Mathematical Methods for Physics III (Hilbert Spaces)

• Lecturer:
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• Info on the course:
  - www.ugr.es/~jsantiago/Docencia/MMIIIen/
Mathematical Methods for Physics III (Hilbert Spaces)

• Main Literature:

• Lecture notes are very succinct: examples, proofs and relevant comments on the blackboard (take your own notes)
Motivation

• Postulates of Quantum Mechanics

1st Postulate: Every physical system is associated to a complex separable Hilbert space and every pure state is described by a ray $|\Psi\rangle$ in such space.

2nd Postulate: Every observable in a system is associated to a self-adjoint linear operator in the Hilbert space whose eigenvalues are the possible outcomes of a measure of the observable.

3rd Postulate: The probability of getting a value $\alpha$ when measuring an observable $A$ in a pure state $|\Psi\rangle$ is $\langle\Psi|P_{A,\alpha}|\Psi\rangle$ where $P_{A,\alpha}$ is the projector on the eigenvalue proper subspace.

• But not only QM, also differential and integral equations, ...
Motivation

• But not only QM, also differential and integral equations, ...

• More generally, Hilbert Spaces are the mathematical structure needed to generalize $\mathbb{R}^n$ (or $\mathbb{C}^n$), including its geometrical features and operations with vectors to infinite dimensional vector spaces
Outline

- Linear and metric spaces
- Normed and Banach spaces
- Spaces with scalar product and Hilbert spaces
- Spaces of functions. Eigenvector expansions
- Functionals and dual space. Distribution theory
- Operators in Hilbert Spaces
- Spectral theory
Why Hilbert Spaces?

- They generalize the properties of $\mathbb{R}^n$ to spaces of infinite dimension
  - (Finite) linear combinations of vectors. Linear independence. Linear basis.
  - Infinite linear combinations require limits: notion of distance
  - Translational invariant distance: it is enough with distance to the origin (norm)
  - Generalization of $\mathbb{R}^n$ we need geometry (orthogonality, angles). Scalar product

Linear Space
Metric Space
Normed Space
(pre)Hilbert Space
Linear Space

- Definition: Linear (or vector) space over a field $\Lambda$ is a triad $(L, +, .)$ formed by a non-empty set $L$ and two binary operations (addition and scalar multiplication) that satisfy:

  \[ + : L \times L \rightarrow L \quad \cdot : \Lambda \times L \rightarrow L \]

  (i) $(L, +)$ additive group
  
  \[ x + y = y + x \] \[ (x + y) + z = x + (y + z) \] \[ \exists 0 \in L \mid x + 0 = x \] \[ \forall x \in L, \exists (-x) \in L \mid x + (-x) = 0 \]

  (ii) $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$

  (iii) $\lambda \cdot (\mu \cdot x) = (\lambda \mu) \cdot x$

  (iv) $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$

  (v) $1 \cdot x = x$

\[ \forall x, y, z \in L \]

\[ \forall \lambda, \mu \in \Lambda \]
Linear Space

• Trivial properties:

(i) $\alpha \cdot 0 = 0$

(ii) $0 \cdot x = 0$

(iii) $-x = (-1) \cdot x$

(iv) $x + y = x + z \Rightarrow y = z$

(v) $\alpha \cdot x = \alpha \cdot y, \alpha \neq 0 \Rightarrow x = y$

(vi) $\alpha \cdot x = \beta \cdot x, x \neq 0 \Rightarrow \alpha = \beta$

(vii) $\alpha \cdot x = 0 \Rightarrow \alpha = 0 \circ x = 0$

• Notation:

$A + B = \{x + y, \forall x \in A, \forall y \in B\}$

$\lambda A = \{\lambda \cdot x, \forall x \in A\}$

$\Lambda x = \{\lambda \cdot x, \forall \lambda \in \Lambda\}$

$\Lambda A = \{\lambda \cdot x, \forall \lambda \in \Lambda, \forall x \in A\}$
Linear Space

- **Definition**: Linear subspace. Non-trivial subset of a linear space with the structure of a linear space.

  \[ M \subset L \text{ (} L \text{ linear space, } M \neq \emptyset \text{)} \text{ linear subspace if} \]

  \[ \alpha x + \beta y \in M, \quad \forall \alpha, \beta \in \Lambda, \forall x, y \in M \]

- **Properties**:

  \[ \{ M_\alpha \}_{\alpha \in A} \text{ subsp. } \Rightarrow \bigcap_\alpha M_\alpha, \sum_{i=1}^{n} M_i \text{ subsp.} \]

  \[ \sum_{i=1}^{n} \lambda_i x_i \in M, \forall n \text{ finite}, \forall x_1, \ldots x_n \in M \]

- **Definition**: Linear span:

  let \( S \subset L \)

  \[ [S] = \{ \sum_{i=1}^{n} \alpha_i x_i, \forall n \text{ finite}, \forall x_i \in S, \forall \alpha_i \in \Lambda \} \text{ (it is linear subsp.)} \]

- **Properties**:

  \[ [S] \text{ is the smaller subsp. that contains } S \]

  \[ [S] = \bigcap_i M_i, \{ M_i \} \text{ set of subsp. that contain } S \]
Linear Space

- **Definition:** Linear independence.
  
  \[ X \subset L \text{ is linearly independent (l.i.) if} \]
  
  \[ \sum_{i=1}^{n} \alpha_i x_i = 0, \ x_i \in X, \alpha_i \in \Lambda \Rightarrow \alpha_1 = \ldots = \alpha_n = 0 \]

- **Definition:** Hamel (or linear) basis. Maximal l.i. set (i.e. that it is not contained in any other l.i. set).

- **Properties:**
  
  Every l.i. set can be extended to a Hamel basis
  
  Every Hamel basis of \( L \) has the same number of elements (linear dimension)
  
  \( L = [B], \ \forall B \text{ Hamel basis of } L \)
  
  B Hamel basis of \( L \) \( \Rightarrow \ x = \sum_{i=1}^{n} \alpha_i x_i, \alpha_i \in \Lambda, \ x_i \in B \) is unique
Linear Space

- Definition: Subspace direct sum. Let \( \{M_i\}_{i=1}^{n} \) be subsps. of \( L \)

\[ L = M_1 \oplus \ldots \oplus M_n \] (L is direct sum of \( M_i \)) if
\[ \forall x \in L \ \exists! \ x_1 \in M_1, \ldots x_n \in M_n \ / \ x = x_1 + \ldots + x_n \]

- Theorem: Let \( L = M_1 + M_2 \)

\[ L = M_1 \oplus M_2 \iff M_1 \cap M_2 = \{0\} \]

[\( M_2 \) is the linear complement of \( M_1 \) in \( L \)]

- More generally, if \( L = M_1 + \ldots + M_n \)

\[ L = M_1 \oplus \ldots \oplus M_n \iff M_i \cap \sum_{j \neq i} M_j = \{0\} \]
Linear Space

• Summary:

  ● Linear (sub)space: (L,+,..)
  ● Linear span: \([S]=\{\text{FINITE linear combinations of elements of S}\}\)
  ● Linear independence: finite linear combination=0 \(\Rightarrow\) all coeffs=0
  ● Hamel basis: Maximal l.i. set. Unique cardinal (linear dimension). Unique linear expansion of elements of \(L\) in terms of elements of \(B\).
  ● Direct sum of subspaces: sum of subspaces with null intersection (to the sum of the remaining subspaces).
  ● Other results and definitions (mappings, inverse mapping, isomorphisms, projectors, …) can be defined here but we will postpone it to Hilbert spaces
Metric spaces

- Definition: Metric space is a pair \((X,d)\) where \(X\) is an arbitrary but non-empty set and \(d : X \times X \to \mathbb{R}\) is a function (distance or metric) that satisfies:

  \((i)\) \(d(x, y) \geq 0\)

  \((ii)\) \(d(x, y) = 0 \iff x = y\)

  \((iii)\) \(d(x, y) = d(y, x)\) \(\forall x, y, z \in X\)

  \((iv)\) \(d(x, z) \leq d(x, y) + d(y, z)\)

- Properties

  \((i)\) \(d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n)\)

  \((ii)\) \(|d(x, z) - d(y, z)| \leq d(x, y)\)

  \((iii)\) \(Y \subset X, \ d'(y_1, y_2) = d(y_1, y_2) \Rightarrow (Y, d')\) metric space with induced metric \(d'\)
Metric spaces

- Definitions: Let \((X,d)\) be a metric space.
  - Open ball of radius \(r\) centered at \(x\): \(B(x, r) = \{y \in X \mid d(x, y) < r\}\)
  - Closed ball of radius \(r\) centered at \(x\): \(\bar{B}(x, r) = \{y \in X \mid d(x, y) \leq r\}\)
  - Let \(A \subset X\), \(x \in A\) is an interior point if \(\exists r > 0 / B(x, r) \subset A\)
  - Interior of \(A\): \(\text{int } A = \{x \in X \mid x\) is an interior point of \(A\}\)
  - \(A\) is open \(\text{int } A = A\)
  - Given \(A \subset X\), \(x \in X\) is an adherence point if \(\forall r > 0, \ B(x, r) \cap A \neq \emptyset\)
  - Closure of \(A\): \(\bar{A} = \{x \in X \mid x\) is an adherence point in \(A\}\)
  - Closed subspace: \(A \subset X\) is closed si \(A = \bar{A}\)
  - Dense subspace: \(A \subset X\) is dense in \(X\) if \(\bar{A} = X\)
Metric spaces

