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Abstract

In the present work a generalized Fierz-Pauli action is proposed as an expansion of the Einstein-Hilbert action. More specifically, by analogy with the Minkowski spacetime, it is attempted to define an action whose variation yields directly the first-order Einstein equation, linearized around an arbitrary background metric. First of all, a succinct introduction is presented to explain the problem and the motivation. Subsequently, a suitable theoretical framework is developed in order to contextualize the study. Among other things, the tight connection between the Fierz-Pauli formalism and perturbation theory over the Minkowski spacetime is explained. Then, a preamble chapter is included to study some specific cases; later, the so-called generalized Fierz-Pauli action is computed, by expanding up to second-order the Einstein-Hilbert action, linearized around an arbitrary metric. Thereupon, the validity of this proposal is analyzed, varying this action and comparing its equations of motion with the first-order linearized Einstein equation. The outcome happens to be positive, and thus, the work is concluded by emphasizing the significance of its results.

Notation and Conventions

- Throughout this work we will adopt Einstein's summation convention: *if an index variable appears repeated up and down in a single term, summation of that term over all the possible values of the index is implied.*

- For tensors, upper indices will correspond to the contravariant (vectorial) part, whereas lower ones correspond to the covariant (covectorial) part.

- Lowercase greek indices $\mu, \nu, \rho, \lambda, \dots$ will refer to components in a coordinate basis $\{\partial_\mu\}$ if they are up or its dual basis $\{dx^\mu\}$ if they are down. They will run from 0 to 3.

- Lowercase latin indices $i, j, k, \dots$ will refer to the spatial components; they correspond to the part of the greek indices that run from 1 to 3. Therefore, the temporal component will be identified with the index 0.

- Usual partial derivative will be noted by ∂.

- Covariant derivative with regards to the Levi-Civita connection of the metric g will be noted by ∇.

- Bars over mathematical objects will be used to indicate that they have been exclusively derived from the background metric $\bar{g}$.

- δ^μ_ν will be the usual Kronocker delta symbol.

- A number above an equal (=) or approximate symbol ($\approx$) will be used to indicate that the formula of the same number is being used.

- The signature $(+,-,-,-)$ will be used for the Lorentzian metrics.

1 Introduction and Goals

It is common ground that Einstein's relativity theories shook up the physics to that date. First of all, special relativity (published in 1905 [1]) abolished the classical notions of absolute time and space, arguing that each observer measures them in a different way. On the other hand, general relativity (published in 1915 [2]) introduced the concept of the dynamical spacetime. In fact, up until then, spacetime was just the canvass where the physical theories were built, that is, a static scenario where physics was described. However, Einstein gave it the status of physical entity, putting it on an equal footing of magnitudes such as mass, charge, energy etc. Even further, he described how spacetime influences its content, and vice versa; as simply put by Misner, Thorne and Wheeler [3]: *"Matter tells spacetime how to curve, spacetime tells matter how to move."*

Indeed, curve is a key concept, since general relativity identifies the curvature of the spacetime —the newly defined dynamical variable— with gravity. Hence, within this theory, the gravitational field stops being a field as such, and it is characterized through the metric of the spacetime, a pure mathematical object that describes its geometrical properties. Then, the dynamics of spacetime and its content can be either described by the notorious Einsteins field equations or an action, namely the Einstein-Hilbert action. Nonetheless, purely geometrical objects are always required.

Certainly, this profound relation with differential geometry is another unique feature of general relativity, and also, considerably dissuasive for many physicist. Precisely, apart from gravity, all the other known interactions are described by special-relativistic field theories. Furthermore, this approach enables their quantization, as opposed to the purely geometrical one of Einstein's theory. Undoubtedly, the relativistic quantum field theories have had a major success, and thus, several efforts have been made to describe gravitational interactions in such a formalism[1]. Specially, the Fierz-Pauli theory must be mentioned (published in 1939 [4]), the starting point of most of these attempts, which could be summarized as: gravity is mediated by a free, massless spin-2 particle, also known as the graviton. Based on mere special relativistic concepts and gauge principles, Fierz-Pauli theory manages to reproduce general relativity at leading order, for a perturbation on a flat (Minkowski) spacetime.

[1]For a deeper examination of this approach, please refer to the book based on Feynman's lectures on gravitation [5] and the book of T. Ortín [6].

More specifically, the equation of motion of this spin-2 particle is equal to the first-order Einstein equation, linearized around the Minkowski metric. Hence, the Fierz–Pauli theory can also be considered as the lowest-order perturbation theory of general relativity in the Minkowski spacetime. Therefore, it is natural to wonder whether this theory may be extended to describe the same dynamic but for perturbations in an arbitrary background metric, beyond the flat spacetime. In other words, can the Fierz-Pauli action be generalized to obtain directly from its variation the perturbed Einstein equations, linearized around any spacetime up to leading order?

Based on the principle of general covariance, we may think that it is sufficient to covariantize the mentioned Fierz-Pauli action; however, this procedure is ambiguous[2]. Then, another plausible option would be to somehow associate the generalized Fierz-Pauli action with the Einstein-Hilbert action, since the full Einstein equation is obtained through its variation. For instance, due to the properties of the Fierz-Pauli action, it seems logical to identify its generalization with the second-order weak-field expansion of Einstein-Hilbert, linearized around an arbitrary metric.

