UNIVERSIDAD
DE GRANADA

Facultad de Ciencias

GRADO EN FÍSICA

TRABAJO FIN DE GRADO

VAIDYA METRICS

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Curso Académico 2020/2021

Resumen

Las métricas de Vaidya son soluciones esféricamente simétricas y no-estacionarias de las ecuaciones de Einstein. En este trabajo derivaremos la forma de estas métricas a partir de la solución de Schwarzschild y exploraremos la física que surge al considerarlas. Primero, la estructura causal será estudiada obteniendo el Diagrama de Penrose de esta métrica, lo que nos permitirá estudiar el concepto de horizonte aparente, diferenciándolo del horizonte de sucesos. El hecho de que la masa sea una función de las coordenadas cono de luz introducirá diferencias en el tratamiento de la energía, adentrándonos en el formalismo ADM, contraponiéndolo con la energía Bondi y el tratamiento Misner-Sharp. Por último, presentaremos las generalizaciones conocidas más relevantes de estas métricas. A lo largo del proyecto daremos el contexto necesario para entender la importancia de estas soluciones en la investigación actual en Astrofísica y la Gravedad Cuántica.

Abstract

Vaidya metrics are spherically symmetric and non-static solutions of the Einstein equations. In the present work we will derive the form of these metrics from the Schwarzschild solution and we will explore the Physics that arises when considering them. First, the causal structure will be studied by deriving the Penrose Diagram of this metric, which will allow us to introduce the concept of apparent horizon against event horizon. Furthermore, the fact that mass is a function of the lightcone coordinates will introduce differences in the treatment of energy, allowing us to introduce the ADM formalism in contraposition to both Bondi energy and Misner-Sharp treatment. Finally we will present the known generalizations to these metrics. During the project we will study the necessary context to understand the importance of these metrics in the state of the art research on Astrophysics and Quantum Gravity.

Contents

1 Introduction

The development of Special Relativity in 1905 [1] and General Relativity in 1915 [2] by Albert Einstein supposed, not only a change in the philosophical paradigms of the time, but also a new way of understanding physics. Against Newtonian gravity, the theory of General Relativity put on the table a revolutionary point of view, where gravity was no longer a force but a manifestation of the curvature of spacetime. Using Differential Geometry, Einstein came to the most important result of its theory, the homonym equation

$$\mathcal{R}_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}\mathcal{R} = -8\pi G_N \mathcal{T}_{\mu\upsilon}, \tag{1.1}$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor, $\mathcal{R}$ the Ricci scalar, $g_{\mu\nu}$ the metric tensor, G_N the universal gravitational constant and $T_{\mu\nu}$ the stress-energy tensor. On the left side we have the geometrical description of the spacetime and on the right the energy content of it. Quoting the American physicist John Wheeler [3], we could sumarize equation (1.1) in:

Space-time tells matter how to move;
matter tells space-time how to curve.

Actually, expression (1.1) is not an equation, but a set of ten coupled second order partial differential equations. The nonlinearity and the invariance under diffeomorphisms makes it not possible to find a solution for the general case. Instead of this, we have to impose symmetry restrictions or other arguments.

The first solution non-trivial flat to the Einstein's equations was found by the German physicist Karl Schwarzschild in 1916 [4] and it describes the gravitational field outside a spherical and static mass, located in vacuum. After this, many generalizations were found, adding angular momentum, electric charge or cosmological constant along other features [5]. Kerr metric, for example, is used to introduce angular momentum in a star [6] and the Reissner-Nördstrom solution includes electric charge [7]. Despite the fact that there is a complete 'zoo' of solutions, in the standard textbooks [8–13] only stationary solutions are discussed. A stationary spacetime is one that does not change with time and the above cited metrics are examples of it.

In the present work we will study the physics underlying a non-stationary solution. In 1951, at the age of 24, the Indian physicist and mathematician Prahalad Chunnilal Vaidya found a solution to equation (1.1) generalizing the Schwarzschild metric to incorporate the radiation coming in or out of the star [14, 15]. Despite the simplicity of this change, these metrics introduce many important concepts in the field of General Relativity.

A clear example is the way energy is measured. For assymptotically flat spacetimes[1] there are different ways of measuring it, such as the ADM energy and the Bondi energy. We will develop the formalism in section 5, but the essence of it is the different ways of going to infinity (spatial or null) when measuring it. Schwarzschild metric is characterized by just a parameter, the mass M, which is a constant, thus both treatments are equivalent in this case. With Vaidya metrics this is not the case. As the mass function varies with time the energy content of the spacetime will vary depending on

[1] An assymptotically flat spacetime is one in which, roughly speaking, curvature vanishes at large distances from the singularity. See section 3 for a detailed description.

which formalism we decide to apply. Moreover, the appearance of apparent horizons, in contraposition to event horizons serves us as a motivation in the study of these solutions.

This, along with the fact that they are widely used in the current research, make Vaidya metrics a topic suitable, but challenging, for an undergraduate dissertation. The study of Hawking radiation [16], a model for evaporating black holes [17] and stars in process of gravitational collapse [18] are just a few examples of the importance of this solution.

The fact that Vaidya metrics are a generalization of Schwarzschild metric makes it necessary to devote some time in the introduction of them. Moreover, although it was not initially posed as an objective, we have decided to introduce the conformal transformations and Penrose Diagrams formalism in order to illustrate the differences between apparent and event horizonts.

