Trabajo Fin de Grado en Física

Warped compactifications and braneworlds

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Resumen

Las compactificaciones alabeadas fueron propuestas por L. Randall y R. Sundrum, en el marco de la cosmología de branas, como una alternativa al modelo de dimensiones extra compactas y microscópicas de Kaluza-Klein. En los primeros capítulos discutiremos la posibilidad de dimensiones adicionales, introduciendo dos clases de dimensiones extra, según los modelos de Kaluza-Klein y de Randall-Sundrum II. Justificaremos la importancia del segundo en relación al confinamiento de la gravedad en la brana y la normalización del gravitón. Asimismo, se revisarán conceptos de Relatividad General que aparecerán recurrentemente a lo largo del trabajo. En la segunda parte del proyecto generalizaremos el modelo de Randall-Sundrum, centrándonos primero en la métrica y posteriormente en la dinámica, y prestando especial atención al surgimiento de una dinámica efectiva cuatrodimensional en la brana.

Abstract

Warped compactifications were proposed by L. Randall and R. Sundrum, in the frame of brane cosmology, as an alternative to the Kaluza-Klein model of compact and microscopic extra dimensions. In the first chapters we will discuss the possibilty of additional dimensions, presenting two sorts of them, in relation with the Kaluza-Klein theory and the Randall-Sundrum II model. Besides, we will justify the importance of the latter in the confinement of gravity in the brane and the normalisation of the graviton. Furthermore, we will review certain concepts of General Relativity which will appear throughout the text. In the second half of the project, we will generalise the metrics and dynamics of the Randall-Sundrum model, paying special attention to the emerging four-dimensional dynamics in the braneworld.

Contents

1 Introduction: A Matter of Scale

It is widely assumed that we live in a universe with three spatial (probably infinitely large) dimensions, which evolve in a temporal "direction", this is, a (3+1)-dimensional spacetime. But is it a fair assumption? From our experience, it seems so, as these are the only dimensions we can perceive, and all the well-established physical theories have seemingly managed properly in this frame. For instance, Standard Model fields propagating (a large distance) in extra dimensions would not be consistent with observations.

Experimentally, it is possible to determine the number (and even the geometry) of spatial dimensions by studying how the gravitational field decays, as gravity describes the geometry and dynamics of spacetime. In terms of newtonian mechanics, the gravitational potential $\Phi(r)$ satisfies the Laplace equation, which means that if there are d "conventional" spatial dimensions, the equipotential surfaces are $(d-1)$-dimensional hyperspheres of arbitrary radius r and area

$$S_{d-1}(r) = \frac{2\pi^{d/2} r^{d-1}}{\Gamma(\frac{d}{2})}. \tag{1.1}$$

In consequence, as the gravity flux through such surfaces is constant,

$$\int_{S_{d-1}(r)} \vec{g} \cdot d\vec{S} = g S_{d-1}(r) = \text{const}, \tag{1.2}$$

the gravitational field, $\vec{g}$, necessarily decays as $\sim \frac{1}{r^{d-1}}$. Similarly, in the frame of general relativity, we can study the static spherical solution with mass m in an N-dimensional spacetime ($N = d+1$), this is, the Schwarzschild-Tangherlini solution [2]:

$$ds_N^2 = \left(1 - \frac{2G_N m}{r^{N-3}}\right) dt^2 - \left(1 - \frac{2G_N m}{r^{N-3}}\right)^{-1} dr^2 - r^2 d\Omega_{N-2}^2, \tag{1.3}$$

where G_N is the N-dimensional gravitational constant. In this case, the tt component of the metric is related to the classical gravitational potential when applying the Newtonian limit, $g_{tt} \sim \frac{1}{r^{N-3}} \sim \Phi(r)$, which is coherent with what we mentioned before. As the well-known inverse-square law for the gravitational field can be measured (this is, $\vec{g} \sim \frac{1}{r^2}$), spacetime is necessarily four-dimensional.

All the evidence indicates that we live in (3+1) dimensions, nevertheless, there is a way out of all these arguments. The fact is that the $\sim \frac{1}{r}$ decay of the gravitational potential has been experimentally observed for distances no shorter than centimetres. Thus, as gravity is considerably weaker than the other interactions, the evidence for possible additional dimensions might be "hidden" at shorter scales (submillimetric scales), where there might be deviations from the macroscopic behaviour of gravity. It is a matter of scale.

