Dimensional Reduction in Lovelock Gravity

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Abstract

In this work, the compatibility of Lovelock gravity and Kaluza-Klein dimensional reduction is studied. First, the Lovelock action is presented as a generalisation of the Einstein-Hilbert action for an arbitrary number of dimensions. This theory involves an action that consists of a sum of subsequently higher-order terms but it does not provide a numerical value for the coupling constants. In order to obtain them, the gauge principle is applied giving the value of the constants of the theory in $D$ and $D - 2$ dimensions. With this two actions, Kaluza-Klein dimensional reduction is applied over two dimensions and after comparing the coupling constants, they are found to be different. The dimensional reduction is later applied to an arbitrary number $N$ of compact dimensions in order to find a particular case of compatibility without any successful result.
Chapter 1

Introduction

1.1 Background and Motivation

Einstein’s general relativity is a successful theory of gravitation that has explained a wide range of phenomena since it was published in 1915. In this theory, the Einstein-Hilbert action, which is linear in terms of the curvature, yields second-order differential equations of motion of the metric. Although other terms of higher curvature can be added to the action, generally they involve higher-order derivatives of the metric and therefore they are excluded.

However, in 1938, Lanczos proposed a quadratic term of curvature which leads to second-order equations of motion as well and therefore is not to be excluded when considering a valid action for gravity. This term is known as Gauss-Bonnet term. It turns out that in a four dimensional world with three spatial directions and one temporal direction, the Gauss-Bonnet term is a topological term and therefore it does not contribute to the equations of motion. On the other hand, if the gravitational theory is made to live in a higher dimensional world, five or more, the Gauss-Bonnet term becomes dynamical and then it does modify the equations of motion.

In 1971, Lovelock extended the result of Lanczos to an arbitrary number of dimensions. In his theory, the action was built as a sum of subsequently higher-order curvature terms in a generic number of dimensions, each of them with a coupling constant. He was also able to give a description of which of the terms are dynamical in the equations of motion and which do not contribute to them relating the number of dimensions to the curvature order. A great success of this theory is to include the Einstein-Hilbert action as a four-dimensional particular case. The fact of being a solving, inclusive theory concedes to Lovelock gravity an important argument to be studied with further detail: its implications, predictions, compatibility with other well-established theories, etc. Regrettably, Lovelock gravity suffers from a major problem, it does not provide the values of the coupling constants of each term. A complementary theory must be used in order to obtain their numerical values and this theory it is found in the gauge formulation of gravity.

Previously in 1956, Utiyama applied the gauge principle to the Poincaré algebra and obtained successfully Einstein’s general relativity. This is an impressive result because if Einstein had not discover general relativity geometrically, Utiyama would have done algebraically in a lapse of almost forty years. With a modification of the Poincaré algebra and extending the number of dimensions, the theory can provide the undetermined coupling constants that the Lovelock the-
ory alone was not able to yield.

However, an unresolved question when applying the gauge principle in Lovelock theory is the uniqueness of the coupling constants for each term in different dimensions. Since their value depends on the dimension in which the theory lives, it is unclear whether it will coincide with another higher-dimensional theory in which dimensional reduction has been applied. In other words, if the coupling constants of the \((D - N)\)-dimensional Lovelock action are computed using the gauge principle, will they coincide with the coupling constants of a \(D\)-dimensional Lovelock action in which \(N\) dimensions have been reduced?

The procedure of dimensional reduction is described in Kaluza-Klein theory. In 1919, Kaluza considered a pure gravitational theory in five dimensions and decomposed it in a four dimensional gravitational field coupled to electromagnetic and to a massless scalar field. In 1926, Klein solved some issues present in the theory of Kaluza. His major contribution was its consideration of the fifth dimension to be a compact dimension. With this approach they unified the gravitational and electromagnetic interaction. Even though the theory was discarded because its phenomenological implications could not meet the observations, it has inspired many unification theories due to its simplicity, elegance and mathematical consistence. This last attributes make the dimensional reduction suitable to compare the coupling constants of the two Lagrangians.

When applying dimensional reduction from a higher to a lower dimensional theory a consistent truncation can be made so that the theory keeps only the gravitational part allowing to compare two purely gravitational actions, namely, the Lovelock actions discussed above.

1.2 Objectives and Procedure

The objective of the present work is to provide an answer to the unresolved question posed in the previous section.

In order to do so, the Lovelock action is to be presented as the \(D\)-dimensional generalisation of the Einstein-Hilbert and Gauss-Bonnet actions. When the problem with the coupling coefficients is presented, the gauge formulation of the Einstein-Hilbert action is discussed from Poincaré algebra. Since gauging this algebra the cosmological constant is not included a modification is necessary. The solution is found gauging the (A)dS algebra, and doing so the zeroth-order and the second-order curvature term, i.e. the cosmological constant and the Gauss-Bonnet term, are included in the theory. Once this action is reproduced, a higher dimensional extension is constructed following the structure of the Lovelock gravity presented in the previous chapter using the Levi-Civita tensor.

Since this extension has its coupling constants determined, a \(D\)-dimensional and \((D - N)\)-dimensional actions are considered. After applying dimensional reduction to the former, the two actions will have the same dimensions and then the coupling constants are compared. If they coincide, the uniqueness of the coupling constants is granted and they do not depend on the dimensions in which they are built.
Chapter 2

Lovelock Gravity

In this section, the Lovelock action is to be presented as the \(D\)-dimensional generalisation of the Einstein-Hilbert and Gauss-Bonnet actions. The reason of choosing these two is the familiarity with them so that the construction of Lovelock seems a natural extension.

2.1 Four-Dimensional Home Sweet Home

The action of general relativity is such that, through the principle of least action, yields the Einstein field equations of general relativity. Its expression is given by the sum of the Einstein-Hilbert Lagrangian \(L_{EH}\) plus a term \(L_m\) describing any matter field that appears in the theory,

\[
S_{GR} = \int d^4x \sqrt{|g|} \left( L_{EH} + L_m \right) = \int d^4x \sqrt{|g|} \left( \frac{R}{2\kappa} + L_m \right),
\]

(2.1)

where \(\kappa\) is the Newton’s constant, \(|g|\) is the absolute value of the determinant of the metric \(g_{\mu\nu}\) and \(R\) is the Ricci scalar\(^1\). The Ricci scalar is defined as the trace of the Ricci curvature tensor which in turn is defined as the trace of the Riemann tensor,

\[
R = R_{\mu}^{\mu} = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}g^{\rho\sigma}R_{\mu\rho\nu\sigma}.
\]

(2.2)

Besides, the Riemann tensor is written in terms of the Christoffel symbols as

\[
R_{\mu\rho\nu\sigma} = g_{\alpha\mu}R_{\rho\nu\sigma}^\alpha = g_{\alpha\mu} \left( \partial_{\nu}\Gamma_{\rho\sigma}^{\alpha} - \partial_{\sigma}\Gamma_{\rho\nu}^{\alpha} + \Gamma_{\nu\lambda}^{\alpha}\Gamma_{\rho\sigma}^{\lambda} - \Gamma_{\sigma\lambda}^{\alpha}\Gamma_{\rho\nu}^{\lambda} \right).
\]

(2.3)

and the Christoffel symbols are written in terms of the metric as

\[
\Gamma_{\rho\sigma}^{\alpha} = \frac{1}{2}g^{\alpha\beta} \left( \partial_{\rho}g_{\sigma\beta} + \partial_{\sigma}g_{\rho\beta} - \partial_{\beta}g_{\sigma\rho} \right).
\]

(2.4)

Observe that the Ricci scalar, as (2.3) and (2.4) indicate, includes terms with second-order derivatives of the metric

\[
R = R(g, \partial g, \partial^2 g).
\]

(2.5)

This would imply that the equations of motion incorporate higher than second-order derivatives of the metric. However, instead of such scenario, the variation of the action (2.1) yields, as

\(^1\)From now on the Ricci scalar will be also referred as the linear curvature term of a gravitational theory.
mentioned above, the Einstein field equations,

$$ R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = -\kappa \frac{2}{\sqrt{|g|}} \frac{\partial \left( \sqrt{|g|} \mathcal{L}_m \right)}{\partial g^{\mu\nu}}, $$

(2.6)

which are expressed only in terms of second-order derivatives of the metric. This is because when computing the variation, the higher than second-order terms end up vanishing each other. Equations (2.6) are usually written as

$$ G_{\mu\nu} = -\kappa T_{\mu\nu}, $$

(2.7)

where the two introduced tensors $G_{\mu\nu}$ and $T_{\mu\nu}$ are respectively the Einstein and stress-energy tensors. All these expressions look familiar, they are included in every lecture of general relativity.

