Gravity as Gauge Theory

And Group Theoretical Properties of Fock-Lorentz Transformations

Bachelor Thesis

submitted on: August 06, 2021

Department of Physics at the University of Konstanz

Name:	Daniel Matthias Fröhlich
Matriculation number:	01/851330
Course year:	2021
First Supervisor:	Peter Nielaba
Second Supervisor:	Bert Janssen (Universidad de Granada)

Contents

1 Introduction **3**

2 Mathematical Basics of Group Theory **4**
2.1 Basic Definitions . . . 4
2.2 Continuous Groups, Generators and the Lie Algebra . . . 6
2.2.1 The Rotation Group $SO(2)/U(1)$ . . . 6
2.2.2 The Translation Group T . . . 8
2.2.3 The Group of Three-Dimensional Rotations $SO(3)$ . . . 9
2.2.4 The Special Unitary Group $SU(2)$ . . . 10
2.2.5 The Lorentz Group $SO(3,1)$ . . . 11
2.2.6 The Poincaré group . . . 15

3 The Principles of Gauge Theory **16**
3.1 The Gauge Theory of $U(1)$ - Quantum Electrodynamics . . . 16
3.2 The Covariant Derivative . . . 18
3.3 The Gauge Theory of $SU(2)$ . . . 19

4 The Gauge Theory of the Poincaré Group **22**
4.1 Mathematical Basics of General Relativity . . . 22
4.2 The Vielbein Formalism . . . 23
4.3 Gauging the Poincaré Group . . . 26

5 Fock-Lorentz Transformations **30**
5.1 What are Fock-Lorentz Transformations? . . . 30
5.2 Explicit Form of the Fock-Lorentz Transformations in One Dimension . . . 30
5.3 Fock-Lorentz-Transformations in Three Dimensions . . . 32
5.4 Preservation of Uniform Rectilinear Motion . . . 34
5.5 Group Theoretical Properties of the Fock-Lorentz Transformation . . . 35
5.6 Further Properties of the Fock-Lorentz Transformation . . . 37
5.7 Special Relativity based on Fock-Lorentz Transformations . . . 40
5.7.1 Maxwell Equations in Fock Coordinates . . . 41
5.7.2 Dirac Equation in Fock Coordinates . . . 42

6 Conclusion **44**

7 Literature **45**

8 Appendix A: Explicit Calculation of the Christoffel Symbols and the Spin Connection **46**

9 Appendix B: Spatial Metric in Fock Coordinates **47**

10 Appendix C: Infinitesimal Transformations **48**

11 Appendix D: Summary in German **48**

1 Introduction

Mathematical symmetry principles have become an essential part of modern theoretical physics. The theory of a physical interaction is in modern physics always developed on the basis of an underlying Lagrangian. The Lagrangian determines the dynamics of the observed fields. If it is possible to transform the dynamic quantities, i.e. the fields, locally in such a way that the Lagrangian and thus the physical dynamics remain unchanged, one speaks of a local gauge symmetry of the theory [16]. While the local gauge symmetry of the symmetry group $U(1)$ appears rather as an incidental attribute in the theory of (quantum) electrodynamics, the principle of gauge symmetry is the fundamental starting point for the derivation of the Lagrangian of the weak interaction (symmetry group $U(1) \times SU(2)$) and the strong interaction (symmetry group $SU(3)$) [13]. From work by R. Utiyama, D. Sciama and T.W. Kibble emerged around 1960 that General Relativity is the gauge theory of the Poincaré group, which is the group of all possible transformations between two inertial systems [1]. This means that all fundamental physical interactions known to us today are based on a certain mathematical symmetry. So the concept of gauge theories has proven to be an extremely effective method for constructing consistent physical theories. In this thesis, we first discuss the basic principles of Lie groups and some groups that play an important role in modern physics. Then, the basic concepts of gauge theory are explained using the gauge theory of the group $U(1)$ (electromagnetism) and the gauge theory of the group $SU(2)$. To apply the concept of gauge theories to gravity, the vielbein formalism of differential geometry is introduced first. While in the standard formulation of General Relativity the gravitational field is described by the metric tensor and the associated Christoffel symbols, the vielbein formalism uses so-called vielbeins (or tetrads) and the spin connection to describe the gravitational field [12]. In the gauge theory of the Poincaré group the introduced gauge fields can then be identified with the tetrads and the spin connection, whereby one obtains the curvature scalar, known from General Relativity, as Lagrangian density. Therefore the field equations of General Relativity follow from the gauge theory of the Poincaré group [7].

Furthermore, as an open problem, the goal of this thesis is to investigate the gauge theory of the group of Fock-Lorentz transformations. W. Fock showed in his book "The Theory of Space, Time and Gravitation" that coordinate transformations between two inertial systems do not necessarily have to be linear if only the principle of relativity but not the invariance of the homogeneous wave equation is to be fulfilled [6]. The resulting Fock-Lorentz transformations are linear fractional functions which obey a group structure. Therefore it is aimed in this thesis to derive a theory of gravity on the basis of the Fock-Lorentz group in an analogous way to the construction of the general theory of relativity on the basis of the Poincaré transformations. Due to the non-linearity of the Fock-Lorentz transformations, an interesting new gravitational dynamics is expected. This is of interest insofar as the standard theory of general relativity does not make correct predictions for the rotation curves of spiral galaxies and the deflection of light on extremely massive objects [14].