- Properties of open and closed subspaces:
  - Let \((X, d)\) be a metric space and \(A \subset X\)

\[
\emptyset, X \text{ are closed (and open)}
\]

\[
A \text{ open } \iff A^c \text{ closed}
\]

\[
\bigcap_{i \in I} A_i \text{ closed if } A_i \text{ closed}
\]

\[
\bigcap_{i=1}^n A_i \text{ open if } A_i \text{ open}
\]

\[
\bigcup_{i=1}^n A_i \text{ closed if } A_i \text{ closed}
\]

\[
\bigcup_{i \in I} A_i \text{ open if } A_i \text{ open}
\]
Metric spaces

- **Definition: Convergent sequence**

\[ \{x_n\}_1^\infty \subset X \text{ converges to } x \text{ in } X, \ x_n \to x, \text{ if } \forall r > 0, \exists N / x_n \in B(x, r), \forall n > N \]

(equivalent: the sequence of real numbers \( \{d(x_n, x)\} \) converges to 0)

- **Definition: Cauchy sequence**

\[ \{x_n\}_1^\infty \subset X \text{ is Cauchy if } \forall r > 0, \exists N / d(x_n, x_m) < r, \forall n, m > N \]

- **Property: Every convergent sequence is a Cauchy sequence**

\[ d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) \to 0 \]

- **Definition: A metric space is complete if every Cauchy sequence is convergent. A subspace } S \subset X \text{ is complete if every Cauchy sequence in } S \text{ converges in } S \]

- **Properties: Let } S \subset X, \ x \in X \]

\[ x \in \bar{S} \iff \exists \{x_n\}_1^\infty \subset S / x_n \to x \]

Let } X \text{ be complete: } S \text{ is complete } \iff \ S \text{ closed \}

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Metric spaces

• Summary:
  • Metric (sub)space: \((X,d)\)
  • Open and closed balls
  • Interior point: open ball centered in \(x\) inside \(A\)
  • \(\text{int } A = \text{ set of all interior points of } A\). Open subspace.
  • Adherence point of \(A\), every open ball centered in \(x\) has non-zero intersection with \(A\). Closure of \(A\). Closed subspace. Dense subspace in \(X\)
  • Convergent sequence
  • Cauchy sequence
  • Complete metric space: Cauchy \(\Rightarrow\) convergent
  • Other properties (maps, continuity, boundedness, …) can be defined here but we will do it in Hilbert spaces.
Normed spaces

- Definition: Normed space is a pair $(X, \| \cdot \|)$ where $X$ is a linear space and $\| \cdot \| : X \to \mathbb{R}$ is a function (norm) with the following properties:
  
  (i) $\| x \| \geq 0$
  
  (ii) $\| x \| = 0 \iff x = 0$
  
  (iii) $\| \alpha x \| = |\alpha| \| x \|$
  
  (iv) $\| x + y \| \leq \| x \| + \| y \|$ (triangle inequality)

- Every linear subspace of a normed space $X$ is a normed subspace with the norm of $X$.

- Relation between normed and metric spaces
  
  - Every normed space is a metric space with the distance $d(x,y) = \| x - y \|$
  
  - The associated distance satisfies $d(x + z, y + z) = d(x, y)$, $d(\alpha x, \alpha y) = |\alpha| d(x, y)$
  
  - Every metric linear space with these properties is a normed space with $\| x \| = d(x, 0)$

- Definition: Banach space. Complete normed space.
Normed spaces

- Properties: \((X, \|\cdot\|)\) normed space
  
  \((i)\) \(|\|x\| - \|y\|\| \leq \|x - y\|, \forall x, y \in X\)

  \((ii)\) \(B(x_0, r) = x_0 + B(0, r), \forall x_0 \in X, r > 0\)

  \((iii)\) \(X\) Banach \(\iff\) \(\{a_n\}_1^\infty \in X, \sum_{n} \|a_n\| < \infty \Rightarrow \sum_{n} a_n\) converges in \(X\)

  \((iv)\) Let \(X\) be Banach, a subspace \(Y\) is complete \(\iff\) \(Y\) is closed in \(X\)

- Completion theorem:
  
  - Every normed linear space \(L = (L, \|\cdot\|)\) admits a completion \(\tilde{L}\), Banach space, unique up to norm isomorphisms, such that \(L\) is dense in \(\tilde{L}\) and \(\|x\|_{\tilde{L}} = \|x\|_L\)

- Infinite sums in normed spaces
  
  \(v_n \in X, v = \sum_{n=1}^{\infty} v_n\) \(\iff\) \(\exists v \in X/\sum_{n=1}^{k} v_n - v\| \xrightarrow{k \to \infty} 0\)
Normed spaces

- Hölder inequality (for sums):

\[ \sum_{j=1}^{\infty} |a_j b_j| \leq \left( \sum_{j=1}^{\infty} |a_j|^p \right)^{1/p} \left( \sum_{j=1}^{\infty} |b_j|^q \right)^{1/q} \]

\[ p, q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \{a_j\}_1^\infty \in l_p^\Lambda \quad \{b_j\}_1^\infty \in l_q^\Lambda \]

- Minkowski inequality (for sums):

\[ \left( \sum_{j=1}^{\infty} |a_j + b_j|^p \right)^{1/p} \leq \left( \sum_{j=1}^{\infty} |a_j|^p \right)^{1/p} + \left( \sum_{j=1}^{\infty} |b_j|^p \right)^{1/p} \]

\[ p \geq 1 \quad \{a_j\}_1^\infty, \{b_j\}_1^\infty \in l_p^\Lambda \]
Normed spaces

• Summary:
  • Normed (sub)space: $(X,||.||)$
  • Relation norm $\leftrightarrow$ distance
  • Banach space (complete normed space)
  • Absolute convergence $\Rightarrow$ convergence in Banach spaces
  • A subspace of a Banach space is Banach $\iff$ it is closed
  • Completion theorem: every normed space can be made complete in a unique way
  • An infinite sum converges in $(X,||.||)$ to $v$ if the sequence of partial sums converges to $v$
  • Hölder and Minkowski inequalities
Hilbert Space

- Definition: A pre-Hilbert space is a linear space with an associated scalar product.
  - Scalar product: $\langle ., . \rangle : X \times X \rightarrow \Lambda$ with the following properties
    
    (i) $\langle v, v \rangle \geq 0, \langle v, v \rangle = 0 \Leftrightarrow v = 0$
    
    (ii) $\langle v, v_1 + v_2 \rangle = \langle v, v_1 \rangle + \langle v, v_2 \rangle$
    
    (iii) $\langle v, \lambda w \rangle = \lambda \langle v, w \rangle$
    
    (iv) $\langle v, w \rangle = \overline{\langle w, v \rangle}$
    
    - In particular we have
      
      $\langle \lambda_1 v_1 + \lambda_2 v_2, v \rangle = \overline{\lambda_1} \langle v_1, v \rangle + \overline{\lambda_2} \langle v_2, v \rangle$
      
      $\langle v, w \rangle = 0 \ \forall w \in L \Rightarrow v = 0$
      
      $\langle v_1, w \rangle = \langle v_2, w \rangle \ \forall w \in L \Rightarrow v_1 = v_2$
Hilbert Space

- Property: A pre-Hilbert space is a normed space with the norm associated to the scalar product $\|v\| = +\sqrt{(v, v)}$

- Definition: A Hilbert space is a pre-Hilbert space that is complete with the norm associated to the scalar product (rather the distance associated to the norm).

- Properties: Let $(X, \langle ., . \rangle)$ be a pre-Hilbert space and $\|\cdot\|$ the associated norm:

  $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ (Parallelogram identity)

  \[
  \text{Re} \left[ \langle x, y \rangle \right] = \frac{1}{4} [\|x + y\|^2 - \|x - y\|^2] \\
  \text{Im} \left[ \langle x, y \rangle \right] = -\frac{1}{4} [\|x + iy\|^2 - \|x - iy\|^2] \quad \text{(if $X$ is complex)}
  \]

  Polarization identity

- Relation between scalar product and norm: a normed space $(X, \|\cdot\|)$ that satisfies the parallelogram identity is a pre-Hilbert space with a scalar product that satisfies $\|x\| = +\sqrt{\langle x, x \rangle}$
Hilbert Space

- Properties: Let \((X, \langle ., . \rangle)\) be a pre-Hilbert space and \(\| \cdot \|\) the associated norm:
  - Schwarz-Cauchy-Buniakowski inequality
    \[
    |\langle v, w \rangle| \leq \|v\| \|w\|, \quad \forall v, w \in X, \quad ("\Rightarrow" \iff v, w \text{ lin. dep.})
    \]
  - Triangle inequality
    \[
    \|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in X \quad ("\Rightarrow" \iff y = 0 \circ x = cy, c \geq 0)
    \]
  - Continuity of the scalar product
    \[
    x_n \to x, \quad y_n \to y \Rightarrow \langle x_n, y_n \rangle \to \langle x, y \rangle
    \]
    \[
    \{x_n\}_1^\infty, \quad \{y_n\}_1^\infty \text{ are Cauchy in } X \Rightarrow \{\langle x_n, y_n \rangle\}_1^\infty \text{ is Cauchy in } \Lambda
    \]
Hilbert Space

• Properties: Let \( (X, \langle \cdot, \cdot \rangle) \) be a pre-Hilbert space and \( \| \cdot \| \) the associated norm:
  
  - \( v, w \in X \) are orthogonal if \( \langle v, w \rangle = 0 \)
  
