accepted statement that the metric and the Palatini formalism are equivalent for general Lovelock Lagrangians. Indeed, even though the Levi-Civita connection is always a solution to the metric-affine Lovelock actions, it is now clear that in general, higher-order Lovelock theories can allow for physically distinct connections. It would be interesting to find explicit non-trivial solutions for Lovelock theories in non-critical dimensions.

Chapter 5

Conclusions

Prior to this thesis, the idea that metric and metric-affine formalisms were equivalent for Lovelock Theories was claimed by some authors (see, for example, papers citing [5] or [13]). The confusion arises from the fact that Levi-Civita is a solution of the connection equation in the metric-affine formalism only for Lovelock Theories. However, we have proven in this thesis that it is not the only possible solution, hence the non-equivalence.

First of all, it is worth noting that there are two types of equivalence. We could speak about mathematical equivalence if we compared the sets of solutions and check if they have exactly the same elements. However, the equivalence that we have focused on in this thesis is physical equivalence, where solutions are called equivalent if there is no physical difference measurable between them.

Keeping that criteria in mind, although we have seen that Palatini connections (2.53) are mathematically different from the Levi-Civita connection (1.39), along this thesis we have seen that there is a complete physical equivalence between both solutions for all projective-invariant theories, being indistinguishable one from another. In fact, we have seen that, for every Lovelock Theory in any number of dimensions, we can add the projective term to any solution, ${\Gamma_{\mu\nu}}^{\rho} \to {\Gamma_{\mu\nu}}^{\rho} + A_\mu \delta^\rho_\nu$, getting an equivalent solution due to projective symmetry. We have also discussed the interpretation of this symmetry in terms of the reparameterization of the geodesics.

This fact is very important in the first case that we studied: Einstein-Hilbert. To be more clear, we have studied Einstein-Hilbert in any number of dimensions greater than two and with minimal coupling. With this constraints, we have proven the complete equivalence between metric and metric-affine formalisms, as the most general solution in metric-affine formalism are Palatini connections, that is, Levi-Civita plus the term coming from the projective symmetry, hence non-measurable.

On the contrary, for Gauss-Bonnet in four dimensions, we have found a solution (4.32) of the metric-affine formalism that is different from Levi-Civita and not equivalent to it. Therefore, this refutes the equivalence between both formalisms for Gauss-Bonnet. It is also refuted for every Lovelock theory in critical dimensions in [23], being our result a particular solution. This solution also made us suspect that Gauss-Bonnet in metric-affine formalism was not a total derivative, which has been later demonstrated for every Lovelock Theory in critical dimensions in [23], where this solution was also extended to every Lovelock Theory in its critical dimension.

Summing up, first- and second-order formalisms are not equivalent. They happen to be equivalent for Einstein-Hilbert with some restrictions: more than two dimensions and minimal coupling. We have not studied the theory without those restrictions, so they remain as open questions for future work. The case of two dimensions is already well discussed in [23] as a particular case of critical Lovelock. On the other hand, without minimal coupling, Palatini connections can become measurable, so the equivalence is no longer valid. This happens, for example, for Einstein-Maxwell or when introducing fermions. In those cases, we can measure some quantities related with torsion, measuring the value of the (previously nonphysical) vector A. The coupling with fermions has been discussed in [22].

There are more open questions related to this research. An interesting question that we did not look at when checking the non-measurability of the Palatini connections was the second clock effect. This effect happens when measuring time along trajectories. The first clock effect says that the elapsed time will be different due to the fact that the length of the curves will be different, therefore it is due to the metric. However, the second clock effect says that the elapsed time will be different because the rate at which time goes by is affected by the non-metricity due to the change of the norm of the tangent vector; it is an effect related to the connection. This could be a way for measuring A but, as we know that it comes from a symmetry and cannot be measured, there has to be a way of making this two facts consistent. Maybe, the fact that the non-metricity induced by projective transformations is pure trace is related to the solution, but we have not done a complete demonstration.

