A Study of Anisotropy in Cosmology within the framework of General Relativity

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Abstract

This is a theoretical work in which an anisotropic metric is considered within the framework of general relativity in order to get cosmological solutions. Several results and conclusions are derived for different type of sources of matter and energy: vacuum, dust, cosmological constant, scalar field and electromagnetic field.

Este es un trabajo teórico en el que se considera una métrica anisótropa dentro del marco de la relatividad general con el propósito de obtener soluciones cosmológicas. Se derivan numerosos resultados y conclusiones para diferentes tipos de fuentes de materia y energía: vacío, polvo, constante cosmológica, campo escalar y campo electromagnético.

Keywords: anisotropy, general relativity, cosmological model.

Contents

1 Introduction

The Theory of General Relativity is today one of the pillars of modern physics. It breaks with the old and classical conception of space and time, stating that both of them are in fact part of an unique and inseparable fabric, called spacetime. Spacetime has the property of curving in presence of matter and energy, and, likewise, matter and energy move accordingly to the geometry of spacetime.

Since the theory links matter and energy with the geometry of spacetime, it is possible to consider the whole content of the universe as an entity and study its geometry and its evolution. In other words, we can create cosmological models.

The current standard cosmological model is the ΛCDM model and it implies the cosmological principle, which states that the universe is homogeneous and isotropic at every instant at very large scales (over hundreds of Mpc). It was introduced by Friedmann and Lemaître in an attempt to simplify the hard calculations, arriving at the famous models known as FLRW models [1] [2].

On the other hand, the study of cosmological models which do not satisfy the conditions of homogeneity and isotropy are also part of investigations in physics in order to achieve a better understanding of the universe. During the last decades several solutions that violate the cosmological principle have been presented. Some of them lack homogeneity, others, isotropy and others, both of them. Some of these metrics are Kasner metric [1] [8], Gödel metric [5], Bianchi metric [3], Lemaître Tolman Bondi metric [4], Kantowski-Sachs metric [7] and Szekeres metric [6].

In this work, the interest of considering universes that do not satisfy the cosmological principle is theoretical and not phenomenological, since we derive solutions that exhibit anisotropy at large scale and observations, on the contrary, indicate that inhomegenity and anisotropy can only be considered as perturbations. Thus, our goal is to study the features of spacetime when we relax the symmetries of FLRW metric, what solutions we can get and what conclusions we can derive. We can wonder if an isotropic source of matter and energy can be conceived in an anisotropic universe and, ultimately, we can study the implementation of electromagnetic fields, which is impossible in an isotropic universe.

For that purpose, we present an anisotropic Ansatz and perform the study with several sources of matter and energy: vacuum, dust, cosmological constant, scalar field and electromagnetic field. Our metric is based in the Szekeres metric which is anisotropic and inhomogeneous, however, we erase the inhomogeneity and only keep the anisotropy, which allows us to focus exclusively in the study of anisotropy and simplifies the expressions so that we can arrive at particular solutions, each of them corresponding to a universe or a family of universes. Moreover, we consider a maximally symmetric two-dimensional spatial block where Szekeres metric considers a flat two-dimensional spatial block, which gives us more options to study. Some of the solutions that we obtain are known and they have been obtained in a different way throughout the course of history while others may be new. In any case, we perform a study of the features of the different cosmological models obtained.

This document starts with some preliminaries to present our conventions, basics of the theory of relativity and a brief explanation of the FLRW metric. Following this, we present our metric in detail and try to find cosmological solutions for different sources of matter and energy. Therefore, we try to show different scenarios allowed by general relativity and analyse mathematically solutions that may or may not agree with the observations in general or in a particular stage or region of the universe to a greater or lesser extent. Some of the scenarios, even if they are not consistent with our universe, are significant from a mathematical and physical point of view in the development of a deeper understanding of the theory, gravity and spacetime.