  - \( S = \{ v_\alpha \}_{\alpha \in A} \subset X \) is an orthogonal set if \( \langle v_\alpha, v_\beta \rangle = 0 \) \( \forall \alpha \neq \beta \)
  
  - \( S = \{ v_\alpha \}_{\alpha \in A} \subset X \) is an orthonormal set if \( \langle v_\alpha, v_\beta \rangle = \delta_{\alpha \beta} \)
  
  - Every orthogonal set of non-vanishing vectors is l.i. (the inverse is not true)

• (Generalized) Pythagora’s Theorem: Let \( \{ v_j \}_{j=1}^n \) be orthonormal in \( X \)

\[
\| v \|^2 = \sum_{j=1}^n |\langle v_j, v \rangle|^2 + \| v - \sum_{j=1}^n \langle v_j, v \rangle v_j \|^2, \quad \forall v \in X
\]

• Pythagora’s theorem

\[
\left\| \sum_{j=1}^n v_j \right\|^2 = \sum_{j=1}^n \| v_j \|^2, \text{ si } \langle v_i, v_j \rangle = 0 \quad (i \neq j)
\]
Hilbert Space

• Properties:

  • Finite Bessel inequality: let \( \{v_j\}_{1}^{n} \) be an orthonormal set

    \[ ||v||^2 \geq \sum_{i=1}^{n} |\langle v_j, v \rangle|^2, \ \forall v \in X \]

  • Let \( \{v_\alpha\}_{\alpha \in A} \) be an arbitrary orthonormal set

    \[ A^{(v)} \equiv \{ \alpha \in A / \langle v_\alpha, v \rangle \neq 0 \} \text{ is finite or numerable infinite} \]

  • Infinite Bessel inequality: let \( \{v_\alpha\}_{\alpha \in A} \) be an arbitrary orthonormal set

    \[ ||v||^2 \geq \sum_{\alpha \in A} |\langle v_\alpha, v \rangle|^2, \ \forall v \in X \]

• Completion Theorem:

For any pre-Hilbert space \((X, \langle ., . \rangle)\), there is a Hilbert space \(H\) (unique up to isomorphisms) and an isomorphism \(A : X \to W\) with \(W\) dense en \(H\)
Hilbert Space

- **Definition: orthogonal complement**  Let $H$ Hilbert and $M \subset H$, $M \neq \emptyset$

  $$M^\perp \equiv \{v \in H / v \perp M\} \quad \text{(also } M^\perp = H \ominus M)$$

- **Properties of the orthogonal complement**

  1. $M^\perp$ is a closed linear subspace $\forall M \subset H$, $H$ Hilbert
  2. $M \cap M^\perp \subset \{0\}$
  3. $M^{\perp\perp} \equiv (M^\perp)^\perp \supset M$
  4. $M^\perp = (\overline{M})^\perp = [M]^\perp = (\overline{M})^\perp$
  5. $\{0\}^\perp = H$, $H^\perp = \{0\}$
Hilbert Space

- Theorem of orthogonal projection

Let $M$ be a closed linear subspace of a Hilbert space $H$, then
\[ \forall v \in H : \exists ! v_1 \in M, \exists ! v_2 \in M^\perp / v = v_1 + v_2 \ (v_1: \text{orthogonal projection of } v \text{ over } M) \]

Equivalent:

Let $M$ be a closed linear subspace of a Hilbert space $H$, then
\[ \forall v \in H : \exists ! v_1 \in M \ / \ ||v - v_1|| = \inf\{||v - y||, y \in M\}, \ v - v_1 \in M^\perp \]
Hilbert Space

• Properties:
  
  • Definition: Orthogonal direct sum
    
    Let $M, N$ be closed linear subspaces of $H$ Hilbert
    
    \[ H = M \oplus N \text{ si } H = M \bigoplus N \text{ y } M \perp N \]
    
    • $H = M \oplus M^\perp$, ∀ closed linear subspace $M \subset H$
    
    • Orthogonal projector over $M$: $P_M : H \rightarrow M$
      
      \[ P_M v = v_1, \ v = v_1 + v_2 \ \text{con } v_1 \in M, \ v_2 \in M^\perp \]
      
      \[ P_M + P_M^\perp = 1_H, \ P_M P_M^\perp = P_M^\perp P_M = 0, \ P_M^2 = P_M, \ P_M^{2\perp} = P_M^\perp \]
      
    • $S^{\perp \perp} = [S] \ \forall S \subset H, \ S \neq \emptyset$ ($S$ closed subspace $\Rightarrow S^{\perp \perp} = S$)
    
    • $S$ linear subspace of $H$ is dense in $H$ $\Leftrightarrow S^\perp = \{0\}$
Hilbert Space

- **Theorem:** Let \( \{x_n\}_1^\infty \) be an orthonormal set in \( H \) (Hilbert) and \( \{\lambda_n\}_1^\infty \subset \Lambda \), then:

\[
\sum_{1}^{\infty} \lambda_n x_n \text{ converges} \iff \sum_{1}^{\infty} |\lambda_n|^2 < \infty
\]

- **Theorem:** Let \( S = \{x_\alpha\}_{\alpha \in A} \) be an orthonormal set in \( H \) (Hilbert). Let \( M \equiv \overline{S} \)

\begin{enumerate}
  \item \( x_M \equiv \sum_{\alpha \in A} \langle x_\alpha, x \rangle x_\alpha \in M \)
  \item \( x_M \) is the only vector that satisfies \( x - x_M \perp M \)
  \item \( x \in M \Rightarrow x = x_M \)
  \item \( d(x, M) \equiv \inf_{y \in M} ||x - y|| = d(x, x_M) \)
\end{enumerate}

The best approximation of a vector \( x \) by elements of \( M = \overline{\{x_\alpha\}_{\alpha \in A}} \) orthonormal is given by \( P_M x \)
Hilbert Space

- **Orthonormalization theorem: Gram-Schmidt method**

  \[
  \text{Let } \{v_j\}_{j \in J} \subset H \text{ a l.i. set, with } J \text{ finite or numerable infinite (N)} \\
  \exists \{u_j\}_{j \in J} \text{ orthonormal such that:} \\
  (i) \ u_i \in [\{v_j\}_{j \in J}], \ v_i \in [\{u_j\}_{j \in J}] \\
  (ii) \ [\{u_j\}_{j \in J}] = [\{v_j\}_{j \in J}]
  \]

  \[
  \text{Solution:} \\
  u_m \equiv \frac{w_m}{||w_m||}, \ \text{con} \ w_m \equiv v_m - \sum_{k=1}^{m-1} \langle u_k, v_m \rangle u_k
  \]

- **Definition: Orthonormal basis**

  Maximal orthonormal set \( S = \{v_\alpha\}_{\alpha \in A} \subset H \)

- **Theorem: Existence of orthonormal basis**

  Every Hilbert space \( \neq \{0\} \) has an orthonormal basis
Hilbert Space

- Theorem: Characterization of orthonormal basis:

Let $S = \{v_\alpha\}_{\alpha \in A} \subset H \neq \{0\}$ an orthonormal set. The following statements are equivalent:

(i) $S$ is an orthonormal basis of $H$

(ii) $[S] = H$

(iii) $v \perp v_\alpha, \forall \alpha \in A \Rightarrow v = 0 \quad S^\perp = \{0\}$

(iv) $\forall v \in H \Rightarrow v = \sum_\alpha \langle v_\alpha, v \rangle v_\alpha$ (Fourier expansion)

(v) $\forall v, w \in H \Rightarrow \langle v, w \rangle = \sum_\alpha \langle v, v_\alpha \rangle \langle v_\alpha, w \rangle$ (Parseval identity)

(vi) $\forall v \in H \Rightarrow ||v||^2 = \sum_\alpha |\langle v_\alpha, v \rangle|^2$ (Parseval identity)
Hilbert Space

• Definition: Separable topological (and metric) space:
  • A topological space $X$ is separable if it contains a numerable subset dense in $X$.
  • A metric space $M$ is separable if and only if it has a numerable basis of open subsets.

• Separability criterion in Hilbert spaces

| A Hilbert space $H \neq \{0\}$ is separable | $\iff$ it admits a numerable orthonormal basis (finite or numerable infinite) |

• Proposition:
  • All orthonormal basis of a Hilbert space $H$ have the same cardinal (Hilbert dimension of $H$).
Hilbert Space

• Theorem of Hilbert Space classification

Definition: Two Hilbert spaces, $H_1, H_2$ over $\Lambda$ are isomorphic if

$$\exists U : H_1 \rightarrow H_2, \ U \text{ linear isomorphism } \langle Ux, Uy \rangle_{H_2} = \langle x, y \rangle_{H_1}, \ \forall x, y \in H_1$$

Theorem:

Every Hilbert space $H \neq \{0\}$ is isomorphic to $l^2(\Lambda)$
where the cardinal of $\Lambda$ = the Hilbert dimension of $H$

Corollaries:

• A Hilbert space of finite Hilbert dimension, $n$, is isomorphic to $\mathbb{C}^n$
• A separable Hilbert space of infinite Hilbert dimension is isomorphic to $l^2(\mathbb{N})$
• Let $H$ be a separable Hilbert space of Hilbert dimension $h$ and linear dimension $l$
  - $h < \infty \Rightarrow l = h$ and any orthonormal basis is a linear basis
  - $h = \infty \Rightarrow l > h$ and no orthonormal basis is a linear basis
Hilbert Space

• Summary:
  • (Pre-)Hilbert space: Complete linear space with scalar product
  • Hilbert Normed
  • Parallelogram and polarization identities
  • Schwarz and triangle inequality, continuity of scalar product
  • Orthonormality. Pythagora's theorem and Bessel inequality
  • Completion theorem
  • Orthogonal complement and orthogonal projector. Best approximation to a vector.
  • Gram-Schmidt orthonormalization method
  • Orthonormal basis. Separable space
  • Theorem of Hilbert Space classification
Space of functions

- Some of the most important Hilbert spaces are spaces of functions.