Note that this hypothesis translates to: the variation and the weak-field expansion commute for any spacetime. This property might be expected, yet, it is certainly not obvious. Indeed, if coupling to matter is considered, the problem is not that trivial[3]. Hence, we will address the present work to assess the validity of this assumption.

For this purpose, we will approximate to leading order the Einstein equation for a perturbation around an arbitrary spacetime. On the other hand, we will linearize up to second order the Einstein-Hilbert equation —including a matter Lagrangian— and apply a variational principle. Thereupon, we will compare the obtained equations with the first-order Einstein equations, to see whether the process commutes. Surely, we will operate within the framework of general relativity, however, we will only employ mathematical arguments, mostly based on linearization theory. It must be noted that there is virtually no research regarding this issue[4], especially working on very general terms and considering coupling to matter, as we will. Broadly, the structure of the work will be the following:

- **Chapter 2**: In this chapter we will develop the theoretical framework of our work. Specifically, we will introduce basic concepts in order to contextualize our calculations, mainly fundamental notions of general relativity and the Fierz-Pauli formalism.

- **Chapter 3**: This chapter will be a preamble for the next one. Before performing the calculations for arbitrary metrics, we will study two specific cases: the trivial flat met-

[2] For a further discussion on this issue, please refer to [7].

[3] The coupling to matter of the Fierz-Pauli action, for instance, is a notorious issue. This problematic is extensively examined in [6], among others.

[4] Linearization up to first order of Einstein's equations is certainly very common in every introductory book of general relativity; still, it is very rare to find a development up to second order, even less of the action.

ric and a non-trivial one, with a non-vanishing curvature and matter and energy content.

- **Chapter 4**: In this chapter we will carry out the entirety of the calculations, without any computational support and in very general terms. Ultimately, we will analyze the possible commutativity of the linearization process. Indeed, it is the kernel of the work and our most personal contribution, as opposed to the more bibliographical second chapter.

- **Chapter 5**: We will end the work with this final chapter, where we will examine the significance of our results.

The final appendices are not a part of the work per se; due to the extension of the calculations, they just contain some longer developments and the consistency analysis of our results.

5 Conclusions

In this final chapter, we conclude the present work by emphasizing the relevance of the obtained results. Fort this purpose, we will recapitulate the motivation and summarize the process of our calculations.

As studied in Section 2.2, the variation of the Einstein-Hilbert action gives raise to the Einstein equation, the fundamental equation of general relativity. This action is of second order in the metric $g^{\mu\nu}$ and highly non-lineal in $g_{\mu\nu}$. As a consequence, the Einstein equation is also highly non-lineal, and thus, extremely complex to solve. Therefore, as explained in Section 2.3, it can be very convenient to approximate it by its linear expansion around a certain known solution, *i.e.*, a background metric. By construction, the resulting equation is linear and of second-order.

In Section 2.4, we have proven that the variation of the Fierz-Pauli action yields exactly this linearized equation at leading order for the Minkowski spacetime. Actually, it is a curious coincidence, since the Fierz-Pauli formalism is a pure special-relitivistic field theory, which has nothing to do with differential geometry as opposed to general relativity. Then, based on this fact, it seems logical to expect some kind of connection between the Fierz-Pauli and the Einstein-Hilbert actions. In fact, in Section 3.1 we have determined that, up to boundary terms, the Fierz-Pauli action is the second-order weak-field expansion of Einstein-Hilbert. Indeed, this result is not surprising; however, the following is implied: the variation and the weak-field expansion processes commute for the Minkowski spacetime.

Certainly, we have found truly interesting the possibility to derive the linearized equations directly from an action —the Fierz-Pauli action— instead of varying first the Einstein-Hilbert action and then linearizing it. Therefore, we have tried to expand this idea to other metrics, that is, we have attempted to generalize the Fierz-Pauli action for an arbitrary metric. First of all, in Section 3.2 we have analyzed this problem in a “trial” background metric, namely the FLRW spacetime. This is a non-trivial exact solution, and furthermore, the dynamic of its perturbations is very much studied in cosmology. Hence, it seemed a suitable choice before moving on to a more general study. For this mentioned background metric, by analogy with the Minkowsky spacetime, we have associated the generalized Fierz-Pauli action with the second-order weak-field expansion of Einstein-Hilbert. After doing the proper calculations, we have concluded that the variation of this action yields exactly the first-order Einstein equation.

Then, after this positive outcome, in Chapter 4 we have proceed to address this problem on general terms, that is, for an arbitrary background metric. Based on the previous results, we have once again associated the generalized Fierz-Pauli action with the second-order Einstein-Hilbert action, but in this case linearized around an arbitrary solution. After long and exhaustive calculations, we have found that the equations of motion obtained through the variation of this action are exactly identical to the first-order linearized Einstein equation, for *any* background metric. Therefore, we have managed to propose a general expression for an action —the so-called generalized Fierz-Pauli action— that describes directly and at leading order the dynamic of perturbations in any spacetime.

Actually, what we have truly proven is that the variation and the weak-field expansion always commute, regardless of the background metric. That is, the outcome of this work may be summarized as: the diagram of Figure 4.1 commutes. This result might be expected, however, it is quite remarkable. In fact, prior to our calculations it did not seem so obvious, and furthermore, based on the long and arduous calculations of Chapter 4 it was certainly non-trivial.

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