Dimensional Reduction in Lovelock Gravity

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Abstract

In this work, the compatibility of Lovelock gravity and Kaluza-Klein dimensional reduction is studied. First, the Lovelock action is presented as a generalisation of the Einstein-Hilbert action for an arbitrary number of dimensions. This theory involves an action that consists of a sum of subsequently higher-order terms but it does not provide a numerical value for the coupling constants. In order to obtain them, the gauge principle is applied giving the value of the constants of the theory in D and $D-2$ dimensions. With this two actions, Kaluza-Klein dimensional reduction is applied over two dimensions and after comparing the coupling constants, they are found to be different. The dimensional reduction is later applied to an arbitrary number N of compact dimensions in order to find a particular case of compatibility without any successful result.

Chapter 1

Introduction

1.1 Background and Motivation

Einstein's general relativity is a successful theory of gravitation that has explained a wide range of phenomena since it was published in 1915. In this theory, the Einstein-Hilbert action, which is linear in terms of the curvature, yields second-order differential equations of motion of the metric. Although others terms of higher curvature can be added to the action, generally they involve higher-order derivatives of the metric and therefore they are excluded.

However, in 1938, Lanczos proposed a quadratic term of curvature which leads to second-order equations of motion as well and therefore is not to be excluded when considering a valid action for gravity. This term is known as Gauss-Bonnet term. It turns out that in a four dimensional world with three spatial directions and one temporal direction, the Gauss-Bonnet term is a topological term and therefore it does not contribute to the equations of motion. On the other hand, if the gravitational theory is made to live in a higher dimensional world, five or more, the Gauss-Bonnet term becomes dynamical and then it does modify the equations of motion.

In 1971, Lovelock extended the result of Lanczos to an arbitrary number of dimensions. In his theory, the action was built as a sum of subsequently higher-order curvature terms in a generic number of dimensions, each of them with a coupling constant. He was also able to give a description of which of the terms are dynamical in the equations of motion and which do not contribute to them relating the number of dimensions to the curvature order. A great success of this theory is to include the Einstein-Hilbert action as a four-dimensional particular case. The fact of being a solving, inclusive theory concedes to Lovelock gravity an important argument to be studied with further detail: its implications, predictions, compatibility with other well-established theories, etc. Regrettably, Lovelock gravity suffers from a major problem, it does not provide the values of the coupling constants of each term. A complementary theory must be used in order to obtain their numerical values and this theory it is found in the gauge formulation of gravity.

Previously in 1956, Utiyama applied the gauge principle to the Poincaré algebra and obtained successfully Einstein's general relativity. This is an impressive result because if Einstein had not discover general relativity geometrically, Utiyama would have done algebraically in a lapse of almost forty years. With a modification of the Poincaré algebra and extending the number of dimensions, the theory can provide the undetermined coupling constants that the Lovelock theory alone was not able to yield.

However, an unresolved question when applying the gauge principle in Lovelock theory is the uniqueness of the coupling constants for each term in different dimensions. Since their value depends on the dimension in which the theory lives, it is unclear whether it will coincide with another higher-dimensional theory in which *dimensional reduction* has been applied. In other words, if the coupling constants of the $(D-N)$-dimensional Lovelock action are computed using the gauge principle, will they coincide with the coupling constants of a D-dimensional Lovelock action in which N dimensions have been reduced?

The procedure of dimensional reduction is described in Kaluza-Klein theory. In 1919, Kaluza considered a pure gravitational theory in five dimensions and decomposed it in a four dimensional gravitational field coupled to electromagnetic and to a massless scalar field. In 1926, Klein solved some issues present in the theory of Kaluza. His major contribution was its consideration of the fifth dimension to be a *compact dimension.* With this approach they unified the gravitational and electromagnetic interaction. Even though the theory was discarded because its phenomenological implications could not meet the observations, it has inspired many unification theories due to its simplicity, elegance and mathematical consistence. This last attributes make the dimensional reduction suitable to compare the coupling constants of the two Lagrangians.

When applying dimensional reduction from a higher to a lower dimensional theory a consistent truncation can be made so that the theory keeps only the gravitational part allowing to compare two purely gravitational actions, namely, the Lovelock actions discussed above.

1.2 Objectives and Procedure

The objective of the present work is to provide an answer to the unresolved question posed in the previous section.

In order to do so, the Lovelock action is to be presented as the D-dimensional generalisation of the Einstein-Hilbert and Gauss-Bonnet actions. When the problem with the coupling coefficients is presented, the gauge formulation of the Einstein-Hilbert action is discussed from Poincaré algebra. Since gauging this algebra the cosmological constant is not included a modification is necessary. The solution is found gauging the (A)dS algebra, and doing so the zeroth-order and the second-order curvature term, i.e. the cosmological constant and the Gauss-Bonnet term, are included in the theory. Once this action is reproduced, a higher dimensional extension is constructed following the structure of the Lovelock gravity presented in the previous chapter using the Levi-Civita tensor.