In fact, **compact and microscopic** additional **dimensions** of scale R_0 might be acceptable provided that R_0 is sufficiently small compared to the minimum distance for which the inverse-square law has been verified. This kind of extra dimensions are associated to a factorizable geometry, which essentialy means that the metric of the four common dimensions does not depend on the additional ones. For instance,

$$d\hat{s}^2 = g(x) dx^\mu dx^\nu - h(z) dz^2, \tag{1.4}$$

where x^μ are the familiar dimensions and z represents a fifth aditional dimension. The number of dimensions observed and their geometry depends on the scale, L, which is directly related to the energy $E \sim \frac{1}{L}$ studied in an experiment, and the Planck mass, which defines the scale of energies of the graviton. For n extra dimensions with topology $\mathcal{C}^n$:

- $L \gg R_0 \Rightarrow$ 4 dimensions with the topology of a Minkowski space $\mathcal{M}^{1,3}$.
- $L \ll R_0 \Rightarrow$ (4+n) dimensions with topology $\mathcal{M}^{1,n}$.
- $L \sim R_0 \Rightarrow$ (4+n) dimensions with topology $\mathcal{M}^{1,3} \times \mathcal{C}^n$.

Therefore, the four-dimensional spacetime is a low-energy image, and the n additional dimensions and their geometry will be noticeable at energies $E \sim \frac{1}{R_0}$.

A traditional example of these additional dimensions is the **Kaluza-Klein theory** (KK theory) [2, 9]. In 1921, Kaluza proposed an extension of general relativity to five dimensions explaining how electromagnetism and gravity can be unified under the condition that the five-dimensional metric does not depend on the fifth dimension. Although this idea was very interesting, there was an objection: there was not a solid justification for supressing the dependence on the extra coordinate. In 1926, Klein came to the conclusion that such hypothesis could be dropped as long as the additional dimension was compact and microscopic (this is, with topology $\mathbb{S}^1$). Hence, the topology of the spacetime would be $\mathcal{M}^{1,3} \times \mathbb{S}^1$, rather than $\mathcal{M}^{1,4}$. The main achievement of the KK theory was explaining electromagnetism as a consequence of pure gravity in five dimensions so that a four-dimensional observer, this is, which is able to measure at scales $L \gg R_0$, would perceive the five-dimensional metric as three independent fields: an effective four-dimensional metric, a gauge vector potential (which we identify as the electromagnetic potential) and a scalar field (Kaluza-Klein scalar).

Nevertheless, we find a major inconvenience when studying the effective theory in four dimensions associated to the five-dimensional Einstein-Hilbert action: the gravitational constant in four dimensions, G_N, is not a fundamental quantity, but is related to the five-dimensional one, $\hat{G}_N$, and to the compactification radius, R_0, [2]

$$G_N = \frac{\hat{G}_N}{2\pi R_0}. \tag{1.5}$$

This could be a possible way to explain why gravity is $\sim 10^{32}$ times weaker than the weak interaction, which is the so-called *hierarchy problem* [12]. The solution given by the Kaluza-Klein theory suggests that this discrepancy between the two interactions might be associated to the fact that a four-dimensional observer would measure the effective constant, G_N, in stead of the fundamental one, which would be of the order of the weak coupling constant. But in that case, R_0 would have to be macroscopic, and therefore the fifth dimension would be noticeable. Thus, it is evident that the Kaluza-Klein scenario in five dimensions has important phenomenological issues.

As an alternative to compact additional dimensions, in 1999, L. Randall and R. Sundrum proposed in their article [1] the first model of warped compactification, and proved that non-compact dimensions are compatible with the experimental observations if the assumption of a factorizable geometry is dropped. They considered that the observable universe is restricted to a (3+1)-dimensional hypersurface, a **brane**, embedded in a five-dimensional space, called the **bulk**. Unlike the Kaluza Klein theory, here, it is the curvature of the bulk which determines the Planck mass and confines gravity to a small region near the brane, and not the size of the additional dimension. Plus, in this model, the effective gravitational constant is a fundamental quantity, avoiding the phenomenological inconsistencies that we found for the KK theory. In section 3 we will study this model more deeply.

