Operators

Ejercicio 1: Bounded operators

Let A be the folloing operator in $\mathbb{R}^2$ defined by the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -7 \end{pmatrix},$$

in an orthonormal basis. Is it bounded? Compute its norm if it is.

Ejercicio 2: Bounded operators

Let U be the following operator in $\mathbb{R}^3$ with a representative matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

in an orthonormal basis. Compute its norm.

Ejercicio 3: Bounded opertors

Let $H = L^2[a,b]$ and $A : H \to H$ the operator defined by

$$Af(x) = \cos(x) f(x).$$

For which values of a and b is A bounded? Compute its norm.

Ejercicio 4: Continuous and bounded operators

Let $H = \mathbb{R}$ and the following operators

i) $A(x) = x^2$

ii) $B(x) = \begin{cases} 1, & \text{if } |x| > 1 \\ 0, & \text{if } |x| \leq 1 \end{cases}$

Are they continuous? Are they bounded?

Ejercicio 5: Linear operators

Which of the following operators are linear?

i) $A(\alpha_1, \alpha_2, \ldots) = (\alpha_1^2, \alpha_2^2, \ldots) \in l^2_{\mathbb{C}}$.

ii) $A(\alpha_1, \ldots, \alpha_n) = (\alpha_1 + \alpha_2 + \ldots + \alpha_n, 0, \ldots, 0) \in \mathbb{C}^n$.

iii) $A(v) = v + v_0 \in \mathbb{R}^3$, with v_0 a non vanishing vector of $\mathbb{R}^3$.

iv) $A(f) = \chi_{[0,2]}(x)\chi_{[1,6]}(x)f(x) \in L^2$.

v) $A(f) = f(x+1) \in L^2(\mathbb{R})$.

Ejercicio 6: Commutator of operators

Compute the commutator of the following operators

$$\begin{aligned} A &: \quad f(x) \in \mathcal{S} \in L^2(\mathbb{R}) \to x f'(x), \\ B &: \quad f(x) \in \mathcal{S} \in L^2(\mathbb{R}) \to x f(x) + 3 f''(x), \end{aligned}$$

Ejercicio 7: Inverse operator

Check if the following operators are bounded in their domain, if the inverse operator exists and if it is bounded.

i) a, a^+, N in l^2_Λ.

ii) R rotation operator in $L^2(\mathbb{R}^3)$.

iii) $Q : f(x) \to x f(x)$ in $L^2[a,b]$.

iv) $Q : f(x) \to x f(x)$ in $L^2(\mathbb{R})$.

Ejercicio 8: Adjoint operator

Let $H = L^2[-1,1]$ and the operator

$$(Tf)(x) = \int_{-1}^{1} \mathrm{d}y \, (x^2 + y^3 + 3\mathrm{i}xy) f(y).$$

Show that it is a continuous linear operator and compute its adjoint.

Ejercicio 9: Parity operator

Let $H = L^2(\mathbb{R})$ and the parity operator defined by

$$(Pf)(x) = f(-x).$$

Show that it is a bounded linear operator and compute its norm. Is it a projector over any direction? Is it Unitary? Is it self-adjoint? Compute its spectrum.

Ejercicio 10: Operator spectrum

Consider a Hilbert space of dimension 2 and the linar operator T defined in an orthonormal basis by

$$T|e_1\rangle = |e_1\rangle, \qquad T|e_2\rangle = |e_1\rangle + |e_2\rangle.$$

Is it a bounded operator? Compute its norm if it is. Compute its adjoint operator and its spectrum.

Ejercicio 11: Adjoint operator

Consider the operator $A \in \mathcal{A}(l^2_{\mathbb{C}})$ defined by

$$Ax = A(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_{2n}, \ldots) = (0, \alpha_1, 0, \alpha_2, \ldots, \alpha_n, \ldots).$$

Compute:

i) $||A||$, $A^\dagger$, $A^{\dagger\dagger}$ y $||A^\dagger||$.

ii) Whether any of the following points $\lambda = 2$, $\lambda = 0$, $\lambda = \mathrm{i}/2$ y $\lambda = 1$ belong to the spectrum (and to which).