In section 2 we expose our sign convention, define some basic Differential Geometry concepts and explain what a Penrose Diagram is. In section 3 we review Schwarzschild metric and in section 4 Vaidya Metrics are derived and characterized. We do not derive these metrics in the same way that Vaidya did. For the sake of completeness we give an insight into the original derivation in appendix A. Additionaly, the Diagrams for the linear Vaidya metrics are given in appendix B. In section 5 we discuss different ways to define energy, enhancing the exposition with the Penrose Diagrams we derived in 2. We end the work by introducing some generalizations to these metrics in section 6.

Throughout the text we have used [19] to expose the basic concepts of General Relavitiy and [14, 15, 20] for those relating to Vaidya metrics. All the figures are originals and have been made with [21] and [22], except figure 3, which is from [19]. To compute some useful tensor quantities we have used the Python package [23].

2 Key concepts in General Relativity

We start by presenting the convention used through the project in subsection 2.1 and introducing the formalism related to Penrose Diagrams in 2.2. To enrich this discussion we derive the Penrose Diagram for Minkowski spacetime while we introduce concepts such as the different 'infinities'.

2.1 Definitions and convention

The Theory of General Relativity is built on the grounds of a mathematical object called *manifold*. A *N-dimensional manifold* $\mathcal{M}^N$, is, roughly speaking, a space that locally behaves like $\mathbb{R}^N$. At each point p of the manifold we can define the *tangent space at the point* p, noted as $T_p\left(\mathcal{M}^N\right)$. The set of all tangent spaces of the manifold, built each of them around each of the points p is called the *tangent bundle*. See [24, 25] for a rigurous treatment.

The *metric* $g_{\mu\nu}$, is the tensor that allows us to establish a way of measuring distances and angles between points of the manifold. The convention we will follow assigns the $(+)$ sign to the timelike component and $(-)$ to the spacelike components, thus the lorentzian metric is written as

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \tag{2.1}$$

This solution is called **Vaidya-Kottler** and it takes the form:

$$ds^2 = \left(1 - \frac{2m(v)}{r} - \frac{\Lambda r^2}{3}\right) dv^2 - 2dvdr - r^2 d\Omega_2^2 \tag{6.12}$$

2. If we add **electrostatic charge**, we would just have to add the electromagnetic stress energy tensor, equation (2.12), with the following $\mathcal{F}$:

$$\mathcal{F}_{\mu\nu} = \frac{q}{r^2}\left(\delta_\mu^0 \delta_\nu^1 - \delta_\mu^1 \delta_\nu^0\right). \tag{6.13}$$

This solution receives the name of **Vaidya-Reissner-Nordstrom** and it is:

$$ds^2 = \left(1 - \frac{2m(v)}{r} + \frac{q^2}{r^2}\right) dv^2 - 2dvdr - r^2 d\Omega_2^2 \tag{6.14}$$

3. If we allow **an injection of charge** apart from the outgoing mass the solution is called the **Vaidya-Bonnor**:

$$ds^2 = \left(1 - \frac{2m(v)}{r} + \frac{q(v)^2}{r^2}\right) dv^2 - 2dvdr - r^2 d\Omega_2^2 \tag{6.15}$$

4. If we introduce the effect of **angular momentum**, the Kerr-Vaidya metric arises. It was proposed only fifteen years ago, see [52, 53], and it describes the spacetime outside a rotating and emitting(absorbing) null dust star.

7 Conclusions

In this work we have presented an extensive review on Vaidya metrics. We have derived them in two different ways: a very intuitive one, from the Schwarzschild solution and for historical purposes the one that P.C. Vaidya himself did seventy years ago. We began by reviewing the basic concepts from Schwarzschild metric, such as event horizon or gravitational redshift, and by computing the geodesics in Eddington-Finkelstein coordinates. By letting the mass to be a function of one of the lightcone coordinates we arrive at the Vaidya solution. This way of deriving the metrics allowed us to draw conclusions very easily, by just applying energy conditions. The ingoing Vaidya metric represents a star/black hole absorbing null dust and the outgoing represents the emission of null radiation.

We have obtained a variety of results that are not usually covered in the standard textbooks. The formalism related to Penrose Diagrams has been presented in a concise way, giving us the required knowledge to draw conclusions from them and to enrich the discussion on event and apparent horizons when describing the behaviour of lightcones. With respect to the energy and mass discussion, we have discussed the Misner-Sharp and Bondi mass against the ADM formalism. We have shown the difference in measuring the energy along wordlines of constant r, null and future infinity respectively. While the ADM formalism gives a constant as solution, the Bondi formalism and the Misner-Sharp treatment give us a function, thus the Bondi mass will depend on at what w we measure it. Being these two ways of measuring mass not conserved in spacetime

Despite the fact that Vaidya metrics themselves are a generalization of Schwarzschild metric we have been able to find solutions that generalize them. The easiest case, the generalized Vaidya metric follows the same philosophy that Vaidya metrics themselves, we just have to let the mass function be a function of both w and r and study the matter content of the spacetime. Generalizations including charge or cosmological constant have been also discussed and they arise when considering concrete values for the function $m(w, r)$.

To conclude, Vaidya metrics are often forgotten in textbooks and we have shown through this project how they could be used in the introduction of much more complex concepts. Although we have not stated it here explicitly, killing vectors, the formalism of hypersurfaces or singularity theorems are the underlying points to our exposition. Bondi mass, apparent horizons and some generalizations used in the study of black hole evaporation or simple gravitational collapse are just a few examples of everything one can learn when allowing the mass in Schwarzschild geometry to be a function of one of the lightcone coordinate.

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