  - Examples:

    \[(C_A[a, b], \| \cdot \|_\infty)\] complete, not pre-Hilbert

    \[(C_A[a, b], \| \cdot \|_p), \ p \geq 1\] not complete \((p = 2\) pre-Hilbert\)

    \[(B(\mathbb{R}), \| \cdot \|_\infty)\] complete, not pre-Hilbert

    \[(L^p(\mathbb{R}), \| \cdot \|_p), \ p \geq 1\] not complete \((p = 2\) pre-Hilbert\)

- Example of not completeness of \((C_A[a, b], \| \cdot \|_2)\)

  \[
f_n(x) = \begin{cases} 
  0, & x \leq \frac{1}{2} - \frac{1}{n}, \\
  nx - \frac{n}{2} + 1, & \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2}, \\
  1, & \frac{1}{2} \leq x,
  \end{cases}
\]

  is Cauchy but does not converge in \((C_A[0, 1], \| \cdot \|_2)\)

- We can enlarge the space with the limits of all Cauchy sequences to complete it. We need a new concept of integral for that.
Space of functions

- Riemann integral:
  - Partition of the “x axis” and common convergence of upper and lower integrals

\[ \int_{a}^{b} f(x) \, dx = I \]

\[ \text{si } I = \lim_{|\pi| \to 0} \sum_{1}^{n} R_{k}^{\inf} = \lim_{|\pi| \to 0} \sum_{1}^{n} R_{k}^{\sup} < \infty \]

- Lebesgue integral:
  - Partition of the “y axis” and measure of subsets of the “x axis”

\[ \int_{\mathbb{R}} f(x) \, dx \equiv \lim_{|\pi| \to 0} \Sigma_{\pi} (f) \]

\[ \Sigma_{\pi} (f) \equiv \sum_{j=1}^{n} y_{j-1} \mu \{ f^{-1} ([y_{j-1}, y_{j}]) \} \]
Space of functions

• We need a new concept of “measure”

• Borel set: Element of $\mathcal{B}$, minimal family of subsets of $\mathbb{R}$ that contains all the open intervals $(a, b)$ and satisfies:

$$\begin{align*}
(i) \quad \{B_j\}_{j=1}^{\infty} \subset \mathcal{B} \Rightarrow \bigcup_{j=1}^{\infty} B_j \subset \mathcal{B} \\
(ii) \quad B \subset \mathcal{B} \Rightarrow \mathbb{R} - B \subset \mathcal{B}
\end{align*}$$

• Borel-Lebesgue measure (of a borel set $B$): $\mu(B) \equiv \inf_{I \supset B} l(I)$

$I = \bigcup_{j=1}^{\infty} (a_j, b_j)$ (union of disjoint open intervals) \quad l(I) \equiv \sum_{j=1}^{\infty} |b_j - a_j|

- Properties:

$B \in \mathcal{B} \Rightarrow \mu(B) = \inf\{\mu(A), A \text{ open} \supset B\} = \sup\{\mu(C), C \text{ compact} \subset B\}$

$B_n \in \mathcal{B}, \ n \geq 1$, disjoint in pairs $\Rightarrow \mu(\bigcup_{i=1}^{\infty} B_n) = \sum_{1}^{\infty} \mu(B_n)$
Space of functions

- We need a new concept of “measure”
  - Borel measurable function: \( f : \mathbb{R} \to \mathbb{R} \) is Borel measurable if \( f^{-1}(B) \in \mathcal{B} \), \( \forall B \in \mathcal{B} \)
  - \( f \) complex is Borel if both its real and imaginary parts are
  - Let \( f, g \), be real: \( f + g, \lambda f (\lambda \in \mathbb{R}), fg, |f| \) are borel

- Characterization of Borel measurable functions:
  a) \( f : \mathbb{R} \to \mathbb{R} \) is Borel \( \iff f^{-1}\{(a, b)\} \in \mathcal{B}, \forall a, b \)
  b) \( f_n(x) \to f(x), \forall x, f_n \text{ Borel} \Rightarrow f \text{ Borel} \)
  c) \( f : \mathbb{R} \to \mathbb{R} \) is Borel \( \iff \{x/f(x) < b\} \in \mathcal{B}, \forall b \)

- Lebesgue integral
  let \( f \geq 0 \), bounded and Borel measurable. Its Lebesgue integral is

\[
\int_{\mathbb{R}} f \, dx \equiv \lim_{|\pi| \to 0} \Sigma_{\pi}(f)
\]

\[
\pi : 0 = y_0 < y_1 < \ldots < y_n = \sup f \text{ partition of the range of } f
\]

\[
\Sigma_{\pi}(f) \equiv \sum_{j=1}^{n} y_{j-1} \mu\{f^{-1}([y_{j-1}, y_j])\}
\]

Easy to extend to more general functions
Space of functions

- Lebesgue integrable functions

$$f \in \mathcal{L}^1_\mathbb{R}(\mathbb{R}) \text{ if } \int_{\mathbb{R}} |f| \, dx < +\infty, \quad \int_{\mathbb{R}} f \, dx \equiv \int_{\mathbb{R}} \frac{|f| + f}{2} \, dx - \int_{\mathbb{R}} \frac{|f| - f}{2} \, dx$$

$$f \in \mathcal{L}^1_C(\mathbb{R}) \text{ if } \int_{\mathbb{R}} |f| \, dx < +\infty, \quad \int_{\mathbb{R}} f \, dx \equiv \int_{\mathbb{R}} \text{Re}(f) \, dx + i \int_{\mathbb{R}} \text{Im}(f) \, dx$$

- Properties almost everywhere (a.e.).

A property $P(x), \ x \in \mathbb{R}$ is satisfied almost everywhere (a.e.) if the set $\{x/P(x) \text{ false}\}$ has vanishing measure.

For instance $f_1 = f_2 \text{ a.e. } \Leftrightarrow \int_{\mathbb{R}} |f_1 - f_2| \, dx = 0$

- $L^1$ Spaces.

$L^1(\mathbb{R})$ is the set of equivalence classes of functions in $\mathcal{L}^1(\mathbb{R})$ with the equivalence relation: $f_1 = f_2 \text{ a.e.}$
Space of functions

- **L^p spaces:**
  \[ f \in L^p(B) \text{ if } ||f||_p \equiv \left( \int_B |f|^p \, dx \right)^{1/p} < +\infty, \quad 1 \leq p < +\infty \]

- **Definition:** \( L^p(B) \) set of equivalence classes of functions \( f \in L^p(B) \) with equivalence relation \( f = g \) a.e.

- **Properties:**
  1. \((L^p(\mathbb{R}), ||.||_p), (L^p(B), ||.||_p)\), are Banach
  2. \( C[a, b] \) is dense in \((L^p([a, b]), ||.||_p)\)
  3. \((L^p([a, b]), ||.||_p)\) is the completion of \( C[a, b] \) (same \([a, b] \to \mathbb{R}\))
  4. \( L^2(\mathbb{R}) \) is Hilbert with the scalar product
  \[ \langle f, g \rangle \equiv \int_{\mathbb{R}} \bar{f}(x)g(x) \, dx, \text{ (same for } [a, b]) \]
Space of functions

- (Integral) Hölder and Minkowski inequalities

Let $f, h \in L^p(X), \ g \in L^q(X), \ 1 < p < \infty, \ \frac{1}{p} + \frac{1}{q} = 1$

Hölder inequality

$$\int_X |fg| \, dx \leq \left( \int_X |f|^p \, dx \right)^{1/p} \cdot \left( \int_X |g|^q \, dx \right)^{1/q}$$

Minkowski inequality

$$\left\{ \int_X |f + h|^p \, dx \right\}^{1/p} \leq \left\{ \int_X |f|^p \, dx \right\}^{1/p} + \left\{ \int_X |h|^p \, dx \right\}^{1/p}$$
Space of functions

- Some relevant orthonormal bases in \( L^2 \):
  - Legendre's basis
    \[
    P_n(x) \equiv \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \text{ (Legendre's Polynomials)}
    \]
    \[
    \left\{ \sqrt{n + 1/2} P_n \right\}_0^\infty \text{ is an orthonormal basis of } L^2[-1, 1]
    \]
    \[
    (1 - x^2) P_n'' - 2x P_n' + n(n + 1) P_n = 0, \ n = 0, 1, \ldots \text{ (Legendre's eq.)}
    \]
  - Hermite's basis
    \[
    H_n(x) \equiv (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \text{ (Hermite's polynomials)}
    \]
    \[
    \left\{ (\sqrt{\pi} 2^n n!)^{-1/2} e^{-x^2/2} H_n \right\}_0^\infty \text{ is an orthonormal basis of } L^2(\mathbb{R})
    \]
    \[
    H_n'' - 2x H_n' + 2n H_n = 0, \ n = 0, 1, \ldots \text{ (Hermite's eq.)}
    \]
Space of functions