The action (2.1) was modified shortly after published to include an additional term, the cosmological constant, making it to look like

$$ S_{\text{GRA}} = \int d^4x \sqrt{|g|} \left( \frac{R}{2\kappa} - \Lambda + \mathcal{L}_m \right), $$

(2.8)

making the Einstein field equations to be

$$ G_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}. $$

(2.9)

Despite the modification, action (2.1) still conserves the interesting property mentioned above and keeps yielding second-order differential equations of motion.

### 2.2 Going Out a Bit to Explore

Although actions with higher-order curvature terms yield in general equations of motion with higher than second-order derivatives, Lanczos discovered that the second-order curvature term $\alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ can yield valid equations of motion when the constants take the values $(\alpha, \beta, \gamma) = (1, -4, 1)$. This term is known as the Gauss-Bonnet term

$$ \mathcal{L}_{\text{GB}} = R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}. $$

(2.10)

When choosing these particular values for the constants $\alpha, \beta$ and $\gamma$, the higher than second-order derivative terms cancel each other in the same way as the action of general relativity (2.8). If this is the case, there is no reason to exclude the Gauss-Bonnet term from the action and therefore it should be borne in mind when computing the equations of motion. The interesting fact is that when the theory lives in four dimensions the Gauss-Bonnet term does not contribute to the equations of motion because it is a topological term. However, when considering a theory in five dimensions or higher, this term does become dynamical and therefore must be included in the action,

$$ S_5 = \int d^5x \sqrt{|g|} \left[ \frac{R}{2\kappa} - \Lambda + \mathcal{L}_m + \lambda_2 \left( R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right) \right]. $$

(2.11)

Notice that at this point the coupling constant $\lambda_2$ of the Gauss-Bonnet term remains unknown because of the lack of means to determine it. Later, a discussion about its determination will be
done. For the time being, a quick look to expression (2.11) allows to separate the terms in the Lagrangian for subsequently higher-order curvatures, from now, each term will be denoted $\mathcal{R}^d$, where $d$ indicates the order. Using this convention, the zero-curvature term can be identified as the cosmological constant, the Ricci scalar as the first order and the Gauss-Bonnet term as the second one, respectively,

$$
\begin{align*}
\mathcal{R}^0 &= \Lambda, \\
\mathcal{R}^1 &= R, \\
\mathcal{R}^2 &= R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}.
\end{align*}
$$

Expressions (2.12b) and (2.12c) can also be written in terms of only Riemann tensors and the antisymmetric product of Kronecker deltas as

$$
\begin{align*}
\mathcal{R}^1 &= \delta^{\alpha}_{[\mu} \delta^{\beta}_{\nu]} R_{\alpha\beta} = R, \\
\mathcal{R}^2 &= 6\delta^{\alpha}_{[\mu} \delta^{\beta}_{\nu} \delta^{\gamma}_{\rho} \delta^{\delta}_{\sigma]} R_{\alpha\beta}^{\mu\nu} R_{\gamma\delta}^{\rho\sigma} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma},
\end{align*}
$$

allowing to infer how the third order curvature term $\mathcal{R}^3$ is to be constructed as well as other higher curvature terms. It would be very convenient for the future if these relations are written in terms of contractions of Levi-Civita tensors instead of antisymmetric products of Kronecker. In appendix A.1, the relation between these two objects is discussed and using it, the two curvature terms yield

$$
\begin{align*}
\mathcal{R}^1 &= \frac{|g|}{4} \epsilon_{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} R_{\alpha\beta}^{\mu\nu} = R, \\
\mathcal{R}^2 &= -\frac{|g|}{4} \epsilon_{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} R_{\alpha\beta}^{\mu\nu} R_{\gamma\delta}^{\rho\sigma} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}.
\end{align*}
$$

Notice that the absolute value of the determinant in these two expressions comes from the contraction of Levi-Civita tensor.

## 2.3 You Came to the Wrong Neighborhood?

As commented before, it is very natural to try to extend the curvature terms when looking at (2.14a) and (2.14b). The Lovelock Lagrangian is constructed taking into account this property considering an arbitrary number terms of curvature,

$$
\mathcal{L}_L = \sum_{d=0}^{\lfloor D/2 \rfloor} \lambda_d \mathcal{R}^d,
$$

where $\lfloor D/2 \rfloor$ is the integer part of $D/2$ and $\lambda_d$ is the coupling constant\(^2\) of the curvature term $\mathcal{R}^d$, which according to the previous discussion takes the form

$$
\mathcal{R}^d = \frac{(-1)^{d-1}|g|}{2^d} \epsilon_{\mu_1\nu_1...\mu_{D/2}\nu_{D/2}} \epsilon^{\alpha_1\beta_1...\alpha_{D/2}\beta_{D/2}} \prod_{i=1}^{d} R_{\alpha_i\beta_i}^{\mu_i\nu_i}
$$

\(^2\)Notice that now makes sense to identify $\lambda_2$ as the coupling constant of the Gauss-Bonnet term since it is the term with second-order curvature ($d = 2$).
for \( d \geq 1 \) and identifying the cosmological constant term of zero curvature,

\[
R^0 = \Lambda.
\]  

(2.17)

In this Lagrangian, along with the zero-curvature term already discussed, the linear curvature term \( d = 1 \) which is completely identified with the Einstein-Hilbert term (2.1) as well as the second-order curvature term \( d = 2 \), which is the Gauss-Bonnet are present. Remember that for \( D = 4 \) Gauss-Bonnet term is only a topological term and therefore it does not contribute to the equations of motion. The higher-order curvature terms are identically zero and this can be checked from expression (2.16). If a term of order three is considered, the Levi-Civita must have six indices whereas there are only four different. Thus, any higher-order curvature vanishes.

The whole reasoning of the previous chapter can be extended as well for an arbitrary number of dimensions yielding a really interesting result. The Lagrangian proposed by Lovelock proved that if \( D \) dimensions are considered, there are exactly \( \lfloor D/2 \rfloor \) terms constructed with respectively higher-curvature tensors such that, when applying the minimal action principle, only equations of motion with second order derivatives are obtained. This means that each term of order \( d \) is unique and vanishes if \( D < 2d \), is a topological term if \( D = 2d \) and if \( D > 2d \) it contributes to the equations of motion.

This is an outstanding result because the Lovelock action is completely able to include the Einstein-Hilbert action plus the Gauss-Bonnet term explaining its properties. However, there is a problem with this theory. It is insufficient to construct the whole Lagrangian, since it does not provide the numerical values of the coefficients \( \lambda_d \), which in (2.15) represent the coupling constant of each term of the series.

In order to compute the value of the coefficients, a complementary theory must be used. Since a gravitational theory can be constructed as a gauge theory, Lagrangian (2.15) is to be reproduced following its principle fixing the numerical value of the undetermined coupling constants.
Chapter 3

Gravitation as a Gauge Theory

Once Lovelock gravity has been presented as the generalisation of Einstein-Hilbert Lagrangian, it is time to compute the coupling constants. In order to do so, the Lovelock Lagrangian is to be constructed as a gauge theory, but first it would be convenient to show that this is possible. In the present chapter, the Einstein-Hilbert action (2.1) is to be constructed applying the gauge principle to the Poincaré algebra in $D = 4$ dimensions. Afterwards, since gauging the Poincaré algebra does not include the cosmological constant term, it will be modified to include so, this modification is the (anti-)de Sitter algebra. When the Einstein-Hilbert action plus a cosmological constant is reproduced, the generalisation for $D$ dimensions is to be done following the discussion on the previous chapter.

3.1 Poincaré Gauge Theory

When a given Lagrangian is invariant under certain group of constants transformations, the theory is said to show a global symmetry. When the transformations are made to be local, the Lagrangian is no longer invariant. However, if new fields are introduced in the Lagrangian to cancel the extra terms, the invariance is restored. These fields introduce interactions to the original theory. If this is the case, from a matter field Lagrangian and a global symmetry, general relativity could be constructed with the procedure described. It turns out that it actually is because the matter fields living in a manifold are invariant under the Poincaré group.

3.1.1 Poincaré Algebra

First, a notion of the Poincaré transformations is to be presented. The Poincaré group $\text{ISO}(3,1)$ is the isometry group of the Minkowski space-time $\eta_{ab}$ and it consists on the semidirect product of the translations group $\text{T}^{1,3}$ and the Lorentz group $\text{SO}(3,1)$,

$$\text{ISO}(3,1) = \text{T}^{1,3} \rtimes \text{SO}(3,1).$$

An generic element of this group can be written as

$$x^b \mapsto x'^b = \Lambda^b_c x^c + a^b,$$

where $\Lambda^b_c$ is a Lorentz transformation and $a^b$ is an arbitrary constant vector. The infinitesimal version of this transformation is given by

$$x'^b = x^b + \xi^b (x^c) = x^b + \theta^b_c x^c + \epsilon^b,$$
where \( \epsilon^b \) are the infinitesimal translations, and the infinitesimal Lorentz transformations \( \theta^b_c \) are antisymmetric \( \theta^b_c = -\theta^c_b \). This relation is found when substituting the infinitesimal Lorentz transformation \( \Lambda^b_c = \delta^b_c + \theta^b_c + \ldots \) in the condition of leaving invariant the Minkowskian space-time,

\[
\eta_{ab} = \Lambda^c_a \Lambda^d_b \eta_{cd}. \tag{3.4}
\]

Because of this antisymmetry relation, in a four-dimensional space-time there are only six independent components (trace components are zero). Three of them are the rotations around the axis and the other three are boosts. Together with the four translations, the infinitesimal transformations are ten constant parameters.