From a phenomenological point of view, it would be interesting to see the effects induced in four dimensions by using the metric-affine formalism and Gauss-Bonnet plus a known theory, for example, Einstein-Hilbert or a phenomenological theory. No one has investigated in this direction because it was thought that Gauss-Bonnet was a total derivative, but now we know that when using the metric-affine formalism, it is not. There should be corrections to the Levi-Civita connection and they might explain things as dark matter or dark energy.

It would be also very nice to find the general solution for the connection for all critical Lovelock Theories, as there are less degrees of freedom than in more dimensions. Finding the general solution to the connection equation in an arbitrary number of dimensions seems to be impossible, though, as all equations of motion are coupled. As a first step in this direction, it would be useful to find some other connections nonequivalent to Levi-Civita in an arbitrary number of dimensions.

Concluding, we think that this thesis has made several improvements to the field, clarifying some aspects apparently mistaken by the community, and it has also led to some other improvements by other people, like [22] and [23], and opened more interesting questions to follow.

Appendix A

Derivatives of the metric determinant

First of all, we will demonstrate

$$\frac{1}{\sqrt{|g|}}\partial_\mu \sqrt{|g|} = \frac{1}{2} g^{\sigma\eta}\partial_\mu g_{\sigma\eta}. \tag{A.1}$$

We will switch temporarily to intrinsic notation. Let A be an invertible matrix that depends on a parameter t. There are three options:

- It is diagonal.
- It is diagonalizable.
- It is not diagonalizable.

Let's recall that even when a matrix is not diagonalizable, we can transform it to its normal Jordan form, so let M be the diagonal or normal Jordan form, respectively, of A. Then, the following will hold,

$$\log(\det A) = \mathrm{tr}(\log A), \tag{A.2}$$

due to

$$\log(\det A) = \log(\det M) = \log\left(\prod m_{ii}\right) = \sum \log m_{ii} = \mathrm{tr}(\log M) = \mathrm{tr}(\log A), \tag{A.3}$$

where m_{ii} are the diagonal elements of M. The second last equality is derived from the diagonal or normal Jordan form of M, and the last equality is guaranteed by the properties of the trace of the logarithm after a change of base (see [47] for further reference).

Thus, we can derive the right side of (A.2) with respect to the parameter t,

$$\frac{d}{dt}\operatorname{tr}(\log A) = \operatorname{tr}\left(A^{-1}\frac{dA}{dt}\right) = \operatorname{tr}\left(A^{-1}A'\right), \tag{A.4}$$

where it is possible that A^{-1} and A' do not commute, but it does not matter in this case due to the cyclic property of the trace. Deriving the left side of (A.2),

$$\frac{d}{dt}\log(\det A) = \frac{1}{\det A}\cdot\frac{d}{dt}\det A. \tag{A.5}$$

So, joining these two results,

$$\frac{1}{\det A}\frac{d}{dt}\det A = \operatorname{tr}\left(A^{-1}\frac{dA}{dt}\right), \tag{A.6}$$

or, applied to the metric, in index notation,

$$\frac{1}{g}\partial_\mu g = g^{\sigma\eta}\partial_\mu g_{\sigma\eta}. \tag{A.7}$$

From that, one could derive the expression with the square root (given that the determinant is positive),

$$\begin{aligned}\frac{d}{dt}\sqrt{\det A} &= \frac{1}{2\sqrt{\det A}}\cdot\frac{d}{dt}\det A\\ &= \frac{1}{2\sqrt{\det A}}\cdot\det A\cdot\operatorname{tr}\left(A^{-1}\frac{dA}{dt}\right)\\ &= \frac{1}{2}\sqrt{\det A}\cdot\operatorname{tr}\left(A^{-1}\frac{dA}{dt}\right),\end{aligned} \tag{A.8}$$

and, so, equivalently,

$$\frac{1}{\sqrt{\det A}}\cdot\frac{d}{dt}\sqrt{\det A} = \frac{1}{2}\operatorname{tr}\left(A^{-1}\frac{dA}{dt}\right). \tag{A.9}$$

If we apply this to the metric and write it in index notation, we get (A.1), as we wanted to demonstrate. We added the absolute value because the determinant of the metric can be negative as we are using mostly-minus convention.