- Some relevant orthonormal bases in $L^2$:
  - Laguerre's basis
    
    
    $$L_n(x) \equiv \frac{1}{n!} e^x \frac{d^n}{dx^n}(e^{-x}x^n) \text{ (Laguerre's polynomial)}$$

    $$\left\{ e^{-x/2} L_n \right\}_0^\infty \text{ is an orthonormal basis of } L^2[0, \infty)$$

    $$xL''_n + (1-x)L'_n + nL_n = 0, \ n = 0, 1, \ldots \text{ (Laguerre's eq.)}$$

  - Orthonormal bases of polynomial associated to a weight function
    
    Let $0 \neq \rho \in L^1(R)$, non-negative / $\exists \alpha > 0$, for which $\int_{\mathbb{R}} e^{\alpha|t|} \rho(t) \ dt < \infty$

    If $\{p_n(t)\}_0^\infty$ are orthonormal polynomial with respect to the scalar product

    $$\langle f, g \rangle_\rho \equiv \int_{\mathbb{R}} f \bar{g} \rho, \text{ obtained from } \{t^n\}_0^\infty \text{ through the Gram-Schmidt method, then}$$

    $$\{p_n(t)\rho^{1/2}(t)\}_0^\infty \text{ is an orthonormal basis of } L^2(\text{sop } \rho)$$
Space of functions

- Some relevant orthonormal bases in $L^2$:
  - Fourier's basis

$$\left\{ e^{\frac{2\pi nx}{L}}/\sqrt{L} \right\}_{-\infty}^{+\infty} \text{ is an orthonormal basis in } L^2[a, a+L]$$

$$\left\{ \frac{1}{\sqrt{L}} \cos \left( \frac{2\pi nx}{L} \right), \sqrt{\frac{2}{L}} \sin \left( \frac{2\pi nx}{L} \right) \right\} \text{ } (n = 1, 2, \ldots) \text{ is an orthonormal basis in } L^2[a, a+L]$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi nx}{L}} = a_0 + \sum_{n=1}^{\infty} \left[ 2a_n \cos \left( \frac{2\pi nx}{L} \right) + 2b_n \sin \left( \frac{2\pi nx}{L} \right) \right]$$

$$c_n = \frac{1}{L} \int_{a}^{a+L} e^{-\frac{2\pi nx}{L}} f(x) \, dx$$

$$a_n = \frac{1}{L} \int_{a}^{a+L} \cos \left( \frac{2\pi nx}{L} \right) f(x) \, dx, \quad b_n = \frac{1}{L} \int_{a}^{a+L} \sin \left( \frac{2\pi nx}{L} \right) f(x) \, dx$$
Space of functions

- Some relevant orthonormal bases in $L^2$:
  - Fourier's basis
    
    $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{2\pi nx}{L}} = a_0 + \sum_{n=1}^{\infty} \left[ 2a_n \cos \left( \frac{2\pi nx}{L} \right) + 2b_n \sin \left( \frac{2\pi nx}{L} \right) \right]
  
    Convergencia en $L^2$ (c.d.)

- Jordan convergence criterion

  Let $f \in L_C^2[a, b]$ with bounded variation in $(a, b)$, then the Fourier series converges at every point $x \in (a, b)$ to
  $\lim_{\epsilon \to 0} \frac{f(x+\epsilon)+f(x-\epsilon)}{2}$

- Bases with only sines or cosines

  $f \in L_C^2[a, b]$ can be expanded in Fourier series using only sines or only cosines by expanding antisymmetric or symmetric extension of the function
Space of functions

- Expansion in eigenvectors
  - Consider the following differential operator

\[ \mathcal{O} = \frac{d^2}{dx^2} \]

Every function \( f \in L^2[a, a + L] \) can be expanded in eigen-functions of \( \mathcal{O} \)

\[ f(x) = \sum_{n=-\infty}^{\infty} c_n f_n(x) \]

with

\[ f_n(x) = e^{i \frac{2\pi n x}{L}}, \quad \mathcal{O} f_n = -\left(\frac{2\pi n}{L}\right)^2 f_n \]

Eigenvalues
Space of functions

- Summary:
  - Lebesgue integral.
  - Lebesgue integrable functions. $L^1$ Spaces
  - Properties almost everywhere. $L^p$ Spaces
  - $L^2(B)$ is a Hilbert space (completion of $C(B)$)
  - Hölder and Minkowski integral inequalities
  - Orthonormal polynomials in $L^2(B)$
  - Fourier basis. Fourier expansion.
  - Expansion in eigenvectors.
Linear forms

- **Definitions:** Let $L$ be a linear space over the field $\Lambda$
  - A linear form (or functional) is a linear mapping $F : L \to \Lambda$
    \[
    F(x + y) = F(x) + F(y), \quad F(\alpha x) = \alpha F(x), \quad \forall x, y \in L, \quad \forall \alpha \in \Lambda
    \]
  - A linear form in a normed space is continuous if
    \[
    \forall \{x_n\} \to x \Rightarrow \{F(x_n)\} \to F(x), \quad \forall x \in L
    \]
    \[
    \forall \varepsilon > 0 \exists \delta > 0 / \|x - y\| < \delta \Rightarrow |F(x) - F(y)| < \varepsilon
    \]
  - A linear form in a normed space is bounded if
    \[
    \exists M \geq 0 / |F(x)| \leq M \|x\|, \quad \forall x \in L
    \]
    \[
    \|F\| = \sup_{x \neq 0} \frac{|F(x)|}{\|x\|} = \sup_{\|x\| = 1} |F(x)| = \inf \{M \geq 0 / |F(x)| \leq M \|x\|}\}
    \]
- **Theorem:** Let $F$ be a linear form in a normed space
  - $F$ is bounded $\Leftrightarrow$ $F$ is continuous
Linear forms

- Definition: Dual space of a Hilbert space \((H, \langle, \rangle)\) is the set of all continuous functional forms in \(H\).

\[
\tilde{H} = \{F : H \to \Lambda/F \text{ linear and continuous}\} \equiv A(H, \Lambda)
\]

It is a Hilbert space (as we will see)

- Proposition: Let \((H, \langle, \rangle)\) be a Hilbert space of finite dimension:
  - All functionals in \(H\) are continuous
  - \(\dim \tilde{H} = \dim H\)

- Riesz-Fréchet representation theorem: Let \((H, \langle, \rangle)\) be a Hilbert space (separable or not)

\[
\forall F : H \to \Lambda \text{ linear and continuous}
\]

\[
\exists! f \in H/F(g) = \langle f, g \rangle, \forall g \in H
\]
Linear forms

- Properties:
  - Let $F \neq 0 \Rightarrow \dim(M_0^\perp) = 1$ \hspace{1cm} ($M_0 \equiv \{h \in H/F(h) = 0\}$)
  - Let $\{e_j\}_{1}^{n}$ be an orthonormal basis of $\Lambda^n$, $\forall \phi : H \rightarrow \Lambda^n$ linear and continuous
    \[ \exists x_1, \ldots, x_n \in H/\phi(y) = \sum_{1}^{n} \langle x_j, y \rangle e_j \]
  - $\|F_x\|_{A(H,\Lambda)} = \|x\|_H$
  - $F$ linear form in a Hilbert space is continuous $\Leftrightarrow$ its kernel $M_0$ is closed in $H$
  - $\tilde{H}$ is a Hilbert space with the scalar product associated to $H$
    \[ \langle ., . \rangle : \tilde{H} \times \tilde{H} \rightarrow \Lambda \]
    \[ F_f, F_g \rightarrow \langle F_f, F_g \rangle \equiv \langle g, f \rangle \]
  - The mapping $f \in H \rightarrow F_f \in \tilde{H}$ with $F_f(g) = \langle f, g \rangle$,
    is an anti-linear isometric bijection
Linear forms

- **Bilinear forms:** let \((H, \langle ., . \rangle)\) be a Hilbert space over \(\Lambda\)

  - **Bilinear form** (rather sesquilinear): mapping \(\phi : H \times H \to \Lambda\) such that
    
    \[(i) \phi(\alpha x, \beta y) = \alpha \beta \phi(x, y), \forall \alpha, \beta \in \Lambda, \forall x, y \in H\]
    
    \[(ii) \phi(x_1 + x_2, y) = \phi(x_1, y) + \phi(x_2, y)\]
    
    \[(iii) \phi(x, y_1 + y_2) = \phi(x, y_1) + \phi(x, y_2)\]

  - A bilinear form is bounded if \(\exists k \geq 0/|\phi(x, y)| \leq k ||x|| ||y||, \forall x, y \in H\)

    \[||\phi|| = \sup_{x \neq 0 \neq y} \frac{|\phi(x, y)|}{||x|| ||y||} \text{ (it is a norm)}\]

  - **Theorem:** let \(\phi : H \times H \to \Lambda\), be a bilinear form bounded in \(H\) (Hilbert).