The generators of this group are the operators that yield the transformations (3.3). For the rotations part, the generator \( M_{ab} \) must verify

\[
x^c = e^{\frac{i}{2} \theta^{ab} M_{ab}} x^c \approx \left( 1 + \frac{i}{2} \theta^{ab} M_{ab} \right) x^c = \theta^c_a x^a. \tag{3.5}
\]

This happens when

\[
M_{ab} = i (x_a \partial_b - x_b \partial_a). \tag{3.6}
\]

Now that the generators of the rotations are found, it is turn for the generators \( P_a \) of the translations. They must verify

\[
x^b = e^{i \epsilon^c P_c} x^b \approx (1 + i \epsilon^c P_c) x^b = x^b + \epsilon^b \tag{3.7}
\]

and this holds when

\[
P_a = -i \partial_a. \tag{3.8}
\]

The commutation relations between the generators of the Poincaré group form the Poincaré algebra

\[
[M_{ab}, M_{cd}] = i \eta_{ad} M_{bc} - i \eta_{bd} M_{ac} - i \eta_{ac} M_{bd} + i \eta_{bc} M_{ad}, \tag{3.9a}
\]

\[
[P_a, M_{bc}] = i \eta_{ab} P_c - i \eta_{ac} P_b, \tag{3.9b}
\]

\[
[P_a, P_b] = 0, \tag{3.9c}
\]

where, in order to obtain them, the relation \([\partial_a, x_b] = \eta_{ab}\) has been used.

### 3.1.2 Global Poincaré Symmetry

Once the Poincaré algebra has been defined, it is time to see if a matter Lagrangian is, indeed, invariant under the Poincaré group. Consider a Lagrangian depending on the coordinates \( x^\mu \) of a manifold \( \mathcal{M} \), the field \( \chi(x^\mu) \) and its first derivative term \( \partial_\mu \chi(x^\mu) \), this is

\[
\mathcal{L}_m = \mathcal{L}_m [\chi(x^\mu), \partial_\mu \chi(x^\mu), x^\mu]. \tag{3.10}
\]

The interest now is to see how does the Lagrangian change under infinitesimal variations of these terms.

A Poincaré transformation acts on the coordinates \( y^b \) of a tangent space \( T_P(\mathcal{M}) \) of the manifold exactly as (3.2), thus, infinitesimally

\[
y'^b = y^b + \xi^b(y^c) = y^b + \theta^b_c y^c + \epsilon^b. \tag{3.11}
\]
Notice that the notation has been changed from $x^b$ to $y^b$ in order to distinguish from the coordinates $x^\mu$ of the manifold. Regarding the field $\chi (x^\mu)$, it is infinitesimally transformed like

$$\chi (x^\mu) \rightarrow \chi' (x^\mu) = \chi (x^\mu) + \delta \chi (x^\mu),$$  \hspace{1cm} (3.12)

where

$$\delta \chi (x^\mu) = \left(-\frac{i}{2} \theta^{ab} M_{ab} + i \epsilon^a P_a\right) \chi (x^\mu) \equiv \mathcal{P} \chi (x^\mu),$$  \hspace{1cm} (3.13)

with $M_{ab}$ supposed to be expressed in its corresponding representation. Notice that in expression (3.13) the term $P_a \chi (x^\mu)$ vanishes because the operator is differentiating with respect to the tangent space coordinates whereas the field depends on the manifold coordinates, it is kept though, for future convenience. Regarding the variation of the terms which include a partial derivative $\partial_\mu \chi (x^\nu)$, they verify

$$\delta \left[ \partial_\mu \chi (x^\nu) \right] = \partial_\mu \left[ \delta \chi (x^\nu) \right] = \mathcal{P} \partial_\mu \chi (x^\nu).$$  \hspace{1cm} (3.14)

The total variation of Lagrangian (3.10) is given by

$$\bar{\delta} L_m = L'_m (x^\prime) - L_m (x) = \delta L_m + \xi^\mu \partial_\mu L_m + \partial_\mu \xi^\mu L_m = 0,$$  \hspace{1cm} (3.15)

where

$$\delta L_m = \frac{\partial L_m}{\partial \chi} \delta \chi + \frac{\partial L_m}{\partial (\partial_\mu \chi)} \delta (\partial_\mu \chi) = 0,$$  \hspace{1cm} (3.16)

are the Euler-Lagrange equations, which are assumed to hold. The second term of (3.15) is a total derivative term which does not modify the equations of motion and the last term vanishes as well when recalling the expression of $\xi^\mu$ from (3.3). This obtained result means that the Poincaré transformations are a global symmetry of the constructed Lagrangian. However, if now the parameters of the transformation are made to be local, this is no longer the case.

### 3.1.3 Local Poincaré Symmetry

When the Poincaré transformations are functions of the coordinates of the manifold,

$$y^b \mapsto y'^b = \Lambda^b_c (x^\mu) y^c + a^b (x^\mu),$$  \hspace{1cm} (3.17)

they are said to be local. Their local character changes the infinitesimal transformation of coordinates (3.11) to

$$y'^b = y^b + \theta^b_c (x^\mu) y^c + \epsilon^b (x^\mu).$$  \hspace{1cm} (3.18)

When this is the case, invariance (3.15) is violated. Even though the transformation rule presented in (3.13) of the field $\chi (x^\mu)$ remains unchanged,

$$\delta \chi (x^\mu) = \left[-\frac{i}{2} \theta^{ab} (x^\mu) M_{ab} + i \epsilon^a P_a\right] \chi (x^\mu) = \mathcal{P} (x^\mu) \chi (x^\mu),$$  \hspace{1cm} (3.19)

the transformation of the derivative of the field $\partial_\mu \chi (x^\nu)$ is different to the previous one (3.14) arising a new term,

$$\delta \left[ \partial_\mu \chi (x^\nu) \right] = \partial_\mu \left[ \delta \chi (x^\nu) \right] = \partial_\mu \mathcal{P} (x^\nu) \chi (x^\nu) + \mathcal{P} (x^\nu) \partial_\mu \chi (x^\nu).$$  \hspace{1cm} (3.20)
This term \( \partial_\mu \mathcal{P}(x^\nu) \) breaks the symmetry that the theory enjoyed when the parameters were global. However, if a new derivative, called *gauge covariant derivative*, is defined as

\[
\mathcal{D}_\mu \chi(x^\nu) = (\partial_\mu - A_\mu) \chi(x^\nu),
\]

where \( A_\mu \) are some introduced gauge fields of the form

\[
A_\mu(x^\nu) = i \frac{1}{2} \omega_{\mu}^{\ ab}(x^\nu) M_{ab} - i e_{\mu}^{\ a}(x^\nu) P_a,
\]

the transformation rule (3.20) recovers the expression of the global symmetry case,

\[
\delta [\mathcal{D}_\mu \chi(x^\nu)] = \mathcal{P}(x^\nu) \mathcal{D}_\mu \chi(x^\nu).
\]

Therefore, a Lagrangian which depends on this covariant derivative instead of the previous regular derivative,

\[
\mathcal{L}_m = \mathcal{L}_m [\chi(x^\mu), \mathcal{D}_\mu \chi(x^\mu), x^\mu],
\]

is again invariant under the localised Poincaré transformations. Regarding the variation of the fields \( A_\mu \), they are found to be

\[
\delta A_\mu = -i \mathcal{D}_\mu \theta_{ab} M_{ab} + i \left( \mathcal{D}_\mu e_{\mu}^{\ b} + \theta_{\mu}^{\ a} e_{\mu}^{\ a} P_b \right),
\]

where

\[
\mathcal{D}_\mu \theta_{ab} = \partial_\mu \theta_{ab} + \omega_{\mu c}^{\ a} \theta_{cb} + \omega_{\mu c}^{\ a} \theta_{ac},
\]

\[
\mathcal{D}_\mu e_{\mu}^{\ b} = \partial_\mu e_{\mu}^{\ b} + \omega_{\mu e}^{\ a} e_{\mu}^{\ a}.
\]