We can also follow the same steps until (A.7) to get the relation

$$\frac{1}{g}\delta g = g^{\eta\sigma}\delta g_{\sigma\eta}, \tag{A.10}$$

as the variation and the derivative share all properties with respect to linearity and product.

Here, we can use (2.6),

$$\begin{aligned} \delta g &= g g^{\eta\sigma} \delta g_{\sigma\eta} \\ &= -g g_{\eta\sigma} \delta g^{\sigma\eta}, \end{aligned} \tag{A.11}$$

and, finally,

$$\begin{aligned} \delta\sqrt{|g|} &= \frac{1}{2\sqrt{|g|}} \delta|g| \\ &= -\frac{|g|}{2\sqrt{|g|}} g_{\sigma\eta} \delta g^{\sigma\eta} \\ &= -\frac{1}{2}\sqrt{|g|} g_{\sigma\eta} \delta g^{\sigma\eta}, \end{aligned} \tag{A.12}$$

which is another useful relation that we will use.

Lastly, we will demonstrate

$$\frac{1}{|g|} \nabla_\mu |g| = g^{\sigma\eta} \nabla_\mu g_{\sigma\eta}. \tag{A.13}$$

First of all, we should take into account that $|g|$ is a tensor density (see [46] for further reading on tensor densities) of weight -2, so its covariant derivative has the form

$$\nabla_\mu |g| = \partial_\mu |g| - 2\Gamma_{\mu\sigma}{}^{\sigma} |g|. \tag{A.14}$$

Using (A.7), we can write it as

$$\nabla_\mu |g| = |g| \left(g^{\sigma\eta} \partial_\mu g_{\sigma\eta} - 2\Gamma_{\mu\sigma}{}^{\sigma} \right). \tag{A.15}$$

Independently, let's compute

$$\begin{aligned} g^{\sigma\eta} \nabla_\mu g_{\sigma\eta} &= g^{\sigma\eta} \partial_\mu g_{\sigma\eta} - g^{\sigma\eta} \Gamma_{\mu\sigma}{}^{\theta} g_{\theta\eta} - g^{\sigma\eta} \Gamma_{\mu\eta}{}^{\theta} g_{\sigma\theta} \\ &= g^{\sigma\eta} \partial_\mu g_{\sigma\eta} - \Gamma_{\mu\sigma}{}^{\theta} \delta^{\sigma}_{\theta} - \Gamma_{\mu\eta}{}^{\theta} \delta^{\eta}_{\theta} \\ &= g^{\sigma\eta} \partial_\mu g_{\sigma\eta} - 2\Gamma_{\mu\sigma}{}^{\sigma}, \end{aligned} \tag{A.16}$$

and, as we can see, that is what we had in (A.15) between parenthesis. Then, it is straightforward to get (A.13).

Appendix B

Palatini identity

The Palatini identity was stated by Attilio Palatini in 1919 [38]. It describes the relation between the variation of the Ricci tensor and the variation of the connection. We will deduce it in this appendix.

We start taking into account that the variation of a connection, as it is the difference between two connections, is a tensor, so we can write its covariant derivative as

$$\nabla_\mu(\delta\Gamma_{\nu\rho}{}^\lambda) = \partial_\mu(\delta\Gamma_{\nu\rho}{}^\lambda) - \Gamma_{\mu\nu}{}^\sigma(\delta\Gamma_{\sigma\rho}{}^\lambda) - \Gamma_{\mu\rho}{}^\sigma(\delta\Gamma_{\nu\sigma}{}^\lambda) + \Gamma_{\mu\sigma}{}^\lambda(\delta\Gamma_{\nu\rho}{}^\sigma). \tag{B.1}$$