Since this extension has its coupling constants determined, a D-dimensional and $(D-N)$-dimensional actions are considered. After applying dimensional reduction to the former, the two actions will have the same dimensions and then the coupling constants are compared. If they coincide, the uniqueness of the coupling constants is granted and they do not depend on the dimensions in which they are built.

Chapter 4

Kaluza-Klein Dimensional Reduction

In this chapter the uniqueness of the found coupling constants using the gauge principle is to be analysed through Kaluza-Klein dimensional reduction. The Kaluza-Klein theory describes a procedure to obtain a four dimensional gravitation theory coupled to electromagnetic field when applying dimensional reduction to a five dimensional gravitational theory. The existence of this fifth dimension was justified with the concept of compact, periodic dimension, an imperceptible coordinate with similar length to the Planck scale. In the present work this theory is to be used as a tool to check the uniqueness of the coupling constants for each term of the Lovelock Lagrangian. Instead of reducing from five dimensions to four dimensions, the D-dimensional Lovelock action is to be reduced to $D-N$ dimensions.

The structure of the present chapter is the following: First, one dimensional reduction is applied and the results are discussed and interpreted focussing on the gravitational part of the decomposition. Afterwards, dimensional reduction is again applied over another dimension so that the total number of reductions is even to fulfil the requirement mentioned in the previous chapter. Once the action has been reduced, it would be straight-forward to reduce a higher even number of dimensions, this is N times. Finally, the constants of the $(D-N)$-dimensional Lagrangian are compared with the ones fixed using the gauge theory.

A final comment is necessary. Since the radius of the compact dimension is similar to the Planck length, when working at scales much greater than it, all the fields lose their dependency on the compact coordinate ω_n. Because of this, the partial derivative of any field with respect to this coordinate is identically zero,

$$\partial_{\omega_n} = 0. \tag{4.1}$$

This limit is often referred to as the *low-energy limit*.

Regarding the notation of this chapter, from now on, D-dimensional objects will be written with a hat

$$\hat{x}^{\hat{\mu}}, \hat{g}_{\hat{\mu}\hat{\nu}}, \hat{R}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^{\hat{\sigma}}, \ldots$$

whereas $(D-N)$-dimensional objects without it

$$x^{\mu}, g_{\mu\nu}, R_{\mu\nu\rho}{}^{\sigma}, \ldots$$

with $\hat{\mu} = 0, 1, \ldots, D$ and $\mu = 0, 1, \ldots, D-N$. As mentioned above, the compact dimensions are noted as $\omega_n = \hat{x}_n^{D-n}$ with $n = 1, \ldots, N$. Every periodic dimension may have a different radius to each other, forming a N-torus, this is

$$\mathcal{M}^{1,D-1} = \mathcal{M}^{1,D-N-1} \times \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{N \text{ times}}.$$

4.1 One-Dimensional Compactification

The aim is to compactify the Lovelock action that results from Lagrangian (3.63), this is

$$\hat{S}_D = \int \mathrm{d}^D\hat{x}\sqrt{|\hat{g}_D|}\sum_{d=0}^{D/2}\hat{\lambda}_d\hat{\mathcal{R}}^d. \tag{4.2}$$

The D-dimensional metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ from which the square root determinant $\sqrt{|\hat{g}_D|}$ appears in the previous expression can be decomposed in a $(D-1)$-dimensional metric $\hat{g}_{\mu\nu}$, a vector $\hat{g}_{\mu\omega}$ and a scalar $\hat{g}_{\omega\omega}$. The usual ansatz for it is given by

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \mathrm{e}^{2\alpha\phi+2\Omega_0}g_{\mu\nu} - \gamma^2\mathrm{e}^{2\beta\phi}A_\mu A_\nu & -\gamma\mathrm{e}^{2\beta\phi} \\ -\gamma\mathrm{e}^{2\beta\phi} & -\mathrm{e}^{2\beta\phi} \end{pmatrix} \tag{4.3}$$

with α, β, γ and Ω_0 arbitrary constants that are usually determined conveniently[1]. Regarding the fields, the scalar field ϕ is called the dilaton and the electromagnetic potential A_μ. In the original work of Kaluza and Klein, the electromagnetic potential would yield the electromagnetic field when compactifying. However, as previously mentioned, the interest is to compare two purely gravitational theories, therefore A_μ is set to zero. Under this assumption, metric (4.3) is diagonal with the line element being

$$\mathrm{d}\hat{s}^2 = \mathrm{e}^{2\alpha\phi+2\Omega_0}g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu - \mathrm{e}^{2\beta\phi}\mathrm{d}\omega^2. \tag{4.4}$$