It is important to notice that the defined gauge fields \( \omega_{\mu a}^{\ b} \) and \( e_{\mu}^{\ a} \) must be taken into account in the total Lagrangian \( \mathcal{L}_T \) of the theory as they will be part of an additional kinetic term,

\[
\mathcal{L}_T = \mathcal{L}_k + \mathcal{L}_m.
\]

The term \( \mathcal{L}_k \) is given by the field strengths which are computed with the commutator of two covariant derivatives, this is

\[
[\mathcal{D}_\mu, \mathcal{D}_\nu] \chi(x^\rho) = \left[ \frac{i}{2} R_{\mu \nu}^{\ ab} M_{ab} - i R_{\mu \nu}^{\ a} P_a \right] \chi(x^\rho),
\]

where the quantities \( R_{\mu \nu}^{\ ab} \) and \( R_{\mu \nu}^{\ a} \) have been defined as

\[
R_{\mu \nu}^{\ ab} = \partial_\mu \omega_{\nu}^{\ ab} - \partial_\nu \omega_{\mu}^{\ ab} + \omega_{\mu c}^{\ a} \omega_{\nu c}^{\ b} - \omega_{\nu c}^{\ a} \omega_{\mu c}^{\ b},
\]

\[
R_{\mu \nu}^{\ a} = \mathcal{D}_\mu e_{\nu}^{\ a} - \mathcal{D}_\nu e_{\mu}^{\ a}.
\]
proportional to the Einstein-Hilbert Lagrangian.

In order to check that the gauge fields are indeed the geometric fields mentioned, the transformation rule of the introduced gauge fields $\omega_{\mu}^{ab}$ is computed. It is particularly easily to do so since it coincides with the covariant derivative of the infinitesimal Lorentz transformation (3.26), this is,

$$\delta \omega_{\mu}^{ab} = \partial_{\mu} \theta^{ab} + \omega_{\mu c}^{a} \theta^{cb} + \omega_{\mu c}^{b} \theta^{ac}. \quad (3.30)$$

This expression can be compared with the transformation of the spin connection from (B.16). It turns out that they are exactly the same and therefore they can be treated as the same tensor. If this is the case, there are strong indications to think that the gauge field $e_{\mu}^{a}$ is actually the Vielbein and assuming this equivalence (3.29b) vanishes when it is written in terms of the covariant derivative $D_{\mu}$, this is,

$$R_{\mu\nu}^{a} = D_{\mu} e_{\nu}^{a} - D_{\nu} e_{\mu}^{a} = 0. \quad (3.31)$$

In order to prove this condition, the existence of inverse gauge fields $e_{\mu}^{a}$ are required and they must verify

$$e_{\mu}^{a} e_{\mu}^{b} = \delta_{a}^{b}, \quad (3.32a)$$

$$e_{\mu}^{a} e_{\nu}^{c} = \delta_{\nu}^{a} \quad (3.32b)$$

similarly to the Vielbein. When this condition is fulfilled, the relation

$$\Omega_{ab}^{c} + \omega_{ab}^{c} - \omega_{ba}^{c} = 0 \quad (3.33)$$

is found, where $\omega_{ab}^{c} = e_{\mu}^{a} \omega_{\mu b}^{c}$ and the new coefficients have been defined as

$$\Omega_{ab}^{c} = e_{\mu}^{a} e_{\nu}^{c} \left( \partial_{\mu} e_{\nu}^{c} - \partial_{\nu} e_{\mu}^{c} \right). \quad (3.34)$$

Combining (3.33) and (3.34), the spin connection can be expressed in terms of the fields $e_{\mu}^{a}$ in the same manner as (B.28), where the $\Omega_{ab}^{c}$ are the coefficients of anholonomy. In the light of the results, it is natural to identify the field $e_{\mu}^{a}$ as the Vielbein.

This is a remarkable result that allows to identify the nature of the field strength $R_{\mu\nu}^{ab}$ as the curvature tensor. This means that the total Lagrangian $\mathcal{L}_{T}$ from (3.27) is actually

$$\mathcal{L}_{T} = \lambda_{ab}^{\mu\nu} R_{\mu\nu}^{ab} + \mathcal{L}_{m}, \quad (3.35)$$

where the factors $\lambda_{ab}^{\mu\nu}$ represent the coupling constant of the Lagrangian and a tensorial quantity so that the product with the field strength is a scalar. When choosing

$$\lambda_{ab}^{\mu\nu} = \frac{1}{2\kappa} e_{\mu}^{a} e_{\nu}^{b}, \quad (3.36)$$

the first term of Lagrangian (3.35) is completely identified to the Einstein-Hilbert Lagrangian plus the matter field term (2.1),

$$\mathcal{L}_{T} \equiv \mathcal{L}_{GR}, \quad (3.37)$$

yielding the correct action from the previous chapter,

$$S_{GR} = \int d^{4}x \sqrt{|g|} \left( \frac{1}{2\kappa} e_{\mu}^{a} e_{\nu}^{b} R_{\mu\nu}^{ab} + \mathcal{L}_{m} \right). \quad (3.38)$$
With this derivation, it has been shown that general relativity can be successfully reproduced using the gauge principle. However, as it may be noticed in action (3.38), the cosmological constant is absent. In order to include it another algebra different from (3.9) must be considered, this is the (anti-)de Sitter algebra.

3.2 The Quest for the Cosmological Constant in (A)dS

The (anti-)de Sitter (also written (A)dS) algebra is quite similar to the Poincaré algebra. The first two commutators \([M_{ab}, M_{cd}]\) and \([M_{ab}, P_c]\) remain exactly the same whereas the last one (3.9c) is modified to be

\[
[P_a, P_b] = \frac{i\epsilon}{L^2} M_{ab}, \tag{3.39}
\]

The parameter \(\epsilon\) in this expression can take the values \(\epsilon = \pm 1\) differentiating between (A)dS and therefore the signature of the cosmological constant. Regarding the other introduced parameter \(L\), it is the radius of the (A)dS space. Notice that in the limit when the radius tends to infinity,

\[
\lim_{x \to \infty} [P_a, P_b] = 0, \tag{3.40}
\]

the whole algebra of Poincaré (3.9) is recovered.

The (A)dS space is invariant under the symmetry group \(\text{SO}(1,4)\) (or \(\text{SO}(2,3)\) respectively). These groups are really similar to the Lorentz group \(\text{SO}(1,3)\) with the only difference of an extra spatial (or temporal) direction. Because of this similarity, in analogy to the Lorentz algebra the (A)dS algebra can be written as

\[
\tilde{\left[M_{\tilde{a}\tilde{b}}, M_{\tilde{c}\tilde{d}}\right]} = i\tilde{\eta}_{\tilde{a}\tilde{d}}\tilde{M}_{\tilde{b}\tilde{c}} - i\tilde{\eta}_{\tilde{b}\tilde{d}}\tilde{M}_{\tilde{a}\tilde{c}} - i\tilde{\eta}_{\tilde{a}\tilde{c}}\tilde{M}_{\tilde{b}\tilde{d}} + i\tilde{\eta}_{\tilde{b}\tilde{c}}\tilde{M}_{\tilde{a}\tilde{d}}. \tag{3.41}
\]

In this expression, the metric \(\tilde{\eta}_{\tilde{a}\tilde{b}} = \text{diag}(1, -\epsilon, -1, -1, -1)\) has the extra direction of the space group, with the value of \(\epsilon\) determining its spatial or temporal character. Using the expression (3.41) the generators are to be expressed as

\[
\tilde{M}_{ab} = M_{ab}, \tag{3.42a}
\]

\[
\tilde{M}_{a\star} = LP_a, \tag{3.42b}
\]

and along with \(\tilde{\eta}_{a\star} = -\epsilon\), algebra (3.41) reduces to (3.9). With the only difference of the last commutator being different from zero, the procedure from section 3.1 can be re-written taken it into account. This time the variation of the generic field has the form

\[
\delta \chi = -\frac{i}{2} \tilde{\theta}^{\tilde{a}\tilde{b}} \tilde{M}_{\tilde{a}\tilde{b}} \chi = -\frac{i}{2} \tilde{\theta}^{ab} M_{ab} - i\tilde{\theta}^{a\star} M_{a\star} = -\frac{i}{2} \theta^{ab} M_{ab} - i\omega^a P_a, \tag{3.43}
\]

where the identification

\[
\tilde{\theta}^{ab} = \theta^{ab}, \tag{3.44a}
\]

\[
\tilde{\theta}^{a\star} = \omega^a, \tag{3.44b}
\]

holds. The gauge fields \(\tilde{\omega}_{\mu} \tilde{a}^{\mu}\), can also be decomposed into

\[
\tilde{\omega}_{\mu}^{ab} = \omega_{\mu}^{ab}, \tag{3.45a}
\]

\[
\tilde{\omega}_{\mu}^{a\star} = -\frac{\epsilon^a}{L}, \tag{3.45b}
\]
which yields the same gauge covariant derivative as in the Poincaré case

\[ \tilde{D}_\mu \chi = \partial_\mu \chi + \frac{i}{2} \tilde{\omega}_\mu \tilde{a}^b \tilde{M}_{ab} \chi = \partial_\mu \chi + \frac{i}{2} \tilde{\omega}_\mu \tilde{a}^b \tilde{M}_{ab} \chi - ie_\mu P_a \chi. \]  (3.46)