We can then calculate the variation of the Ricci tensor as

$$\begin{aligned}\delta R_{\mu\nu} &= \partial_\mu(\delta\Gamma_{\sigma\nu}{}^\sigma) - \partial_\sigma(\delta\Gamma_{\mu\nu}{}^\sigma) + (\delta\Gamma_{\mu\eta}{}^\sigma)\Gamma_{\sigma\nu}{}^\eta + \Gamma_{\mu\eta}{}^\sigma(\delta\Gamma_{\sigma\nu}{}^\eta) \\ &\quad - (\delta\Gamma_{\mu\nu}{}^\sigma)\Gamma_{\eta\sigma}{}^\eta - \Gamma_{\mu\nu}{}^\sigma(\delta\Gamma_{\eta\sigma}{}^\eta),\end{aligned} \tag{B.2}$$

or, rearranging addends,

$$\begin{aligned}\delta R_{\mu\nu} &= \partial_\mu(\delta\Gamma_{\sigma\nu}{}^\sigma) - \Gamma_{\mu\nu}{}^\sigma(\delta\Gamma_{\eta\sigma}{}^\eta) \\ &\quad - \partial_\sigma(\delta\Gamma_{\mu\nu}{}^\sigma) + \Gamma_{\sigma\mu}{}^\eta(\delta\Gamma_{\eta\nu}{}^\sigma) + \Gamma_{\sigma\nu}{}^\eta(\delta\Gamma_{\mu\eta}{}^\sigma) - \Gamma_{\sigma\eta}{}^\sigma(\delta\Gamma_{\mu\nu}{}^\eta) \\ &\quad + T_{\mu\sigma}{}^\eta \cdot \delta\Gamma_{\eta\nu}{}^\sigma.\end{aligned} \tag{B.3}$$

That is, according to (B.1),

$$\delta R_{\mu\nu} = \nabla_\mu(\delta\Gamma_{\sigma\nu}{}^\sigma) - \nabla_\sigma(\delta\Gamma_{\mu\nu}{}^\sigma) + T_{\mu\sigma}{}^\eta \cdot \delta\Gamma_{\eta\nu}{}^\sigma, \tag{B.4}$$

which is called *the Palatini identity*.

Appendix C

Integration by parts in curved space

We will deal with a very simple example of integration by parts in curved space. It will be useful for seeing how it works and that new terms will appear.

Let's start by remembering how to do integration by parts in flat space in Cartesian coordinates,

$$\int d^D x S^{\alpha\beta} \partial_\alpha V_\beta = -\int d^D x V_\beta \partial_\alpha S^{\alpha\beta}, \tag{C.1}$$

plus a term evaluated at the boundary, where we will suppose the tensor fields to vanish.

If we change it to the most similar integral in curved space, let's call it I,

$$I = \int d^D x \sqrt{|g|} S^{\alpha\beta} \nabla_\alpha V_\beta, \tag{C.2}$$

then it is not so trivial how to proceed, especially when not using the Levi-Civita connection. We will do it step by step.

One should start by expanding the covariant derivative into its parts,

$$\begin{aligned} I &= \int d^D x \sqrt{|g|} \left(S^{\alpha\beta} \partial_\alpha V_\beta - S^{\alpha\beta} \Gamma_{\alpha\beta}{}^{\sigma} V_\sigma \right) \\ &= \int d^D x \sqrt{|g|} S^{\alpha\beta} \partial_\alpha V_\beta - \int d^D x \sqrt{|g|} S^{\alpha\beta} \Gamma_{\alpha\beta}{}^{\sigma} V_\sigma. \end{aligned} \tag{C.3}$$