    \(\exists! A \in A(H)\) (bounded linear mapping \(A : H \to H\)) such that

    \[\phi(x, y) = \langle x, Ay \rangle, \forall x, y \in H\]

    and \(||\phi|| = ||A|| \equiv \sup_{0 \neq x \in H} \frac{||Ax||}{||x||} < +\infty\)
Linear forms

- Strong convergence (in norm) \( x_n \xrightarrow{s} n \iff \|x_n - x\| \to 0 \)
- Weak convergence \( x_n \xrightarrow{w} n \iff F(x_n) \to F(x), \forall F \in \tilde{H} \)
- Theorems:

\[
\begin{align*}
x_n \xrightarrow{s} x & \implies x_n \xrightarrow{w} x \\
\left\{ x_n \xrightarrow{w} x, \quad \|x_n\| \to \|x\| \right\} & \iff x_n \xrightarrow{s} x \\
x_n \xrightarrow{w} x, \quad x_n \xrightarrow{s} x' & \implies x_n \xrightarrow{w} x'
\end{align*}
\]
Distributions

- Test function spaces:
  - Test functions of compact support
    \[ D(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) / \text{supp}(f) \text{ bounded on } \mathbb{R} \}, \quad (\text{supp}(f) = \{ x / f(x) \neq 0 \}) \]
    it is a linear space and algebra of functions.
    - Convergence
      \[ f_n \overset{D}{\to} f \text{ if } \begin{cases} 
      i) & \text{supp}(f_n) \subset K \text{ bounded and independent of } n \\
      ii) & \| f_n^{(p)} - f^{(p)} \|_\infty \to 0, \ \forall p \geq 0 
    \end{cases} \]
  - Test functions of rapid decrease
    \[ S(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) / \sup_{k, m \in \mathbb{N}} \| x^k f^{(m)} \|_\infty < \infty \} \]
    it is a semi-normed space \((\| f \|_{km} = \| x^k f^{(m)} \|_\infty \text{ is semi-norm})\)
    - Convergence
      \[ f_n \overset{S}{\to} f \text{ if } \| x^k f_n^{(m)}(x) - x^k f^{(m)}(x) \|_\infty \to 0, \ \forall k, m \in \mathbb{N} \]
  - Properties
    \[ f_n \overset{D}{\to} f \Rightarrow f_n \overset{S}{\to} f, \quad D \text{ is dense in } S \]
Distributions

• Definitions and properties:
  • Distribution: \( T : \mathcal{D}(\mathbb{R}) \to \Lambda \) linear and continuous (in the sense of \( \mathcal{D} \))
    \[
    T(\alpha_1 \phi_1 + \alpha_2 \phi_2) = \alpha_1 T(\phi_1) + \alpha_2 T(\phi_2), \quad \forall \alpha_{1,2} \in \Lambda, \quad \forall \phi_{1,2} \in \mathcal{D}
    \]
    \[
    \phi_n \xrightarrow{\mathcal{D}} \phi \Rightarrow T(\phi_n) \to T(\phi)
    \]
  • Space of distributions: \( \widehat{\mathcal{D}(\mathbb{R})} = \{ T/T \text{ distribution} \} \)
  • Sufficient condition for \( T \) to be continuous
    \( \exists M > 0 \) indep. of \( \phi \mid |T(\phi)| \leq M \|\phi\|_{\infty}, \quad \forall \phi \in \mathcal{D}(\mathbb{R}) \Rightarrow T \) continuous in the sense of \( \mathcal{D} \)
  • Tempered distribution: \( T : \mathcal{S}(\mathbb{R}) \to \Lambda \) linear and continuous (in the sense of \( \mathcal{S} \))
  • Space of tempered distributions: \( \widehat{\mathcal{S}(\mathbb{R})} \)
  • The sufficient condition for continuity applies the same.
  • Property:
    \[
    \widehat{\mathcal{S}(\mathbb{R})} \subset \widehat{\mathcal{D}(\mathbb{R})}
    \]
Distributions

- Operations with distributions
  - Multiplication by a function:
    \[ \rho T : \phi \rightarrow T(\rho \phi) \text{ is an element of } \overline{D(\mathbb{R})}, \quad \forall \rho \in C^\infty \]
    \[ \text{is an element of } \mathcal{S}(\mathbb{R}), \quad \forall \rho \in C^\infty \text{ of slow growth} \]
  - Derivative of a distribution:
    \[ T^{(m)} : \phi \rightarrow T((-1)^m \phi^{(m)}) \]
  - Shift:
    \[ T_a : \phi \rightarrow T(\phi - a) \text{ with } \phi_a(x) \equiv \phi(x - a) \]
- These operations are continuous with respect to the following definition of convergence of distributions
  \[ T_n \rightarrow T \iff T_n(\phi) \rightarrow T(\phi), \quad \forall \phi \in \mathcal{D}(\mathcal{S}) \]
  With this notion of convergence \( \tilde{D} \) and \( \tilde{S} \) are complete and \( \tilde{S} \) is dense in \( \tilde{D} \)
Distributions

- Examples of distributions:
  - Dirac's delta \( \delta_{x_0} : \phi \rightarrow \phi(x_0) \) (tempered distribution)

  Normally introduced as a ”function”: \( \delta_{x_0}(\phi) = \int \delta(x - x_0)\phi(x) \, dx \)

  \( \delta(x - x_0) = \begin{cases} 
  \infty, & x = x_0 \\
  0, & x \neq x_0 
\end{cases} \)

  and as the limit of a sequence of functions

  \( \delta_0 = \lim_{\lambda \to \infty} \frac{\lambda}{\pi} e^{-\lambda x^2} = \lim_{\epsilon \downarrow 0} (\pi i \epsilon)^{-1/2} e^{ix^2/\epsilon} = \lim_{\lambda \to \infty} \frac{\sin \lambda x}{\pi x} \)

  \( \delta(x - x_0) = \frac{d}{dx} \theta(x - x_0), \quad \theta(x) = \begin{cases} 
  1, & x > 0, \\
  0, & x < 0, 
\end{cases} \) (Heaviside step function)

  Let \( f(x) \) be a function with a finite number of simple zeroes, then

  \( \delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|}, \quad f(x_i) = 0 \)
Distributions

• Examples of distributions:
  
  • Principal value of $\frac{1}{x}$ (tempered distribution) \[ \text{PV} \frac{1}{x}(\phi) = \lim_{\epsilon \downarrow 0} \int_{|x| \geq \epsilon} dx \frac{\phi}{x} \]

  We have \[ \text{PV} \frac{1}{x} = \frac{d}{dx} \ln |x| \]

  Taking derivatives of \[ \lim_{\epsilon \downarrow 0} \ln(\epsilon + i x) = \ln |x| - \frac{\pi}{2} + i \pi \theta(x), \] we find \[ \frac{1}{x \pm i \epsilon} \equiv \lim_{\epsilon \downarrow 0} \frac{1}{x \mp i \epsilon} = \text{VP} \frac{1}{x} \pm i \pi \delta(x) \]

  • Characteristic distribution (distribution) \[ \chi_X : \phi \rightarrow \chi_X(\phi) = \int_X \phi(x) \, dx \]

  Usually presented as a "function" \[ \chi_X(x) = \begin{cases} 0, & x \not\in X, \\ 1, & x \in X \end{cases} \]
Distributions

- Regularity theorem

\[ \forall T \in \mathcal{D}(\mathbb{R}), \exists f \text{ continuous in } \mathbb{R}, \exists n \in \mathbb{N} \cap T = T_f^{(n)} \]

where \( T_f(\phi) \equiv \int_{\mathbb{R}} \bar{f}(x) \phi(x) \, dx \)

- Fourier transform

\[
\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} f(x) \, dx, \quad \text{(direct transform)}
\]

\[
\tilde{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} f(y) \, dy, \quad \text{(inverse transform)}
\]

we have \( \hat{\hat{f}} = f \)

- Fourier transform of distributions

\[ \hat{T}(\phi) \equiv T(\tilde{\phi}), \quad \forall T \in \mathcal{D}(\mathbb{R}) \]
Linear forms and distributions

• Summary:
  • Linear forms $T : H \to \Lambda$, bounded and continuous
  • Dual space: bounded linear forms
  • Riesz-Fréchet theorem: representation of linear forms in Hilbert spaces
  • Bilinear forms and their representation in Hilbert spaces
  • Spaces of test functions (bounded support and rapid decrease)
  • (Tempered) distribution: linear form in spaces of test functions
  • Operations with distributions: multiplication by a function, derivative, shift
  • Examples of distributions: delta, step, PV(1/x), characteristic distribution
  • Regularity theorem
  • Fourier transform (of distributions).
Operators in Hilbert spaces

• Definition:

(Anti)linear operator. (anti)linear univalued mapping between Hilbert spaces

\[ T : D(T) \subset H_1 \to R(T) \subset H_2 \]

\[ T(\alpha x + \beta y) = \begin{cases} 
\alpha T(x) + \beta T(y), \text{ (linear)} \\
\bar{\alpha}T(x) + \bar{\beta}T(y), \text{ (anti-linear)} 
\end{cases} \quad \forall x, y \in D(T), \ \forall \alpha, \beta \in \Lambda \]

• Properties:

- \( D(T), \ R(T), \ \text{Ker}(T) \) are linear subspaces
- \( M \) linear subspace of \( H_1 \Rightarrow TM \equiv \{Tx/x \in M\} \) is a linear subspace \( H_2 \)
- \( \mathcal{L}(H_1, H_2) \equiv \{T : D(T) \subset H_1 \to H_2/T \text{ linear}\} \) is a linear space with

\[ (T_1 + T_2)x = T_1x + T_2x, \quad (\alpha T)x = \alpha(Tx) \]

- \( \mathcal{L}(H) \equiv \mathcal{L}(H, H) \)
Operators in Hilbert spaces

- **Definition:** Bounded operator. Let $T \in \mathcal{L}(H_1, H_2)$, $D(T) = H_1$
  
  $T$ is bounded if $\exists M \geq 0/\|Tx\|_{H_2} \leq M\|x\|_{H_1}$, $\forall x \in H_1$

- $\mathcal{A}(H_1, H_2) = \{T : H_1 \to H_2 / T \text{ bounded linear}\}$ is a normed space
  
  with the norm $\|T\| \equiv \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$

- **Definición:** Operador continuo.
  