In order to simplify the calculations, the Vielbein of the metric are to be computed using expression (B.4), yielding

$$\hat{e}^a{}_\mu = \mathrm{e}^{\alpha\phi+\Omega_0}e^a{}_\mu, \tag{4.5a}$$
$$\hat{e}^a{}_\omega = \hat{e}^z{}_\mu = 0, \tag{4.5b}$$
$$\hat{e}^z{}_\omega = \mathrm{e}^{\beta\phi}, \tag{4.5c}$$

where the same notation for the compact coordinate applies this time for the flat index $\hat{x}^{\hat{a}} = (x^a, z)$. If the components of the inverse Vielbein are computed, they are found to be

$$\hat{e}^\mu{}_a = \mathrm{e}^{-\alpha\phi-\Omega_0}e^\mu{}_a, \tag{4.6a}$$
$$\hat{e}^\mu{}_z = \hat{e}^\omega{}_a = 0, \tag{4.6b}$$
$$\hat{e}^\omega{}_z = \mathrm{e}^{-\beta\phi}, \tag{4.6c}$$

where $e^a{}_\mu$ are the Vielbein of metric $g_{\mu\nu}$ and $e^\mu{}_a$ are, respectively, the Vielbein of the inverse metric $g^{\mu\nu}$. Now, with the Vielbein perfectly computed, it is really easy to express the volume element from (4.2) in terms of the $(D-1)$- dimensional metric as

$$\sqrt{|\hat{g}|} = \left|\hat{e}^{\hat{a}}{}_{\hat{\mu}}\right| = \mathrm{e}^{\beta\phi}\left|\mathrm{e}^{\alpha\phi+\Omega_0}e^a{}_\mu\right| = \mathrm{e}^{\beta\phi}\mathrm{e}^{(D-1)(\alpha\phi+\Omega_0)}\left|e^a{}_\mu\right| = \mathrm{e}^{[(D-1)\alpha+\beta]\phi+(D-1)\Omega_0}\sqrt{|g|}. \tag{4.7}$$

[1]Notice that, in this section, the compact dimension is noted as ω instead of ω_1 for simplicity.

Appendix A

Conventions

A.1 The Levi-Civita Symbol

The D-dimensional Levi-Civita symbol is a completely antisymmetric object, defined as

$$\varepsilon_{\mu_1 \dots \mu_D} = \begin{cases} 1 & \text{if } (\mu_1 \dots \mu_D) \text{ is an even permutation of } (0\ 1\ \dots\ D-1)\,, \\ -1 & \text{if } (\mu_1 \dots \mu_D) \text{ is an odd permutation of } (0\ 1\ \dots\ D-1)\,, \\ 0 & \text{otherwise,} \end{cases} \tag{A.1}$$

which is equivalent to

$$\varepsilon_{\mu_1 \dots \mu_D} = \delta^1_{[\mu_1} \cdots \delta^D_{\mu_D]}. \tag{A.2}$$

The Levi-Civita symbol is a fundamental symbol: it has these values in all coordinate systems. In order for this to be true, the Levi-Civita symbol has to transform as a pseudo-tensor density of weight $w = +1$ under general coordinate transformations

$$\varepsilon_{\alpha_1 \dots \alpha_D} = \operatorname{sgn}\left(\frac{\partial y}{\partial x}\right) \left|\frac{\partial y}{\partial x}\right| \prod_{i=1}^{D} \frac{\partial x^{\mu_i}}{\partial y^{\alpha_i}} \varepsilon_{\mu_1 \dots \mu_D}. \tag{A.3}$$

Using these transformation rules the invariant volume element is given by

$$\sqrt{|g|}\mathrm{d}^D x = \sqrt{|g|}\varepsilon_{\mu_1 \dots \mu_D} \prod_{i=1}^{D} \mathrm{d}x^{\mu_i}. \tag{A.4}$$

An alternating symbol with upper indices can also be defined as

$$\varepsilon^{\nu_1 \dots \nu_D} = \left(\prod_{i=1}^{D} g^{\mu_i \nu_i}\right) \varepsilon_{\mu_1 \dots \mu_D}, \tag{A.5}$$

such that

$$\varepsilon^{01 \dots D-1} = (-1)^{D-1} |g|^{-1}. \tag{A.6}$$

The contraction of two Levi-Civita symbols is given by

$$\varepsilon_{\mu_1 \dots \mu_d \nu_1 \dots \nu_{D-d}} \varepsilon^{\mu_1 \dots \mu_d \rho_1 \dots \rho_{D-d}} = (-1)^{D-1} |g|^{-1} d!\,(D-d)! \delta^{\rho_1}_{[\nu_1} \cdots \delta^{\rho_{D-d}}_{\nu_{D-d}]}, \tag{A.7}$$

from which yields two interesting cases, namely

$$\varepsilon_{\mu_1 \dots \mu_D} \varepsilon^{\mu_1 \dots \mu_D} = (-1)^{D-1} D!\, |g|^{-1}, \tag{A.8a}$$

$$\varepsilon_{\mu_1 \dots \mu_D} \varepsilon^{\nu_1 \dots \nu_D} = (-1)^{D-1} D!\, |g|^{-1} \delta^{\nu_1}_{[\mu_1} \cdots \delta^{\nu_D}_{\mu_D]}. \tag{A.8b}$$