Then, it is possible to integrate by parts in the first integral, as if we were in flat space,

$$
\begin{aligned}
I &= -\int d^D x V_\beta \partial_\alpha \left(\sqrt{|g|} S^{\alpha\beta}\right) - \int d^D x \sqrt{|g|} S^{\alpha\beta} \Gamma_{\alpha\beta}{}^{\sigma} V_\sigma \\
&= -\int d^D x V_\beta S^{\alpha\beta} \partial_\alpha \sqrt{|g|} - \int d^D x \sqrt{|g|} V_\beta \partial_\alpha S^{\alpha\beta} - \int d^D x \sqrt{|g|} S^{\alpha\beta} \Gamma_{\alpha\beta}{}^{\sigma} V_\sigma,
\end{aligned} \tag{C.4}
$$

and, taking into account (A.7), we obtain

$$
\begin{aligned}
I = &-\int d^D x \sqrt{|g|} V_\beta S^{\alpha\beta} \cdot \frac{1}{2} g^{\sigma\eta} \partial_\alpha g_{\sigma\eta} \\
&- \int d^D x \sqrt{|g|} V_\beta \partial_\alpha S^{\alpha\beta} - \int d^D x \sqrt{|g|} S^{\alpha\beta} \Gamma_{\alpha\beta}{}^{\sigma} V_\sigma.
\end{aligned} \tag{C.5}
$$

Now, we can use (A.16) to change the partial derivative to a covariant derivative,

$$
\begin{aligned}
I = &-\int d^D x \sqrt{|g|} V_\beta S^{\alpha\beta} \cdot \frac{1}{2} g^{\sigma\eta} \nabla_\alpha g_{\sigma\eta} - \int d^D x \sqrt{|g|} V_\beta S^{\alpha\beta} \Gamma_{\alpha\sigma}{}^{\sigma} \\
&- \int d^D x \sqrt{|g|} V_\beta \partial_\alpha S^{\alpha\beta} - \int d^D x \sqrt{|g|} S^{\alpha\beta} \Gamma_{\alpha\beta}{}^{\sigma} V_\sigma.
\end{aligned} \tag{C.6}
$$

and the same for the other partial derivative,

$$
\begin{aligned}
I = &-\int d^D x \sqrt{|g|} V_\beta S^{\alpha\beta} \cdot \frac{1}{2} g^{\sigma\eta} \nabla_\alpha g_{\sigma\eta} - \int d^D x \sqrt{|g|} V_\beta S^{\alpha\beta} \Gamma_{\alpha\sigma}{}^{\sigma} \\
&- \int d^D x \sqrt{|g|} V_\beta \nabla_\alpha S^{\alpha\beta} + \int d^D x \sqrt{|g|} V_\beta \Gamma_{\alpha\sigma}{}^{\alpha} S^{\sigma\beta} + \int d^D x \sqrt{|g|} V_\beta \Gamma_{\alpha\sigma}{}^{\beta} S^{\alpha\sigma} \\
&- \int d^D x \sqrt{|g|} S^{\alpha\beta} \Gamma_{\alpha\beta}{}^{\sigma} V_\sigma.
\end{aligned} \tag{C.7}
$$

Now, let's group all the connections conveniently,

$$
\begin{aligned}
I = &-\int d^D x \sqrt{|g|} \left(V_\beta \nabla_\alpha S^{\alpha\beta} + V_\beta S^{\alpha\beta} \cdot \frac{1}{2} g^{\sigma\eta} \nabla_\alpha g_{\sigma\eta} \right) \\
&- \int d^D x \sqrt{|g|} \left(V_\beta S^{\alpha\beta} \Gamma_{\alpha\sigma}{}^{\sigma} - V_\beta S^{\alpha\beta} \Gamma_{\sigma\alpha}{}^{\sigma} - V_\beta S^{\alpha\sigma} \Gamma_{\alpha\sigma}{}^{\beta} + V_\beta S^{\alpha\sigma} \Gamma_{\alpha\sigma}{}^{\beta} \right).
\end{aligned} \tag{C.8}
$$