  $T \in \mathcal{L}(H_1, H_2)$ es continuo en $x \in H_1$ si

  $\forall \{x_n\} \to x \Rightarrow \{Tx_n\} \to Tx$, $\left[\|x_n - x\| \to 0 \Rightarrow \|Tx_n - Tx\| \to 0\right]$ 

- $T \in \mathcal{L}(H_1, H_2)$ es continuo si es $\forall x \in H_1$ $T \left(\lim_{n \to \infty} x_n\right) = \lim_{n \to \infty} Tx_n$

- **Theorem:** $T \in \mathcal{L}(H_1, H_2)$, $H_{1,2}$ espacios de Hilbert

  $T \in \mathcal{A}(H_1, H_2) \iff T$ continuo $\iff T$ continuo en cualquier punto de $H_1$

- **Dual space:** $\tilde{H} = \mathcal{A}(H, \Lambda)$
Operators in Hilbert spaces

- **Property:** \( T \in \mathcal{A}(H_1, H_2) \Rightarrow \text{Ker}(T) \) is closed

- **Definition:** \( T : D(T) \neq H_1 \rightarrow H_2 \) is bounded in its domain if
  \[
  \exists M \geq 0 / ||Tx|| \leq M||x||, \ \forall x \in D(T), \quad ||T|| = \sup_{0 \neq x \in D(T)} \frac{||Tx||}{||x||}
  \]

- **Theorem (extension of operators bounded in a dense domain):**
  Let \( T \in \mathcal{L}(H_1, H_2) \) bounded in its domain, dense in \( H_1 \) \( \overline{D(T)} = H_1 \)
  \( \exists \tilde{T} \in \mathcal{A}(H_1, H_2) \) that extends \( T \) to all \( H_1 \) and \( ||\tilde{T}|| = ||T|| \)
  \[
  \tilde{T} x = \begin{cases} 
  Tx, & x \in D(T), \\
  \lim_{n \rightarrow \infty} Tx_n, & x_n \in D(T), \ \lim_{n \rightarrow \infty} x_n = x \notin D(T)
  \end{cases}
  \]

- **Properties:**
  - \( \mathcal{A}(H) \) is a Banach space and algebra of functions with \( ST(x) = S(T(x)) \)
  - \( ||ST|| \leq ||S|| \ ||T|| \)
  - Commutator of operators: \([S, T] = ST - TS \neq 0\) in general
Operators in Hilbert spaces

• Definition: Let $T \in \mathcal{L}(H_1, H_2)$, we define the inverse operator (when it exists)

$$T^{-1} : R(T) \subset H_2 \rightarrow D(T) \subset H_1$$

such that

$$T^{-1}Tx = x, \quad \forall x \in D(T) = R(T^{-1})$$

$$TT^{-1}y = y, \quad \forall y \in R(T) = D(T^{-1})$$

• Criterion of existence of the inverse operator

Let $T \in \mathcal{L}(H_1, H_2)$

$$\exists \ T^{-1} \in \mathcal{L}(H_2, H_1) \iff T \text{ is injective} \iff Tx = 0 \Rightarrow x = 0$$

Note: Let $T \in \mathcal{A}(H_1, H_2),$ $R(T) = H_2,$ $T$ injective $\not\Rightarrow T^{-1} \in \mathcal{A}(H_2, H_1)$

• Theorem (criterion of inversion with boundedness):

Let $T \in \mathcal{A}(H_1, H_2),$ $R(T) = H_2,$ $H_{1,2} \neq \{0\}$ then

$$T^{-1} \in \mathcal{A}(H_2, H_1) \iff \exists k > 0 / \|Tv\| \geq k\|v\|, \quad \forall v \in H_1$$

• Corolary: Let $T \in \mathcal{A}(H)$ bijective, with $H \neq \{0\}.$ Then

$$T^{-1} \in \mathcal{A}(H) \iff \exists k > 0 / \|Tv\| \geq k\|v\|, \quad \forall v \in H$$
Operators in Hilbert spaces

- Topologies on $A(H)$: let $\{A_n \in A(H)\}_{1}^{\infty}$
  - Uniform (or norm) topology
    $$A_n \xrightarrow{u} A \iff \lim_{n \to \infty} \|A_n - A\| = 0$$
  - Strong topology
    $$A_n \xrightarrow{s} A \iff A_n v \xrightarrow{n \to \infty} A v, \forall v \in H$$
  - Weak topology
    $$A_n \xrightarrow{w} A \iff \lim_{n \to \infty} \langle w, A_n v \rangle = \langle w, A v \rangle, \forall v, w \in H$$

- In finite dimension (dim of $H$ is finite) they are all equivalent
- In infinite dimension

Uniform top. $\not\supset$ Strong top. $\not\supset$ Weak top.
Operators in Hilbert spaces

• Some interesting operators
  
  • Operators in finite dimension
    
    \[ T \in \mathcal{L}(H) \Rightarrow \text{matrix in } \Lambda^n \]
    
    \[ \mathcal{A}(H_n) = \mathcal{L}(H_n) \text{ [all linear operators are bounded]} \]

  • Destruction, creation and number operators (in \( l_\Lambda^2 \))
    
    \[ a : (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots) \to (\alpha_1, \sqrt{2}\alpha_2, \ldots, \sqrt{n+1}\alpha_{n+1}, \ldots) \]
    
    \[ a^+ : (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots) \to (0, \alpha_0, \sqrt{2}\alpha_1, \ldots, \sqrt{n}\alpha_{n-1}, \ldots) \]
    
    \[ N : (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots) \to (0, \alpha_1, 2\alpha_2, \ldots, n\alpha_n, \ldots) \]

  • Rotation operator (in \( L^2(\mathbb{R}^3) \)). Let \( R \) be a rotation in \( \mathbb{R}^3 \) around the origin
    
    \[ U(R) : f(x) \to f(R^{-1}x) \]

  • Shift operator (in \( L^2(\mathbb{R}^n) \)). Let \( a \) be a vector \( \in \mathbb{R}^n \) fijo
    
    \[ U_a : f(x) \to f(x - a) \]
Operators in Hilbert spaces

- Some interesting operators
  - Position operator (en $L^2(B)$).
    \[ Q : f(x) \rightarrow xf(x) \]
  - Derivative operator. Let $\mathcal{S}(\mathbb{R}) = \{ f \text{ of rapid decrease}\}$, dense in $L^2(\mathbb{R})$.
    \[ P : f(x) \in \mathcal{S}(\mathbb{R}) \rightarrow -i \frac{d}{dx} f(x) \]
  - Properties: let us define the position and momentum operators in $\mathcal{S}(\mathbb{R}^n)$
    \[ Q_j : f(x) \rightarrow x_j f(x) \]
    \[ P_k : f(x) \in \mathcal{S}(\mathbb{R}) \rightarrow -i \frac{\partial}{\partial x_k} f(x) \quad (j, k = 1, 2, \ldots, n) \]
    we have
    \[ Q_j, P_k \text{ are not bounded and satisfy } [Q_j, P_k] = i\delta_{jk}1_S \]
Operators in Hilbert spaces

• Adjoint operator

Given \( A \in \mathcal{A}(H) \) with \( H \) a Hilbert space, the adjoint operator is defined as the only operator \( A^\dagger \) (\( \in \mathcal{A}(H) \)) that satisfies

\[
\langle w, Av \rangle = \langle A^\dagger w, v \rangle \quad \forall v, w \in H
\]

• Properties:

1) The mapping \( A \to A^\dagger \) is an anti-linear isometric bijection of \( \mathcal{A}(H) \)

2) \( (AB)^\dagger = B^\dagger A^\dagger \)

3) \( (A^\dagger)^\dagger = A \)

4) \( A, A^{-1} \in \mathcal{A}(H) \Rightarrow (A^\dagger)^{-1} = (A^{-1})^\dagger \)

5) \( \|A^\dagger A\| = \|A\|^2 \)

6) \( A^\dagger = (A^T)^* \) in finite dimension
Operators in Hilbert spaces

- Equality of operators
  \[ A = B \text{ if } D(A) = D(B) = D, \ A x = B x, \ \forall x \in D \]
  (equiv. if \( D(A) = D(B) = D, \ \langle y, A x \rangle = \langle y, B x \rangle \ \forall x \in D, \ \forall y \in H \))

- Some special types of operators: Let \( T : D(T) \) dense in \( H \to H \)
  - Symmetric or hermitian operator
    \[ T \subset T^\dagger \quad \left[ D(T) \subset D(T^\dagger), \ \langle x, T y \rangle = \langle T x, y \rangle, \ \forall x, y \in D(T) \right] \]
  - Self-adjoint operator
    \[ T = T^\dagger \quad \left[ D(T) = D(T^\dagger), \ \langle x, T y \rangle = \langle T x, y \rangle, \ \forall x, y \in D(T) \right] \]
  - Bounded self-adjoint operator
    \[ A \in \mathcal{A}(H) / A = A^\dagger \quad \left[ A = A^\dagger \iff \langle x, A x \rangle \in \mathbb{R}, \ \forall x \in H \right] \]
Operators in Hilbert spaces

- Properties of bounded self-adjoint operators

Let \( A, B \in \mathcal{A}(H) \), \( A = A^\dagger \), \( B = B^\dagger \)

\( i \) \quad ||A|| = \sup_{x \neq 0} \frac{|\langle x, Ax \rangle|}{||x||^2}

\( ii \) \quad \alpha A + \beta B \) is a bounded self-adjoint operator \( \forall \alpha, \beta \in \mathbb{R} \)