2 Foundations of General Relativity

In this section we intend to review briefly some fundamental concepts that will appear throughout the text, in order to understand the mathematical work behind it.

2.1 Funtamental tensors related to curvature

In General Relativity, it is through the (locally-valid) Equivalence Principle that Einstein introduces heuristically the concept of gravitational interaction as a manifestation of the curvature of spacetime, which is caused by the presence of matter and energy. This is why differential geometry, which describes curved spaces, is the proper mathematical frame to develop Einstein's theory.

One of the main features of a curved space(time) is the fact that if we "parallel transport" a vector from an initial point to a final one, the vector resulting from this operation depends, generally, on the trajectory followed between this two points. When calculating the difference of the differential parallel transport of a vector V^α along two different directions x^μ and x^ν, we obtain,

$$\left[\nabla_\mu, \nabla_\nu\right] V^\alpha = R_{\mu\nu\lambda}^{\ \ \ \ \alpha} V^\lambda, \tag{2.1}$$

where $R_{\mu\nu\lambda}^{\ \ \ \ \alpha}$ is the Riemann curvature tensor:

$$R_{\mu\nu\rho}^{\ \ \ \ \lambda} = \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}, \tag{2.2}$$

and $\Gamma^\sigma_{\mu\nu}$ is the Levi-Civita connection, which can be calculated from the metric $g_{\mu\nu}$:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right). \tag{2.3}$$

Hence, the metric $g_{\mu\nu}$ describes completely the geometry and curvature of the space.[1]

We say a space is of constant curvature when it is maximally symmetric, which happens when the Riemann tensor verifies the following condition [2]:

$$R_{\mu\nu\rho\lambda} = K\left(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}\right), \tag{2.4}$$

with K a constant related to the radius of curvature R_0. For $K < 0$, the spacetime turns out to be of positive curvature, and for $K > 0$, of negative curvature. If $K = 0$, then the spacetime is a Minkowski space.

[1] Thanks to the fact that we are using the Levi-Civita connection, which naturally appears in the frame of General Relativity.

pling a and the cosmological constant of the bulk $\hat{\Lambda}$ (in contrast with the previous chapter). Consequently, the effective Einstein equations for the brane depends, via the effective cosmological constant, on the coupling constant. This expresses how effects from the bulk manifest on the brane. In some way, introducing the dilatonic scalar field in the Ansatz studied in chapter 6 has allowed a "communication" between the cosmological constants.

8 Conclusions

The Randall-Sundrum model introduces warped compactifications as an attractive and original alternative to the traditional Kaluza-Klein model of compact dimensions, which is not able to give a satisfactory explanation of the hierarchy problem if the scale of the extra dimension is microscopic. In this frame, large non-compact dimensions are compatible with the observations provided that there is a properly curved background "hiding" the additional dimension, this is, confining gravity near the brane and allowing a normalisation of the massless graviton.

When generalising the Randall-Sundrum model to non-dilatonic curved braneworlds, we have found a surprising result: a four-dimensional effective dynamics arises in the brane due to the five-dimensional one. The first consequence of this is that an intrinsically four-dimensional observer could develop a four-dimensional theory for gravity based on measurements made from the brane. This is consistent with the possibility of additional dimensions in our universe. A second important consequence is that the effective four-dimensional cosmological constant (usually interpreted as the energy density of the vacuum) is introduced in this context as an integration constant, offering this way a possible explanation to the cosmological constant problem.

Another interesting aspect of this generalisation is that the constant curvature condition for the brane leads to the maximally symmetric solutions for the bulk. Furthermore, we have obtained several combinations of five-dimensional and effective cosmological constants, concluding that, depending on the foliation of the bulk space, an observer in the brane could actually measure an effective constant curvature Λ completely different from the five-dimensional curvature $\hat{\Lambda}$.

Finally, we have noticed that introducing a dilatonic scalar field in the action of the curved braneworlds presents potentially interesting properties. For instance, we have found that the dependence of Λ on the bulk cosmological constant and the coupling dilaton, expresses how the bulk the influences the brane through the effective Einstein equations.

In conclusion, altough the Randall-Sundrum model was initially proposed as an alternative to Kaluza-Klein compactification, its generalisations turn out to be powerful tools to understand some phenomenological problems, as they allow to extract remarkable physical conclusions.

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