2 Preliminaries

2.1 Notation and fundamentals

In this work, conventions from [1] are applied. First of all, natural units are used, this is, $c = 1$, where c is the speed of light. We also employ Einstein notation, that implies summation over repeated lower and upper indexes:

$$A_\mu B^\mu \equiv \sum_{\mu=0}^{3} A_\mu B^\mu. \tag{2.1}$$

The signature used is $(+,-,-,-)$, i.e., the temporal coordinate is positive and the spacial coordinates are negative. Greek indexes $\mu, \nu, ...$ run from 0 to 3, Latin indexes $i, j, ..$ run from 1 to 3 and Latin indexes $a, b, ...$ run from 1 to 2. Our definitions for the curvature tensors are

$$R_{\mu\nu\rho}{}^{\lambda} = \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho} \tag{2.2}$$

for Riemann tensor,

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho} \tag{2.3}$$

for Ricci tensor,

$$R = g^{\mu\nu} R_{\mu\nu} \tag{2.4}$$

for Ricci scalar, and

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu} \tag{2.5}$$

for Einstein tensor, where $g_{\mu\nu}$ is the tensor metric and $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\, g^{\lambda\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right). \tag{2.6}$$

Tensor F^{ab} must be isotropic and homogeneous due to the fact that the spatial section represented by $g_{ab} = e^{2\beta}\tilde{g}_{ab}$ meets those features. We can write tensor F^{ab} as

$$F_{ab} = f(t)\varepsilon_{ab}\sqrt{|\tilde{g}|}. \tag{8.29}$$

where $f(t)$ is a function of time and ε_{ab} is the Levi-Civita tensor density. The presence of $\sqrt{|\tilde{g}|}$ (the determinant of g_{ab}) is required because ε_{ab} is a tensor density.

Using that ε_{ab} transforms with $\tilde{g}_{ab}$ and

$$\varepsilon_{ac}\varepsilon^{bc} = \delta_a^b(|\tilde{g}_{ab}|)^{-1}, \tag{8.30}$$

we get for the Einstein equations

$$-\dot{\beta}^2 - 2\dot{\alpha}\dot{\beta} - \frac{k}{R_0^2}e^{-2\beta} = -\frac{\kappa}{2}f^2(t)e^{-4\beta}, \tag{8.31}$$

$$3\dot{\beta}^2 + 2\ddot{\beta} + \frac{k}{R_0^2}e^{-2\beta} = \frac{\kappa}{2}f^2(t)e^{-4\beta}, \tag{8.32}$$

$$\dot{\beta}^2 + \ddot{\beta} + \dot{\alpha}\dot{\beta} + \dot{\alpha}^2 + \ddot{\alpha} = -\frac{\kappa}{2}f^2(t)e^{-4\beta}. \tag{8.33}$$

As in the electric field case, when $k = 0$ and $\alpha = \beta$, the equations are incompatible, which indicates that no magnetic field can be considered in an isotropic universe, which, as before, is logical since we have a magnetic field in one direction and not in the others.

We can appreciate a huge similarity between the Einstein equations in the electric and magnetic field. In fact, for $f(t) = A$, i.e., $f(t)$ is a constant, $k = 1$ and β also considered as a constant, we would have the exact same metric but with a magnetic field described by [8]

$$F_{ab} = A\varepsilon_{ab}R_0^2 \sin\theta, \tag{8.35}$$

which also satisfies the Maxwell equation.

Therefore, we have proven in this section that an electromagnetic field cannot exist in an isotropic universe, while it can in a anisotropic one, as expected.

9 Conclusions

In this theoretical work, we have started from the metric

$$ds^2 = dt^2 - e^{2\alpha(t)}dz^2 - e^{2\beta(t)}\tilde{g}_{ab}\, dx^a dx^b \tag{9.1}$$

[8]Remember that the metric of a sphere is given by

$$ds^2 = R_0^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{8.34}$$

and obtained several solutions for the Einstein equations. Let us present them all in a schematic way.