Notice that now the variation of the fields changes with respect to Poincaré because the translations do not commute, the variation of the gauge field,

\[ \delta \tilde{\omega}_{\mu} = \tilde{D}_\mu \tilde{\theta}^b_{\mu}, \]  (3.47)

now is decomposed into

\[ \delta \omega_{\mu}^{ab} = D_\mu \theta_{\mu}^{ab} - \frac{\epsilon}{L^2} \left( e^a_\mu \theta^{b}_\mu - e^b_\mu \theta^{a}_\mu \right), \]  (3.48a)

\[ \delta e^a_\mu = D_\mu a^a + \theta^a_\mu e^c_\mu. \]  (3.48b)

Likewise, the curvature tensors are different to the Poincaré case,

\[ \tilde{R}_{\mu\nu}^{ab} = \partial_\mu \tilde{\omega}_{\nu}^{ab} - \partial_\nu \tilde{\omega}_{\mu}^{ab} - \tilde{\omega}_{\mu} \tilde{\omega}_{\nu}^{\cdot b} \tilde{\omega}^{\cdot b}_{\mu} + \tilde{\omega}_{\mu} \tilde{\omega}_{\nu}^{\cdot b} \tilde{\omega}^{\cdot b}_{\mu}, \]  (3.49)

yielding the decomposition

\[ \tilde{R}_{\mu\nu}^{ab} = R_{\mu\nu}^{ab} - \frac{\epsilon}{L^2} \left( e^a_\mu \theta^{b}_\nu - e^b_\nu \theta^{a}_\mu \right), \]  (3.50a)

\[ \tilde{R}_{\mu\nu}^{a*} = - \frac{R_{\mu\nu}^{a}}{L}, \]  (3.50b)

where \( R_{\mu\nu}^{ab} \) and \( R_{\mu\nu}^{a} \) are the same expressions from (3.29).

In order to construct the Einstein-Hilbert Lagrangian plus the cosmological constant term, the discussion from section 2.2 is to be recalled. The Levi-Civita tensor can be used to construct the different terms of the Lagrangian when contracted with a quadratic term of curvature

\[ L = N \varepsilon^{\mu\nu\rho\sigma} \tilde{\varepsilon}_{abcd} \tilde{R}_{\mu\nu}^{ab} \tilde{R}_{\rho\sigma}^{cd} \tilde{V}^{c}. \]  (3.51)

In this expression, \( \tilde{V}^c = \delta^c_\mu \) is a vector that breaks the symmetry SO(1,4) (SO(2,3)). The normalisation constant \( N \) must be chosen in such a way that the term with first-order curvature has the same value for the coupling constant as the Einstein-Hilbert-Lagrangian. If the four- and five-dimensional Levi-Civita tensors are related as

\[ \tilde{\varepsilon}_{abcd*} = \varepsilon_{abcd}, \]  (3.52)

using the contractions between them specified in appendix A.1, Lagrangian (3.51) yields

\[ L = |g|^{-1} \left[ \frac{R}{2\kappa} + \frac{3\epsilon}{L^2 \kappa} + \frac{\epsilon L^2}{8\kappa} \left( R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right) \right], \]  (3.53)

when the normalisation constant has been chosen as

\[ N = \frac{\epsilon L^2}{32\kappa}. \]  (3.54)
When doing so, the coupling constants of the different terms are fixed both for the zero-curvature term,
\[ \Lambda = -\frac{3\epsilon}{L^2\kappa}, \] (3.55)
which is actually the cosmological constant, and for the Gauss-Bonnet term. Notice that in spite of fixing the coupling constant of the Gauss-Bonnet term, since a \( D = 4 \) space-time is being considered it is a topological invariant as previously discussed.

Now that the Einstein-Hilbert Lagrangian with cosmological constant and the Gauss-Bonnet term has been successfully constructed, following the reasoning of Lovelock a \( D \)-dimensional Lagrangian could be constructed as well.

### 3.3 Expanding the Brain

The \( D \)-dimensional extension of Lagrangian (3.53) is such that resembles its structure and includes \( D/2 \) curvature terms contracted with two \( D \)-dimensional Levi-Civita tensors, according to the discussion of the previous chapter this looks like
\[ \mathcal{L}_D = N_D \varepsilon^{\mu_1\nu_1...\mu_{D/2}\nu_{D/2}} \tilde{a}_1 b_1...\tilde{a}_{D/2} b_{D/2} \tilde{c} \prod_{i=1}^{D/2} \tilde{R}^{\mu_i}_{\nu_i} \tilde{a}_i b_i \tilde{V}^{\tilde{c}}, \] (3.56)
where \( N_D \) is the normalisation constant of the Lagrangian for \( D \) dimensions, \( \varepsilon^{\mu_1\nu_1...\mu_{D/2}\nu_{D/2}} \) and \( \tilde{\varepsilon}^{a_1 b_1...a_{D/2} b_{D/2}} \) are the \( D \)- and \( (D + 1) \)-dimensional Levi-Civita tensors, related as
\[ \varepsilon_{a_1 b_1...a_{D/2} b_{D/2}} = \tilde{\varepsilon}^{a_1 b_1...a_{D/2} b_{D/2}} \] (3.57)
analogously to the lower dimensional case (3.52). Notice that if a generic number of dimensions is considered, the commutator (3.39) also changes to
\[ [P_a, P_b] = i\epsilon L_D^2 M_{ab}, \] (3.58)
where \( L_D \) is the (A)dS space radius in the new dimensions but the structure of the algebra remains the same so the description of the previous section still holds.

Lagrangian (3.56) can only be constructed when the number of dimensions \( D \) is an even number because for each curvature tensor introduced, two more indices are added to the Levi-Civita tensor. This is a key feature whose implications will be commented afterwards. For the time being, when the Lagrangian is expanded using the term (3.50a), the following expression is obtained,
\[ \mathcal{L}_D = N_D \sum_{d=0}^{D/2} \binom{D/2}{d} \left( \frac{-2\epsilon}{L_D^2} \right)^{D/2-d} \varepsilon^{\mu_1\nu_1...\mu_d\nu_d\lambda_1...\lambda_{D/2-d}} \tilde{a}_1 b_1...a_{d} b_{d} \lambda_1...\lambda_{D/2-d} \prod_{i=1}^{d} \tilde{R}^{\mu_i}_{\nu_i} a_i b_i. \] (3.59)
This expression can be shortened using the definition of curvature term (2.16) along with the contraction of Levi-Civita tensors to look like
\[ \mathcal{L}_D = -|g_D|^{-1} N_D \sum_{d=0}^{D/2} \binom{D/2}{d} \left( \frac{-\epsilon}{L_D^2} \right)^{D/2-d} (D-2d)! \frac{2^{D/2} \Gamma(D/2-d)}{\Gamma(2(D/2-d))} \mathcal{R}^d. \] (3.60)
Analogously to the previous section, the normalisation $N_D$ is determined with the condition of having the same coupling constant for the first order curvature term (this is $d = 1$) regardless of the dimension considered. This imposes the relation

$$-N_D \left( \frac{D/2}{1} \right) 2^{D/2} (-\epsilon)^{D/2-1}(D-2)! \frac{R}{L_D^{D-2}} = \frac{R}{2\kappa_D}, \quad (3.61)$$

which re-arranging the terms yields

$$N_D = \frac{(-1)^{D/2} \epsilon^{D/2-1} L_D^{D-2}}{2^{D/2} D(D-2)! \kappa_D}. \quad (3.62)$$

Notice that the Newton constant $\kappa_D$ in expression (3.61) is also different from the previous constant in four dimensions and it will be taken into account. Introducing the obtained value of the normalisation constant in the previous expression (3.59), the obtained Lagrangian is

$$\mathcal{L}_D = |g_D|^{-1} \sum_{d=0}^{D/2} \left( \frac{D/2}{d} \right) \frac{(-\epsilon)^{d+1}(D-2d)! L_D^{2(d-1)}}{D(D-2)! \kappa_D} R^d, \quad (3.63)$$

where the coupling constants are completely identified using again expression (2.15) as

$$\lambda_d = \left( \frac{D/2}{d} \right) \frac{(-\epsilon)^{d+1}(D-2d)! L_D^{2(d-1)}}{D(D-2)! \kappa_D}. \quad (3.64)$$