As we can see, the last two addends cancel, and from the others we get a torsion,

$$
\begin{aligned}
I = &-\int d^D x \sqrt{|g|} \left(V_\beta \nabla_\alpha S^{\alpha\beta} + V_\beta S^{\alpha\beta} \cdot \frac{1}{2} g^{\sigma\eta} \nabla_\alpha g_{\sigma\eta} \right) \\
&- \int d^D x \sqrt{|g|} V_\beta S^{\alpha\beta} \left(\Gamma_{\alpha\sigma}{}^{\sigma} - \Gamma_{\sigma\alpha}{}^{\sigma} \right).
\end{aligned} \tag{C.9}
$$

So, the general formulation, for a general connection, results

$$\begin{aligned}&\int d^D x\sqrt{|g|}S^{\alpha\beta}\nabla_\alpha V_\beta \\ &\quad= -\int d^D x\sqrt{|g|}\left(V_\beta\nabla_\alpha S^{\alpha\beta} + V_\beta S^{\alpha\beta}\cdot\frac{1}{2}g^{\sigma\eta}\nabla_\alpha g_{\sigma\eta} + V_\beta S^{\alpha\beta}\cdot T_{\alpha\sigma}{}^{\sigma}\right).\end{aligned} \tag{C.10}$$

However, if we take the connection to be the Levi-Civita one,

$$\int d^D x\sqrt{|g|}S^{\alpha\beta}\mathring{\nabla}_\alpha V_\beta = -\int d^D x\sqrt{|g|}V_\beta\mathring{\nabla}_\alpha S^{\alpha\beta}, \tag{C.11}$$

all the extra terms disappear and the procedure is very similar to doing it in flat space simply ignoring the invariant volume element and taking the covariant derivative as the partial derivative.

Appendix D

Symmetrization and antisymmetrization

Sometimes, we will need to take the symmetric or the antisymmetric part of a tensor. We will notate it with a parenthesis for the symmetric part:

$$Q_{(\mu_1\ldots\mu_n)} = \frac{1}{n!}\sum_{\sigma\in S_n} Q_{\sigma(\mu_1)\ldots\sigma(\mu_n)}, \tag{D.1}$$

where S_n is the group of permutations of n elements. Then, this tensor would be invariant under any exchange of two of these indices. An example that will appear in this thesis is

$$R_{(\mu\nu)} = \frac{1}{2}\left(R_{\mu\nu} + R_{\nu\mu}\right). \tag{D.2}$$

For the antisymmetric part, we will notate it with square brackets,

$$Q_{[\mu_1\ldots\mu_n]} = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) Q_{\sigma(\mu_1)\ldots\sigma(\mu_n)}, \tag{D.3}$$

so this tensor would change its sign under any exchange of two of those indices. An example of that is the field strength in electromagnetism,

$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{D.4}$$

and the completely antisymmetrized delta that we use along this thesis,

$$\delta^{\sigma_1\ldots\sigma_n}_{\eta_1\ldots\eta_n} = \delta^{\sigma_1}_{[\eta_1}\cdots\delta^{\sigma_n}_{\eta_n]}. \tag{D.5}$$

For any of those notations, if there are any indices that should not be symmetrized or antisymmetrized, they will be enclosed within vertical bars. For example,

$$Q_{(\mu|\nu\rho|\lambda)} = \frac{1}{2}\left(Q_{\mu\nu\rho\lambda} + Q_{\lambda\nu\rho\mu}\right), \tag{D.6}$$

or

$$Q_{(\mu|\nu|\rho)\lambda} = \frac{1}{2}\left(Q_{\mu\nu\rho\lambda} + Q_{\rho\nu\mu\lambda}\right). \tag{D.7}$$

Appendix E

The Levi-Civita symbol

The D-dimensional Levi-Civita symbol is defined as

$$\varepsilon_{1\ldots D} = 1, \tag{E.1}$$

being completely antisymmetric, such that

$$\varepsilon_{\mu_1\ldots\mu_D} = D! \cdot \delta^1_{[\mu_1} \cdots \delta^D_{\mu_D]}. \tag{E.2}$$

