Linear, metric and normed spaces

Ejercicio 1: Direct Sum

Show that $\dim(M_1\vec{\oplus}M_1\vec{\oplus}\ldots\vec{\oplus}M_n) = \dim(M_1) + \dim(M_2) + \ldots \dim(M_n)$.

Ejercicio 2: Linear subspaces

Let $M \subset l^2_{\mathbb{C}}$. $M = \{v = (\alpha_1, \alpha_2, \ldots) \in l^2_{\mathbb{C}} / |\alpha_1| \geq |\alpha_2| \geq \ldots\}$. Is it a linear subspace of $l^2_{\mathbb{C}}$?

Ejercicio 3: Properties of the distance

Let d be a metric in X. For which values of the constant κ are κd y $\kappa + d$ also metrics?

Ejercicio 4: Properties of the distance

Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x,y) = |x^2 - y^2|$. Is it a distance? If it is not, is there a subset of $\mathbb{R}$ for which it is? Determine the largest of such subsets.

Ejercicio 5: Norm-distance relation

Let d be the Euclidean metric. Let $\hat{d} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ the mapping defined as:

$$\hat{d}(x,y) \equiv \begin{cases} d(x,y), & \text{si } y = \lambda x, \lambda \in \mathbb{R}, \\ d(x,0) + d(0,y), & \text{otherwise.} \end{cases}$$

Show that $\hat{d}$ is a distance and that it does not come from a norm.

Ejercicio 6: Norm-distance relation

Show that the mapping $\hat{d} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{d}(x,y) = \frac{|x-y|}{1+|x-y|}.$$

is a metric. Does it come from a norm?

Ejercicio 7: Properties of the metric

Let the mapping $\rho : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by:

$$\rho(x,y) = \begin{cases} d(x,O) + d(O,y), & x \neq y, \\ 0, & x = y, \end{cases}$$

where O is the origin of $\mathbb{R}^2$ and d is the Euclidean metric. Is it a metric in $\mathbb{R}^2$? Does it come from a norm? Show that every point in $\mathbb{R}^2$ except for O is open. Show also that every open ball centered at the origin, $B(O,r)$, is also closed.

Ejercicio 8: Sequences

Provide an example of a sequence in $\mathbb{R}$ that is Cauchy with some metric but it is not with the Euclidean metric.

Ejercicio 9: Sequences

Study the convergence of the following sequence $\{x_n\}_{n\in\mathbb{N}}$ in $l^1 = \{(\alpha_1, \alpha_2, \ldots) / \sum_n |\alpha_n| < \infty\}$, where

$$x_n = \left\{ \frac{1}{k^{\frac{n+1}{n}}} \right\}_{k=1}^{\infty} = \left(1, \frac{1}{2^{\frac{n+1}{n}}}, \frac{1}{3^{\frac{n+1}{n}}}, \ldots\right).$$

Ejercicio 10: Sequences

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a metric space X. Show that:

- If $x_n \to x$, then every sub-sequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x.
- If $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy and has any convergent sub-sequence, then $\{x_n\}_{n\in\mathbb{N}}$ converges to the same limit that the convergent sub-sequence.
- Provide an example of a non-convergent sequence that contains convergent sub-sequences.

Ejercicio 11: Closed sets in metric spaces

Show that if X is a metric space, $A \subset X$ and $r > 0$, then $V_r(A) = \{x \in X / d(x, A) \leq r\}$ is a closed set.

Ejercicio 12: Continuity of the metric

Show that if $v_n \to v$, $w_n \to w$ then we have $d(v_n, w_n) \to d(v, w)$.

Ejercicio 13: Closed balls in normed spaces

Show that in a normed space we have $\overline{B}(x, r) = \overline{B(x, r)}$. Is that true in general in a metric space?

Ejercicio 14: Closed subspaces in normed spaces

Let $(X, ||.||)$ be a normed space and S a subspace of X. Show that:

- $\bar{S}$ is a subspace of X.
- If X is complete, $\bar{S}$ is also complete.
- If $S \neq X$, int $S = \emptyset$.