As usual, the term $d = 0$ yields the cosmological constant, which obviously changes its value depending on the dimension considered

$$\Lambda_D \equiv \lambda_0 = -\frac{(D-1)\epsilon}{L_D^2 \kappa_D}. \quad (3.65)$$

Now that the undetermined coupling constant from the previous chapter have been computed using the gauge principle, it is important to remark a difference between Lagrangian (3.63) and the one indicated in (2.15). Expression (2.15) is able to produce a Lagrangian for even and odd dimensions whereas (3.63) is constrained to an even dimension. Since the question of this work is whether the coupling constants match when applying dimensional reduction, the procedure of the next chapter is conditioned by this property. The only possibility to compare the constants is to fix the values of $\lambda_D^D$ and $\lambda_D^{(D-N)}$ (with $N$ an even number) of a $D$-dimensional Lagrangian and $(D - N)$-dimensional Lagrangian respectively.
Chapter 4

Kaluza-Klein Dimensional Reduction

In this chapter the uniqueness of the found coupling constants using the gauge principle is to be analysed through Kaluza-Klein dimensional reduction. The Kaluza-Klein theory describes a procedure to obtain a four dimensional gravitation theory coupled to electromagnetic field when applying dimensional reduction to a five dimensional gravitational theory. The existence of this fifth dimension was justified with the concept of compact, periodic dimension, an imperceptible coordinate with similar length to the Planck scale. In the present work this theory is to be used as a tool to check the uniqueness of the coupling constants for each term of the Lovelock Lagrangian. Instead of reducing from five dimensions to four dimensions, the $D$-dimensional Lovelock action is to be reduced to $D - N$ dimensions.

The structure of the present chapter is the following: First, one dimensional reduction is applied and the results are discussed and interpreted focusing on the gravitational part of the decomposition. Afterwards, dimensional reduction is again applied over another dimension so that the total number of reductions is even to fulfil the requirement mentioned in the previous chapter. Once the action has been reduced, it would be straightforward to reduce a higher even number of dimensions, this is $N$ times. Finally, the constants of the $(D - N)$-dimensional Lovelock action is to be reduced to $D - N$ dimensions.

A final comment is necessary. Since the radius of the compact dimension is similar to the Planck length, when working at scales much greater than it, all the fields lose their dependency on the compact coordinate $\omega_n$. Because of this, the partial derivative of any field with respect to this coordinate is identically zero,

$$\partial_{\omega_n} = 0.$$  \hspace{1cm} (4.1)

This limit is often referred to as the low-energy limit.

Regarding the notation of this chapter, from now on, $D$-dimensional objects will be written with a hat

$$\hat{x}^{\hat{\mu}}, \hat{g}_{\hat{\mu}\hat{\nu}}, \hat{R}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}, \ldots$$

whereas $(D - N)$-dimensional objects without it

$$x^{\mu}, g_{\mu\nu}, R_{\mu\nu\rho}, \ldots$$

with $\hat{\mu} = 0, 1, \ldots, D$ and $\mu = 0, 1, \ldots, D - N$. As mentioned above, the compact dimensions are noted as $\omega_n = \hat{x}_n^{D-n}$ with $n = 1, \ldots, N$. Every periodic dimension may have a different radius
to each other, forming a $N$-torus, this is
\[ M^{1,D-1} = M^{1,D-N-1} \times S^1 \times \cdots \times S^1. \]

### 4.1 One-Dimensional Compactification

The aim is to compactify the Lovelock action that results from Lagrangian (3.63), this is
\[ \hat{S}_D = \int d^D \hat{x} \sqrt{|\hat{g}_D|} \sum_{d=0}^{D/2} \hat{\lambda}_d \hat{R}^d. \] (4.2)

The $D$-dimensional metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ from which the square root determinant $\sqrt{|\hat{g}_D|}$ appears in the previous expression can be decomposed in a $(D-1)$-dimensional metric $\hat{g}_{\mu\nu}$, a vector $\hat{g}_{\mu\omega}$ and a scalar $\hat{g}_{\omega\omega}$. The usual ansatz for it is given by
\[ \hat{g}_{\hat{\mu}\hat{\nu}} = (e^{2\alpha \phi + 2\Omega_0 g_{\mu\nu}} - \gamma e^{2\beta \phi} - \gamma e^{2\beta \phi}) \] (4.3)

with $\alpha, \beta, \gamma$ and $\Omega_0$ arbitrary constants that are usually determined conveniently\(^1\). Regarding the fields, the scalar field $\phi$ is called the dilaton and the electromagnetic potential $A_{\mu}$. In the original work of Kaluza and Klein, the electromagnetic potential would yield the electromagnetic field when compactifying. However, as previously mentioned, the interest is to compare two purely gravitational theories, therefore $A_{\mu}$ is set to zero. Under this assumption, metric (4.3) is diagonal with the line element being
\[ ds^2 = e^{2\alpha \phi + 2\Omega_0} g_{\mu\nu} dx^\mu dx^\nu - e^{2\beta \phi} d\omega^2. \] (4.4)

In order to simplify the calculations, the Vielbein of the metric are to be computed using expression (B.4), yielding
\[ \hat{e}^a_{\hat{\mu}} = e^{\alpha \phi + \Omega_0} e^a_{\mu}, \] (4.5a)
\[ \hat{e}^z_{\hat{\omega}} = \hat{e}^z_{\mu} = 0, \] (4.5b)
\[ \hat{e}^z\omega = e^{2\beta \phi}. \] (4.5c)

where the same notation for the compact coordinate applies this time for the flat index $\hat{x}^a = (x^a, z)$. If the components of the inverse Vielbein are computed, they are found to be
\[ \hat{e}^\mu_{a} = e^{-\alpha \phi - \Omega_0} e^\mu_{a}, \] (4.6a)
\[ \hat{e}^\mu_{z} = \hat{e}^\omega_{a} = 0, \] (4.6b)
\[ \hat{e}^\omega_{z} = e^{-2\beta \phi}, \] (4.6c)

where $e^a_{\mu}$ are the Vielbein of metric $g_{\mu\nu}$ and $e^\mu_{a}$ are, respectively, the Vielbein of the inverse metric $g^{\mu\nu}$. Now, with the Vielbein perfectly computed, it is really easy to express the volume element from (4.2) in terms of the $(D-1)$-dimensional metric as
\[ \sqrt{|g|} = |\hat{e}^\hat{\mu}_{\hat{\mu}}| = e^{2\beta \phi} |e^{\alpha \phi + \Omega_0} e^a_{\mu}| = e^{\beta \phi} e^{(D-1)(\alpha \phi + \Omega_0)} |e^a_{\mu}| = e^{(D-1)\alpha + \beta \phi + (D-1)\Omega_0} \sqrt{|g|}. \] (4.7)

\(^1\)Notice that, in this section, the compact dimension is noted as $\omega$ instead of $\omega_1$ for simplicity.
The next step is to reduce the curvature term \( \hat{R}^d \). Recalling that this term is built up with Riemann tensors (and its contractions) and since the Vielbein are already computed, it is to be done in terms of the spin connection \( \hat{\omega}^c_{\dot{a}\dot{b}} \), whose expression without the hat notation is given in (B.28), in this case it is

\[
\hat{\omega}^c_{\dot{a}\dot{b}} = \frac{1}{2} \left( \Omega^c_{\dot{a}\dot{d}} \eta^{\dot{d}\dot{b}} + \Omega^c_{\dot{b}\dot{d}} \eta^{\dot{d}\dot{a}} - \Omega^c_{\dot{a}\dot{b}} \right). \tag{4.8}
\]

In order to compute them, the components of anholonomy coefficients are needed. According to expression (B.7), they are given by

\[
\hat{\Omega}^c_{\dot{a}\dot{b}} = e^{\dot{e}_\dot{a}} e^{\dot{e}_\dot{b}} \left( \hat{\partial}_{\dot{a}} \hat{e}^{\dot{b}} - \hat{\partial}_{\dot{b}} \hat{e}^{\dot{a}} \right), \tag{4.9}
\]

whose non-trivial components are

\[
\begin{align*}
\hat{\Omega}^{ab}_c &= e^{-\alpha_0} \left( \Omega^{ab}_c + \alpha \partial_a \phi \delta^c_b - \alpha \partial_b \phi \delta^c_a \right), \\
\hat{\Omega}^{az}_z &= \beta e^{-\alpha_0} \partial_a \phi.
\end{align*} \tag{4.10a,b}
\]