The Levi-Civita symbol is a fundamental symbol: it has the same value in every coordinate system. Thus, it transforms as a pseudo-tensor density of weight 1 under general coordinate transformations,

$$\varepsilon_{\mu_1\ldots\mu_D} = \operatorname{sgn}\left(\frac{\partial y}{\partial x}\right)\left|\frac{\partial y}{\partial x}\right| \frac{\partial x^{\alpha_1}}{\partial y^{\mu_1}} \cdots \frac{\partial x^{\alpha_D}}{\partial y^{\sigma_D}} \varepsilon_{\alpha_1\ldots\alpha_D}. \tag{E.3}$$

As these are the transformations, the invariant volume element is given by

$$\sqrt{|g|} d^D x = \sqrt{|g|} \varepsilon_{\sigma_1\ldots\sigma_D} dx^{\sigma_1} \cdots dx^{\sigma_D}. \tag{E.4}$$

We can also define a completely contravariant symbol as

$$\varepsilon^{\mu_1\ldots\mu_D} = g^{\mu_1\sigma_1} \cdots g^{\mu_D\sigma_D} \varepsilon_{\sigma_1\ldots\sigma_D}. \tag{E.5}$$

Then, evaluating it, we get

$$\varepsilon^{1\ldots D} = \det(g^{-1}) = (-1)^{D-1}|g|^{-1}. \tag{E.6}$$

It is also a pseudo-tensor density of weight 1, by construction.

The contraction of two Levi-Civita symbols is given by

$$\varepsilon_{\sigma_1\ldots\sigma_p\mu_1\ldots\mu_{D-p}} \varepsilon^{\sigma_1\ldots\sigma_p\nu_1\ldots\nu_{D-p}} = (-1)^{D-1} p!(D-p)! \cdot |g|^{-1} \delta^{\nu_1}_{[\mu_1} \cdots \delta^{\nu_{D-p}}_{\mu_{D-p}]}, \tag{E.7}$$

being two useful particular cases

$$\varepsilon_{\sigma_1\ldots\sigma_D}\varepsilon^{\sigma_1\ldots\sigma_D} = (-1)^{D-1}D!\cdot|g|^{-1}, \tag{E.8}$$

$$\varepsilon_{\mu_1\ldots\mu_D}\varepsilon^{\nu_1\ldots\nu_D} = (-1)^{D-1}D!\cdot|g|^{-1}\delta^{\nu_1}_{[\mu_1}\cdots\delta^{\nu_D}_{\mu_D]}. \tag{E.9}$$

Finally, the derivatives of these symbols are

$$\nabla_\mu\varepsilon_{\nu_1\ldots\nu_D} = 0, \qquad \nabla_\mu\varepsilon^{\nu_1\ldots\nu_D} = g_{\sigma\eta}\nabla_\mu g^{\sigma\eta}\varepsilon^{\nu_1\ldots\nu_D}. \tag{E.10}$$

When working in the vielbein formalism, the symbol is written as

$$\varepsilon_{a_1\ldots a_D} = e^{\sigma_1}_{a_1}\cdots e^{\sigma_D}_{a_D}\varepsilon_{\sigma_1\ldots\sigma_D}. \tag{E.11}$$

Thus, when calculating the value of the components of this symbol, we obtain the inverse of the determinant of the vielbein,

$$\varepsilon_{1\ldots D} = e^{\sigma_1}_1\cdots e^{\sigma_D}_D\varepsilon_{\sigma_1\ldots\sigma_D} = e^{-1}, \tag{E.12}$$

and, hence,

$$\partial_\mu\left(\sqrt{|g|}\varepsilon_{a_1\ldots a_D}\right) = \partial_\mu\left(|e|\varepsilon_{a_1\ldots a_D}\right) = 0, \tag{E.13}$$

as all the components are constant.

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