6 Conclusion

In this work we have seen that continuous symmetry groups with an infinite number of elements can be represented by a finite number of parameters and the elements can be generated by exponentiating a finite number of generators in a complex exponential function. The Lie algebra of generators provides a method to characterize a continuous group independent of its representation. We then applied this theory of Lie groups to derive the Lagrangian of quantum electrodynamics as well as the Lagrangian of $SU(2)$ gauge theory based on the requirement of invariance under local group transformations. These methods were then applied to develop the gauge theory of the Poincaré group on a manifold. Under certain conditions (4.68),(4.69) to the appearing gauge fields, General Relativity could be identified with the gauge theory of the Poincaré group. In addition it gave us the opportunity to interpret the torsion as the field strength of the vielbein field. On the other hand, it should be noted that the Standard Model of particle physics is the gauge theory of the group $SU(3) \times SU(2) \times U(1)$ [15]. Thus there seem to be two principles underlying all fundamental physical interactions, which are known to us today: The action principle, which determines the dynamics of the physical fields, and the invariance of the associated Lagrangian under a certain local gauge transformation.

Beyond the well known gauge theory of the Poincaré group, one goal of this work was to develop the gauge theory of the group of Fock-Lorentz transformations. Because of its non-linear property, it was expected that an interesting new dynamics for gravitational fields could arise from the associated gauge theory. This expectation was not fulfilled, since after further investigations it was shown that the Fock-Lorentz group is isomorphic to the known Lorentz group. Thus the gauge theory of the Fock-Lorentz group is identical to the gauge theory of the Lorentz group. Therefore no new gravitational dynamics could be developed on the basis of the Fock-Lorentz transformations that could have been applied to the problems of dark matter. The result of the isomorphism of the Fock-Lorentz group was nevertheless used to formulate the special theory of relativity on the basis of these non-linear coordinate transformations. A version of Newton's second axiom, Maxwell equations and the Dirac equation were found, that are invariant under Fock-Lorentz transformations. From the theoretical point of view, it may be interesting to note that the transformations need not necessarily to be linear but still lead to the same theory. From the practical point of view, there is no real benefit in sight for the more complicated formulation of special relativity based on the non-linear Fock-Lorentz transformations. The question arises whether further representations of the Lorentz group in the form of coordinate transformations on a Minkowski space exist.

7 Literature

References

[1] S. A. Ali et al. "On the Poincaré Gauge Theory of Gravitation". In: *International Journal of Theoretical Physics* 48.12 (Sept. 2009), pp. 3426–3448. ISSN: 1572-9575. DOI: 10.1007/s10773-009-0149-0. URL: http://dx.doi.org/10.1007/s10773-009-0149-0.

[2] Jason Bennett. "A pedagogical review of gravity as a gauge theory". In: (2020). URL: https://arxiv.org/abs/2104.02627.

[3] James D. Bjorken and Sidney David Drell. *Relativistic quantum mechanics*. International series in pure and applied physics. New York, NY: McGraw-Hill, 1964. URL: https://cds.cern.ch/record/100769.

[4] Sean M. Carroll. "Lecture Notes on General Relativity". In: (1997). URL: https://arxiv.org/abs/gr-qc/9712019.

[5] Torsten Fließbach. *Allgemeine Relativitätstheorie*. Springer-Verlag Berlin Heidelberg, 2016. ISBN: ISBN 978-3-662-53105-1.

[6] V. A. Fok. *The Theory of Space, Time and Gravitation*. New York: Macmillan, 1964.

[7] Bert Janssen. *Teoría de la Relatividad General*. Universidad de Granada, private communication.

[8] A. Jiménez-Cano. "METRIC-AFFINE GAUGE THEORIES OF GRAVITY". PhD thesis. Universidad de Granada, May 2021.

[9] Lev Davidovich Landau and E. M. Lifshitz. *The classical theory of fields*. 1975.

[10] S. N. Manida. "Fock-Lorentz transformations and time-varying speed of light". In: (2008), pp. 201–213. URL: https://arxiv.org/abs/gr-qc/9905046.

[11] Wolfgang Nolting. *Grundkurs theoretische Physik 4/1 - Spezielle Relativitätstheorie*. Springer-Verlag Berlin Heidelberg, 2016. ISBN: 978-3-662-49031-0.

[12] Tomás Ortín. *Gravity and Strings*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2004. DOI: 10.1017/CBO9780511616563.

[13] Michael E. Peskin and Daniel V. Schroeder. *An introduction to quantum field theory*. Includes exercises. Boulder, CO: Westview, 1995. URL: https://cds.cern.ch/record/257493.

[14] Peter Schneider. *Extragalactic Astronomy and Cosmology*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006. ISBN: 978-3-540-33174-2. DOI: 10.1007/978-3-540-33175-9. URL: http://dx.doi.org/10.1007/978-3-540-33175-9.

[15] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, Mar. 2014. ISBN: 978-1-107-03473-0, 978-1-107-03473-0.

[16] Jakob Schwichtenberg. *Durch Symmetrie die moderne Physik verstehen*. Springer-Verlag GmbH Deutschland 2017, 2016.

[17] Hans Walliser Torsten Fließbach. *Arbeitsbuch zur Theoretischen Physik*. Spektrum Akademischer Verlag Heidelberg 2012, 2012. ISBN: ISBN 978-3-8274-2832-5.

[18] Wu-Ki Tung. *Group Theory in Physics*. World Scientific Publishing Co. Pte.Ltd, 1985. ISBN: 9971-966-56-5.

[19] Kai Walter. "Darstellungstheorie der Lorentz-Gruppe". In: (2008). URL: https://www.uni-muenster.de/Physik.TP/archive/typo3/fileadmin/lehr%20e/teilchen/ss08/Lorentzgruppe.pdf.