Notice that when computing the components of the partial derivative,

\[
\hat{\partial}_{\dot{a}} = e^{\dot{e}_\dot{a}} \partial_{\mu}, \tag{4.11}
\]

the expression (4.1) must be recalled from the discussion of the low energy limit and therefore they are given by

\[
\begin{align*}
\hat{\partial}_{\dot{z}} &= e^{\dot{e}_\dot{z}} \partial_{\mu} + e^{\dot{e}_\dot{a}} \partial_{\nu} = 0, \\
\hat{\partial}_{\dot{a}} &= e^{\dot{e}_\dot{a}} \partial_{\mu} + e^{\dot{e}_\dot{b}} \partial_{\nu} = e^{-\alpha_0} \partial_{\mu} = e^{-\alpha_0} \partial_{\mu}.
\end{align*} \tag{4.12a,b}
\]

With all these ingredients, the components of the spin connection are completely determined,

\[
\begin{align*}
\hat{\omega}^{c}_{ab} &= e^{-\alpha_0} \left( \omega^{c}_{ab} + \alpha \partial_a \phi \delta^c_b - \alpha \partial_b \phi \delta^c_a \right), \\
\hat{\omega}^{c}_{az} &= \beta e^{-\alpha_0} \partial_a \phi, \\
\hat{\omega}^{c}_{zz} &= \beta e^{-\alpha_0} \partial_z \phi,
\end{align*} \tag{4.13a,b,c}
\]

and therefore the Riemann tensor, given by expression (B.33) is computed with the notation of this chapter

\[
\hat{R}^{\dot{d}}_{\dot{a}\dot{b}\dot{c}\dot{e}} = \partial_{\dot{a}} \hat{\omega}^{\dot{d}}_{\dot{b}\dot{c}} - \partial_{\dot{b}} \hat{\omega}^{\dot{d}}_{\dot{a}\dot{c}} + \hat{\omega}^{\dot{e}}_{\dot{b}\dot{c}} \hat{\omega}^{\dot{d}}_{\dot{a}\dot{e}} - \hat{\omega}^{\dot{e}}_{\dot{a}\dot{c}} \hat{\omega}^{\dot{d}}_{\dot{b}\dot{e}} + \hat{\Omega}^{\dot{e}}_{\dot{a}\dot{b}} \hat{\omega}^{\dot{d}}_{\dot{e}\dot{c}} , \tag{4.14}
\]

yielding the components

\[
\begin{align*}
\hat{R}^{\dot{d}}_{\dot{a}\dot{b}\dot{c}\dot{e}} &= e^{-2\alpha_0 - 2\alpha_0} \left( R^{\dot{d}}_{\dot{a}\dot{b}\dot{c}\dot{e}} - \alpha^2 \partial_a \phi \partial_c \phi \delta^d_b + \alpha^2 \partial_a \phi \partial_d \phi \eta_{bc} + \alpha D_a D_c \phi \delta^d_b \\
&\quad - \alpha D_a D^d \phi \eta_{bc} + \alpha^2 \partial_a \phi \partial_d \phi \delta^c_b - \alpha^2 \partial_b \phi \partial_d \phi \delta^c_a - \alpha D_b D_c \phi \delta^d_a + \alpha D_b D^d \phi \eta_{ac} \\
&\quad - \alpha^2 (\partial \phi)^2 \eta_{bc} \delta^d_a + \alpha^2 (\partial \phi)^2 \eta_{ac} \delta^d_b \right),
\end{align*} \tag{4.15a}
\]

\[
\hat{R}^{\dot{z}}_{\dot{a}\dot{z}\dot{c}\dot{e}} = \beta e^{-2\alpha_0 - 2\alpha_0} \left[ \partial_{\dot{a}} \phi \partial_{\dot{e}} \phi (\beta - 2\alpha) + \alpha (\partial \phi)^2 \eta_{ac} + D_a D_c \phi \right]. \tag{4.15b}
\]

Regarding the contractions of the Riemann tensor, namely, the Ricci tensor and the Ricci scalar are given by

\[
\begin{align*}
\hat{R}_{\dot{a}\dot{b}} &= \hat{R}^{\dot{d}}_{\dot{a}\dot{b}\dot{c}\dot{d}}, \tag{4.16a} \\
\hat{R} &= \hat{\eta}^{\dot{a}\dot{c}} \hat{R}_{\dot{a}\dot{c}}, \tag{4.16b}
\end{align*}
\]
which means
\[
\hat{R}_{ab} = e^{-2\alpha \phi - 2\Omega_0} \left\{ R_{ab} + \alpha D^2 \phi \eta_{ab} + [(N - 3) \alpha + \beta] D_a D_b \phi \right. \\
+ \left[ \beta^2 - (N - 3) \alpha^2 - 2\alpha \beta \right] \partial_a \phi \partial_b \phi + \left[ (N - 3) \alpha^2 + \alpha \beta \right] \phi (\partial \phi)^2 \left\}, \right.
\]
\[
(4.17a)
\]
\[
\hat{R}_{zz} = -e^{-2\alpha \phi - 2\Omega_0} \left\{ \beta D^2 \phi + \left[ (N - 3) \alpha \beta + \beta^2 \right] (\partial \phi)^2 \right\},
\]
\[
(4.17b)
\]
\[
\hat{R} = e^{-2\alpha \phi - 2\Omega_0} \left\{ R + 2 [(N - 2) \alpha + \beta] D^2 \phi \right. \\
+ \left[ (N - 2) (N - 3) \alpha^2 + 2(N - 3) \alpha \beta + 2\beta^2 \right] (\partial \phi)^2 \left\}. \right.
\]
\[
(4.17c)
\]
Remember that these expressions are the building blocks of the curvature $\hat{R}^d_\ell$. The length of the expression of each of them means a great complexity when reducing a higher curvature term from a dimension to a lower one. In order to illustrate the complete phenomena, the Einstein-Hilbert term is to be reduced and a solution to the complexity is to be found analysing the result. The Einstein-Hilbert term, after plugging the obtained expression of the reduced Ricci scalar (4.17c) and the metric (4.7), yields
\[
\hat{S}_D \bigg|_{d=1} = \frac{2\pi \ell_1}{2\kappa_D} \int d^{D-1}x \sqrt{|g_D|} e^{[(D-3)\alpha + \beta \phi + (D-3)\Omega_0} \left\{ R + 2 [(N - 2) \alpha + \beta] D^2 \phi \right. \\
+ \left[ (N - 2) (N - 3) \alpha^2 + 2(N - 3) \alpha \beta + 2\beta^2 \right] (\partial \phi)^2 \left\} \right.
\]
\[
(4.18)
\]
Notice that the compact dimension $\omega$ has been integrated over the circumference $S^1$ of radius $\ell$ because the independence with respect to any field as commented before. The second derivative term would yield third-order equations of motion. In order to avoid this situation, it is integrated by parts, thus expression (4.18) is now
\[
\hat{S}_D \bigg|_{d=1} = \frac{2\pi \ell_1 e^{(D-3)\Omega_0}}{2\kappa_D} \int d^{D-1}x \sqrt{|g_{D-1}|} e^{[(D-3)\alpha + \beta \phi \left\{ R \\
- \alpha (D - 2) [(D - 3) \alpha + 2\beta] (\partial \phi)^2 \right\} \right. \right.
\]
\[
(4.19)
\]
The next step is to pay attention to the conformal factor present in the action. Generally speaking, when a conformal factor $e^{f(x)}$ is present in any Lagrangian, for instance
\[
\mathcal{L} = \sqrt{|g_{D-1}|} e^{f(x)} R,
\]
\[
(4.20)
\]
it is said to be expressed in the Jordan frame. Since the usual actions are written in terms of the Einstein frame, this is
\[
\mathcal{L} = \sqrt{|g_{D-1}|} R,
\]
\[
(4.21)
\]
it is enough to conveniently choose the value of the arbitrary constant
\[
\alpha = -\frac{\beta}{D - 3},
\]
\[
(4.22)
\]
making the conformal factor to disappear allowing to work in the usual Einstein frame. Using this, expression (4.19) yields
\[
\hat{S}_D \bigg|_{d=1} = \frac{2\pi \ell_1 e^{(D-3)\Omega_0}}{2\kappa_D} \int d^{D-1}x \sqrt{|g_{D-1}|} \left[ R - \frac{D - 2}{D - 3} \beta^2 (\partial \phi)^2 \right].
\]
\[
(4.23)
\]
All this procedure has been done for the Einstein-Hilbert term but the validity of it is limited. Consider the Gauss-Bonnet term, this is

$$\hat{L}_{GB} = \lambda_2 \left( \hat{R}^2 - 4 \hat{R}_{ab} \hat{R}^{ab} + \hat{R}_{abcd} \hat{R}^{abcd} \right).$$

(4.24)