\( iii \) \quad AB \) is a bounded self-adjoint operator \( \Leftrightarrow [A, B] = 0 \)

\( iv \) \quad ||A^n|| = ||A||^n

- Isometric operator

\( T : D(T) \subset H \rightarrow H/||Tx|| = ||x|| \), \( \forall x \in D(T) \)

property

\[
\begin{array}{c}
\text{T isometric} \quad \Rightarrow \quad T \text{ bounded in its domain with } ||T|| = 1 \\
\end{array}
\]
Operators in Hilbert spaces

- Unitary operator

\[ U \in \mathcal{A}(H) \land U^\dagger = U^{-1} \]

Note:

\[ U \in \mathcal{A}(H) \text{ isometric } \iff U^\dagger U = 1 \quad \text{[} UU^\dagger = 1 \iff R(U) = H \text{]} \]

\[ U \in \mathcal{A}(H) \text{ unitary } \iff U^\dagger U = UU^\dagger = 1 \]

- Characterization of a unitary operator. Let \( U \in \mathcal{A}(H) \). They are equivalent

  i) \( U \) unitary

  ii) \( R(U) = H, \langle Ux, y \rangle = \langle x, y \rangle, \forall x, y \in H \)

  iii) \( R(U) = H, \|Ux\| = \|x\|, \forall x \in H \)

  iv) \( \{e_\alpha\}_{\alpha \in A} \) orthonormal basis of \( H \Rightarrow \{Ue_\alpha\}_{\alpha \in A} \) orthonormal basis of \( H \)

  v) \( U^\dagger \) is unitary
Operators in Hilbert spaces

- Orthogonal projector
  
  \[ P \in \mathcal{A} \text{ is an orthogonal projector if } P^2 = P = P^\dagger \]

- Theorem: Let \( P \) be an orthogonal projector, then
  
  \( \exists M \text{ closed linear subspace in } H \text{ such that } P \text{ is the orthogonal projector over } M \)

- Normal operator
  
  \[ A : D(A) \text{ dense in } H \rightarrow H / D(AA^\dagger) = D(A^\dagger A), \ [A, A^\dagger] = 0 \]

  Note:
  
  \[ A \in \mathcal{A}(H) \Rightarrow A^\dagger \in \mathcal{A}(H) \Rightarrow D(AA^\dagger) = D(A^\dagger A) = H \]

  \[ A \in \mathcal{A}(H) \text{ normal } \Leftrightarrow ||Av|| = ||A^\dagger v||, \ \forall v \in H \]

- Properties:
  
  A self-adjoint \( \Rightarrow A \text{ normal } (AA^\dagger = A^\dagger A = A^2) \)
  
  A hermitian \( \nRightarrow A \text{ normal } (D(AA^\dagger) \neq D(A^\dagger A)) \)
  
  A unitary \( \Rightarrow A \text{ normal } (AA^\dagger = A^\dagger A = 1) \)
  
  A isometric \( \nRightarrow A \text{ normal } (D(AA^\dagger) \neq D(A^\dagger A)) \)
Operators in Hilbert spaces

- **Summary**
  - Operator $\mathcal{L}(H_1, H_2)$, bounded $\mathcal{A}(H_1, H_2)$ and bounded in its domain
  - Continuous operator $\Leftrightarrow$ bounded
  - Theorem of extension of bounded operators with a dense domain
  - Inverse operator. Existence of inverse operator (with boundedness)
  - Uniform, strong and weak topologies in $\mathcal{A}(H)$
  - Examples of operators (finite dim., creation, destruction, number, position, derivative)
  - Adjoint operator
  - Hermitian, self-adjoint, isometric, unitary, normal operator
  - Orthogonal projector
Spectral theory

- Definition: Spectrum and resolvent of linear operators

Let $A \in \mathcal{L}(H)$, with dense domain in $H$, separable Hilbert space over $\mathbb{C}$

- $\mathbb{C}$ can be split in the following subsets, depending on the behavior of the operator $(A - \lambda I)^{-1}$

$$\mathbb{C} = \rho \cup \sigma \equiv \rho \cup \sigma_p \cup \sigma_r \cup \sigma_c, \text{ disjoint in pairs}$$

<table>
<thead>
<tr>
<th>$\lambda \in \mathbb{C}$</th>
<th>$(A - \lambda I)^{-1}$</th>
<th>$R(A - \lambda I)$</th>
<th>$(A - \lambda I)^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_p(A)$</td>
<td>does not exist</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\sigma_r(A)$</td>
<td>exists</td>
<td>not dense in $H$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\sigma_c(A)$</td>
<td>exists</td>
<td>dense en $H$</td>
<td>not bounded in its domain</td>
</tr>
<tr>
<td>$\rho(A)$</td>
<td>exists</td>
<td>dense en $H$</td>
<td>bounded in its domain</td>
</tr>
</tbody>
</table>
Spectral theory

• Properties:

  • Eigenvectors and eigenvalues \( \lambda \in \sigma_p(A) \Leftrightarrow \exists v_\lambda \neq 0 \in D(A)/Av_\lambda = \lambda v_\lambda \)

  • Linear independence of eigenvectors with different eigenvalues

\[
\{\lambda_i\}_{1}^{n} \subset \sigma_p(A), \ Av_i = \lambda_i v_i, \ \lambda_i \neq \lambda_j \ (i \neq j) \Rightarrow \{v_i\} \text{ l.i.}
\]

  • Topological properties of the spectrum and resolvent

\[
\forall A \in \mathcal{L}(H) \Rightarrow \rho(A) \text{ open, } \sigma(A) \text{ closed in } \mathbb{R}^2
\]

  • Spectrum of the adjoint operator: let \( A \in \mathcal{A}(H) \)

\[
\begin{align*}
i) \ & \lambda \in \rho(A) \Leftrightarrow \overline{\lambda} \in \rho(A^\dagger) \\
ii) \ & \lambda \in \sigma_p(A) \Rightarrow \overline{\lambda} \in \sigma_p(A^\dagger) \cup \sigma_r(A^\dagger) \\
iii) \ & \lambda \in \sigma_r(A) \Rightarrow \overline{\lambda} \in \sigma_p(A^\dagger) \\
iv) \ & \lambda \in \sigma_c(A) \Leftrightarrow \overline{\lambda} \in \sigma_c(A^\dagger)
\end{align*}
\]
Spectral theory

• Properties:

  • Spectrum of normal operators: \( A \in \mathcal{A}(H) \) normal

    \begin{align*}
    a) & \quad Av = \lambda v \iff A^\dagger v = \overline{\lambda} v \\
    b) & \quad Av_i = \lambda_i v_i, \quad \lambda_i \neq \lambda_j \implies v_i \perp v_j \\
    c) & \quad \sigma_r(A) = \emptyset
    \end{align*}

  • Spectrum of unitary operators (are normal):

    \[ U \text{ unitary} \implies \sigma(U) = \sigma_p(U) \cup \sigma_c(U) \subset \{ \lambda/|\lambda| = 1 \} \]

  • Spectrum of isometric operators (not normal in general):

    \[ A \text{ isometric} \implies \sigma_p(A) \subset \{ \lambda/|\lambda| = 1 \}, \quad \left[ \text{in general} \sigma(A) \not\subset \{ |\lambda| = 1 \}, \sigma_r \neq \emptyset \right] \]
Spectral theory

- Properties:
  - Spectrum of orthogonal projectors:
    \[ \sigma(0) = \{0\}, \quad \sigma(1) = \{1\}, \text{ all other orthogonal projectors satisfy} \]
    \[ P \in \mathcal{A}(H), \quad P^2 = P = P^\dagger, \quad 0 \neq P \neq 1, \Rightarrow \sigma(P) = \sigma_p(P) = \{0, 1\} \]
  - Spectrum of self-adjoint operators: \(\text{sea } A \in \mathcal{A}(H) \text{ autoadjunto}\)
    1) \(\sigma(A) \subset \mathbb{R}, \quad \sigma_r(A) = \emptyset\)
    2) \(\sigma(A) \subset \left[ \inf_{||v||=1} \langle v, Av \rangle, \sup_{||v||=1} \langle v, Av \rangle \right]\)
    3) \(M_\lambda(A) = \{v \in H : Av = \lambda v\}\) closed linear subspace
    4) \(\forall A \in \mathcal{A}(H), \quad A = A^\dagger \Rightarrow \exists \{v_i\} \text{ orthonormal, maximal } / Av_i = \lambda_i v_i\)
      (not necessarily complete in \(H\))
Spectral theory

- **Definition:** \( A \in \mathcal{L}(H_1, H_2) \) is compact \((A \in \mathcal{C}(H_1, H_2))\)
  \[ \text{if } A(X) \text{ is compact in } H_1, \forall X \subset H_1, \text{ } X \text{ bounded } (\sup_{x \in X} \|x\| < \infty) \]
  
  - If \( \dim(H) < \infty \rightarrow \mathcal{L}(H) = \mathcal{A}(H) = \mathcal{C}(H) \)

- **Theorem:** \( \forall A \in \mathcal{C}(H) \)
  
  1) \( \sum_{\lambda \in \sigma_p / |\lambda| > k} \dim M_\lambda(A) < +\infty, \forall k > 0 \)
  
  2) \( \sigma_p(A) \) is at most numerable, with 0 as the only possible limit point
  
  3) \( \mathbb{C} - \{0\} \subset \sigma_p(A) \cup \rho(A) \)
  
  4) \( 0 \in \sigma(A) \)
  
  5) \( \sigma_r(A) \cup \sigma_c(A) \subset \{0\} \)