Vacuum solutions

- Minkowski space
- Milne space in four dimensions (Minkowski space)
- Kasner metric with repeated coefficients

Dust solutions

- FLRW case for dust and $k = 0$.
- Milne space in three dimensions plus curved direction

Cosmological constant

- De Sitter space
- Anti-De Sitter space
- De Sitter type space in two dimension plus two-dimensional sphere
- Anti-De Sitter type space in two dimension plus two-dimensional hyperboloid
- Anisotropic universe expanding in every direction

Scalar field

- Two expanding flat dimensions plus static dimension
- Isotropic expanding universe (FLRW case)

Electromagnetic field

- Dynamical direction plus two-dimensional sphere for electric field
- Dynamical direction plus two-dimensional sphere for magnetic field

All of the sources considered are isotropic except the electromagnetic field and yet, they all allow the existence of at least one anisotropic universe. On the other hand, an electromagnetic field is not admissible in an isotropic universe, since all the directions of a universe of such features are equivalent. We can conclude that we have answered the question stated in the introduction: Can an isotropic source give rise to an anisotropic source? Yes, even the absence of sources (vacuum) can provide an anisotropic universe. The obtained solutions are also very dissimilar between them, even for a same source, which brings to light how polyvalent Einstein equations are, exhibiting the possibility of Big Bang and different expansions rates.

Several solutions of the FLRW metric are also solutions of our anisotropic metric, as expected, since all of the FLRW solutions discussed in the preliminaries, except

Anti-De Sitter space in four dimensions, can be obtained by setting $e^{\alpha} = e^{\beta}$ and $k = 0$ for different sources, that is, turning our anisotropic metric into an isotropic and homogeneous one.

We have obtained several solutions that show no expansion or contraction in a spatial block, while exhibiting a dynamical behaviour in the other. Other solutions, on the contrary, show a dynamical anisotropic universe that evolves in every direction.

It is remarkable the amount of solutions that have been obtained out of the framework of the cosmological principle, which lays bare the new paths that are opened by relaxing the symmetries of the FLRW metric. This is motivation enough to continue the search of new solutions and explore the limits of the theory of the general relativity and the physics and mathematics behind it, as we have done in the study performed in this work.

References

[1] Janssen, B. (2022). *Gravitación y geometría: Una introducción moderna a la Teoría de la Relatividad General.* Editorial Universidad de Granada. ISBN 978-84-338-6913-5.

[2] Weinberg, S. (1972). *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* John Wiley & Sons.

[3] Pontzen, A. (2016). *Bianchi universes.* Scholarpedia. Retrieved July 30, 2022, from http://www.scholarpedia.org/article/Bianchi_universes.

[4] Kransinski, A. and Hellaby, C. (2001). Structure formation in the Lemaître-Tolman model. *Phys. Rev. D.* doi: 10.1103/PhysRevD.65.023501.

[5] Gödel, K (1949). An Example of a New Type of Cosmological Solutions of Eins- tein's Field Equations of Gravitation. *Reviews of Modern Physics, 21 (3)*, 447-450.

[6] Bolejko, K. , Celerier, M-N. and Krasinski, A. (2011). Inhomogeneous cosmological models: exact solutions and their applications. *Phys. Rev. D.* doi: 10.1088/0264-9381/28/16/164002.

[7] Kantowski, R. and Sachs, R. K. (1966). Some spatially inhomogeneous dust models. *J. Math. Phys. 7 (3)*, 443. doi:10.1063/1.1704952.

[8] Misner, C. W., Thorne, K. S. and Wheeler J. A. (1973). *Gravitation.* San Francisco: W. H. Freeman. ISBN 0-7167-0344-0.

[9] Merriam-Webster. (n.d.). *Ansatz.* Merriam-Webster.com dictionary. Retrieved August 19, 2022, from https://www.merriam-webster.com/dictionary/ansatz.

[10] Carrara, Mark D. (n. d.). *Ansatz*. MathWorld–A Wolfram Web Resource, created by Eric W. Weisstein. Retrieved on August, 19, 2022, from https://mathworld.wolfram.com/Ansatz.html.

[11] Davis, T. and Griffen B. (2010). *Cosmological constant*. Scholarpedia. Retrieved August 21, 2022, from http://www.scholarpedia.org/article/Cosmological_constant.