Linear Forms and distributions

Ejercicio 1: Continuous linear forms

Consider the space of real 2×2 matrices with scalar product

$$\langle A, B \rangle = \mathrm{Tr}[A^T B].$$

Find, if possible, the vector associated to the following functionals:

(i) $F(A) = \mathrm{Tr}[A]$,

(ii) $F(A) = a_{11}a_{12} + a_{21}a_{22}$.

Ejercicio 2: Continuous linear forms

Consider the Hilbert space $L^2(\mathbb{R})$. Discuss which of the following forms are linear and continuous and find the associated vector for those which are:

(a) $F_1(f) = \int_{-\pi}^{\pi} f \, \mathrm{dx}$.

(b) $F_2(f) = \int_0^1 f^2 \, \mathrm{dx}$.

(c) $F_3(f) = 1 + \int_{\mathbb{R}} |f| \, \mathrm{dx}$.

Ejercicio 3: Continuous linear forms

Consider the following functionals acting on $l^2_{\mathbb{R}}$

$$(i)\ F(\{a_n\}_1^\infty) = \sum_1^\infty \frac{a_n}{n}, \qquad (ii)\ F(\{a_n\}_1^\infty) = \sum_1^\infty a_{2n+1}.$$

Check if they are continuous and compute their norms when possible. Show that $M = \left\{\{a_n\}_1^\infty / \sum_{k=1}^\infty a_{2k-1} = 0\right\}$ is not closed.

Ejercicio 4: Continuous linear forms

Let $\mathcal{P} \in H = L^2[0, 2\pi]$ be the set of polynomials $P(x)$ in $[0, 2\pi]$. Provide an example, if it exists, of a continuous linear form F over H such that

$$(i)\ F(P) = 1,\ \forall P \in \mathcal{P}, \qquad (ii)\ F(P) = 0,\ \forall P \in \mathcal{P}.$$

Ejercicio 5: Bilinear forms

Find the operator A_j that represents each of the following continous bilinear forms, defined over $H = l^2_{\mathbb{C}}$. $v = \{\alpha_n\}$, $w = \{\beta_n\}$, in certain orthonormal basis $\{e_j\}$ of H.

$$(i)\ \phi_1(w, v) \equiv \bar{\beta}_2\alpha_1, \qquad (ii)\ \phi_2(w, v) = \sum_1^{10} \bar{\beta}_j\alpha_j, \qquad (iii)\ \phi_3(w, v) = \sum_1^\infty \frac{\bar{\beta}_j\alpha_{j+1}}{j}.$$

Ejercicio 6: Strong and weak convergence

Study the strong and weak convergence of the following sequences in l^2, $v_n = n e_n$ and $w_n = e_n / n$, with e_n the canonical basis.

Ejercicio 7: Distributions

Compute the limit, in the sense of distributions, of

$$\lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \exp(-x^2/\epsilon).$$

Ejercicio 8: Distributions

Compute the third derivative of $x^2\theta(x)/2$ in the sense of distributions.

Ejercicio 9: Distributions

How would you define the complex conjugation of a distribution?

Ejercicio 10: Distributions

Let $f(x)$ be a function that is discontinuous at $x = 0$ with a finite discontinuity given by $\sigma = f(0^+) - f(0^-)$ but continuous and differentiable everywhere else in $\mathbb{R}$. Let $\{f'\}$ be the usual derivative for $x \neq 0$ and 0 in $x = 0$. Show that the derivative in the sense of distributions satisfies

$$f' = \{f'\} + \sigma \delta_0.$$

Ejercicio 11: Linear forms (Problem to hand)

Let $H = l^2_{\mathbb{C}}$ and the following maps

$$A : H \to \mathbb{C}, \quad A(\{\alpha_n\}_{n=1}^{\infty}) = \alpha_1 \sum_{n=1}^{157} \alpha_{3n},$$

$$B : H \to \mathbb{C}, \quad B(\{\alpha_n\}_{n=1}^{\infty}) = \alpha_1 + \frac{1}{300}\alpha_{300} + \frac{1}{5}\alpha_{500},$$

$$C : H \times H \to \mathbb{C}, \quad C(\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \bar{\alpha}_{2n+1}\beta_{2n+1}.$$

Are A and B linear forms? If they are, are they continuous? Find the associated vector when possible. Is C a bounded bilinear (sesquilinear) form? Find the associated linear operator.