Espaces of functions

Ejercicio 1: Best approximation

Compute the distance of the function $f(x) = x^2$ to the subspace of $L^2[-1,1]$ generated by the functions $\{\sin(x), \cos(x)\}$.

Ejercicio 2: Best approximation

Find the best approximation to the function of $L^2[0,\pi]$ $\sin(x)$ within the subspace of polynomials of degree 1 or less. Get an estimate of the improvement of the approximation when degree 2 polynomials are included. Compare the best approximation with the Taylor expansion around $x = 0$ far from that point.

Ejercicio 3: Best approximation

Find the best approximation of $f(x) = \ln(1+x)$ by means of orthonormal polynomials up to second degree in the interval $[0,1]$. Compare with Taylor expansion up to x^2.

Ejercicio 4: Spaces of functions with scalar product

Let $(P_2, ||.||)$ be the space

$$P_2 = \{p(x) = a_0 + a_1 x + a_2 x^2, x \in [0,1], a_{0,1,2} \in \mathbb{C}\},$$

with norm $||p(x)|| = (|a_0|^2 + |a_1|^2 + |a_2|^2)^{1/2}$. Does it satisfy the parallelogram identity? If it does, compute the associated scalar product. Is that the usual scalar product in L^2? Why?

Ejercicio 5: Orthogonal projection in spaces of functions

Let $H = L^2[-1,1]$ and the following subset of H

$$M = [\{q_n\}_{n=1}^{\infty}],$$

where we have defined the vectors $q_n = n\bar{P}_n(x) - \bar{P}_{n-1}(x)$, with $\bar{P}_n$ the normalized Legrendre's polynomials. Is it a linear subspace?

(i) Find $M^\perp$.

(ii) Is there, for any $f \in H$ a sequence of vectors in M that converges to f?

(iii) Can we define the orthogonal projection over M for any function in H? How about over $M^\perp$?

Ejercicio 6: Best approximation

Let $M = [\{1, \ln x, (\ln x)^2\}] \subset L^2[0,1]$. Find the best approximation of x^2 in M and compute its distance to M.

Ejercicio 7: Fourier expansion

Find a Fourier expansion in sines and cosines that converges point-wise in $[-1,1]$ to the function $f(x) = x$.

Ejercicio 8: Fourier expansion

Find the Fourier expansion in sines in $[-\pi, \pi]$ of the function $\sin^3 x$.

Ejercicio 9: Fourier expansion

Use the Fourier expansion in sines and cosines of the function $f(x) = x^2$ to find the value of the following sums

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Ejercicio 10: Fourier expansion (problem to hand)

Find the Fourier expansion in sines and cosines for the function $f(x) = x^2$ in $L^2([-\pi, +\pi])$ and, using Parseval's identity, show that

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$