When plugging the expressions of the Riemann tensors and its contractions (4.15) and (4.17), the conformal factor that previously was found to be $e^f(x) = e^{((D-3)\alpha+\beta)\phi(x)}$ in (4.19) is now

$$e^f(x) = e^{((D-5)\alpha+\beta)\phi(x)},$$

(4.25)

making useless the relation (4.22) imposed to keep working in the Einstein frame. This also happens for higher-curvature terms $d = 3$, $d = 4$ and subsequently. The only possibility to stay always in the Einstein frame is to do a consistent truncation of the dilaton field $\phi(x)$ keeping only the gravitational part of the theory, which is actually the one important for the results of the work. Applying these changes to expression (4.19), it yields a simpler expression,

$$\hat{S}_D \bigg|_{d=1} = \frac{2\pi \ell_1 e^{(D-3)\Omega_0}}{2\kappa_D} \int \, d^{D-1}x \sqrt{|g_{D-1}|} R,$$

(4.26)

in which the first order curvature term $\hat{R}^1$ has been compactified from (4.17c) in a extraordinarily uncomplicated expression

$$\hat{R}^1 = e^{-2\Omega_0} R^1,$$

(4.27)

that allows to easily extend to other curvature terms $d$ in the future. Finally, in order to get rid of the extra factors of the action (4.26), if the Newton constant is chosen to be

$$\kappa_D = 2\pi \ell_1 e^{(D-3)\Omega_0} \kappa_{D-1},$$

(4.28)

the coupling constant of the one-dimensional reduced Einstein-Hilbert action is found to have the same structure in $D$ dimensions and in $D - 1$ dimensions.

Taking advantage of the knowledge achieved with this particular case of the Lovelock action, it is time to come back to the general case (4.2). The general curvature term after applying dimensional reduction yields

$$\hat{R}^d = e^{-2d\Omega_0} R^d.$$  

(4.29)

Using this expression, the action (4.2) yields

$$\hat{S}_D = 2\pi \ell_1 e^{(D-1-2d)\Omega_0} \int \, d^{D-1}x \sqrt{|g_{D-1}|} \sum_{d=0}^{D/2-1} \binom{D/2}{d} (-\epsilon)^{d+1} (D-2d)! \hat{L}_D^{2(d-1)} \hat{R}^d,$$

(4.30)

where the explicit expression of the coupling constants $\lambda_d$ has been recalled from (3.64). The relation between the Newton constant in $D$ dimensions and in $D - 1$ dimension was previously found and must hold whatever the term, therefore, the action is

$$\hat{S}_D = e^{-2d\Omega_0} \int \, d^{D-1}x \sqrt{|g_{D-1}|} \sum_{d=0}^{(D-1)/2} \binom{D/2}{d} (-\epsilon)^{d+1} (D-2d)! \hat{L}_D^{2(d-1)} \hat{R}^d.$$  

(4.31)

At this point, the following step is to relate the radius of the (anti-)de Sitter space of a higher dimension $\hat{L}_D$ with the radius of a lower dimension. Since the construction of the Lovelock action of the previous chapter requires an even dimension, the next radius that can be determined is $L_{D-2}$. This means that compactifying over two dimensions is necessary.


## 4.2 Two-Dimensional Compactification

The necessity of compactifying over two dimensions has been showed. The fact of having applied the consistent truncation and considered only the gravitational part has simplified greatly the procedure and allows to retake the computations from the previous section and extend them in a quite straight-forward way. The difference is that now the compactification is over two dimensions, because of this, the reduction of the metric is now

\[
\sqrt{|g_{D}|} = e^{(D-1)\Omega_0} e^{(D-2)\Omega_0} \sqrt{|g_{D-2}|} = e^{(2D-3)\Omega_0} \sqrt{|g_{D-2}|}. \tag{4.32}
\]

The geometry of these two coordinates is a 2-torus \(S^1 \times S^1\) with the action

\[
\mathcal{S}_D = 4\pi^2 \ell_1^2 \ell_2^2 e^{(2D-3-4d)\Omega_0} \int d^{D-2} x \sqrt{|g_{D-2}|} \sum_{d=0}^{D/2-1} \binom{D/2}{d} (-\epsilon)^{d+1}(D-2d)! \hat{L}_{D-2}^{2(d-1)} \hat{R}^d. \tag{4.34}
\]

As previously, if the Newton constant in \(D\)-dimensions is chosen so that the term \(d = 1\) is equivalent to the Einstein-Hilbert term, a condition to relate it with the constant of two lower dimensions is found,

\[
\hat{k}_D = 4\pi^2 \ell_1 \ell_2 e^{(2D-7)\Omega_0} \kappa_{D-2}. \tag{4.35}
\]

Moreover, unlike before, another condition to relate the value of the (anti-)de Sitter \(\hat{L}_D\) with \(L_{D-2}\) in two lower dimensions is available. The simplest term that allows to relate them is the \(d = 0\) one, this is, the cosmological constant. The cosmological constant in \(D\) dimensions is given by

\[
\hat{\Lambda}_D \equiv \frac{(D-1)\epsilon}{\hat{L}_D^2 \hat{k}_D}, \tag{4.36}
\]

while the cosmological constant in \(D - 2\) dimensions is

\[
\Lambda_{D-2} \equiv \frac{(D-3)\epsilon}{L_{D-2}^2 \hat{k}_{D-2}}. \tag{4.37}
\]

When comparing the first term of the two actions in \(D\) and \(D - 2\) dimensions, a relation between the two (anti-)de Sitter radii is given by

\[
\frac{\hat{L}_D}{L_{D-2}} = \frac{D - 1}{D - 3} e^{4\Omega_0}, \tag{4.38}
\]

and plugging this expression in action (4.34), it yields

\[
\hat{S}_D = \int d^{D-2} x \sqrt{|g_{D-2}|} \sum_{d=0}^{D/2-1} \binom{D/2}{d} (-\epsilon)^{d+1}(D-2d)! \left( \frac{D-1}{D-3} L_{D-2} \right)^{2(d-1)} \hat{R}^d. \tag{4.39}
\]

Now that finally the action has been reduced from \(D\) to \(D - 2\) dimensions, it can be compared with the action

\[
S_{D-2} = \int_{\mathcal{M}^{D-2}} d^{D-2} x \sqrt{|g_{D-2}|} \sum_{d=0}^{D/2-1} \binom{D/2-1}{d} (-\epsilon)^{d+1}[D-2(d+1)]! \hat{L}_{D-2}^{2(d-1)} \left( \frac{D-2}{D-4} \right) \hat{R}^d, \tag{4.40}
\]
obtained gauging the (A)dS algebra in $D - 2$ dimensions. The two expressions (4.39) and (4.40) match when $d = 0$ and $d = 1$ (expected because imposed with the two conditions for $\hat{\kappa}_D$ and $\hat{L}_D$), but the terms with higher curvature $d \geq 2$ do not, independently of the dimension $D$ of the theory. A natural question arises now about the choice of compactifying two dimensions. It is uncertain whether compactifying other even number of dimensions the coupling constant would coincide, some of them or all of them. Since the assumptions made allow to extend easily the reduction to a generic number of dimensions, it is to be done in next section.

4.3 N-Dimensional Compactification

In order to study the compatibility of the coupling constants after $N$ dimensional reductions, the first task is to compactify the geometric objects involved in the action. Previously done for the two-dimensional case, this time the Riemann tensor, Ricci tensor and Ricci scalar take the form

\begin{align}
\hat{R}_{abc} &= e^{-2N\Omega_0} R_{abc}, \\
\hat{R}_{ab} &= e^{-2N\Omega_0} R_{ab}, \\
\hat{R} &= e^{-2N\Omega_0} R. 
\end{align}

This results can be used to generalise the introduced scalar $\hat{R}^d$, which happens to be reduced as

\begin{equation}
\hat{\mathcal{R}}^d = e^{-dN\Omega_0} \mathcal{R}^d.
\end{equation}

Likewise, for this case the volume element yields

\begin{equation}
\sqrt{|\hat{g}_D|} = \prod_{n=1}^{N} e^{(D-n)} \sqrt{|g_{D-N}|} = e^{N \left(N - \sum_{n=1}^{N} n\right) \Omega_0} \sqrt{|g_{D-N}|} = e^{N \left(D - \frac{N+1}{2}\right) \Omega_0} \sqrt{|g_{D-N}|}.
\end{equation}

Taking into account these increase in the number of compact dimensions the relation between the Einstein constants $\hat{\kappa}_D$ and $\kappa_{D-N}$, which for the case $N = 2$ was computed in (4.35), is now required to be

\begin{equation}
\frac{\hat{\kappa}_D}{\kappa_{D-N}} = (2\pi)^N e^{N \left(D - \frac{N+1}{2}\right) \Omega_0} \prod_{n=1}^{N} \ell_n.
\end{equation}

Respectively, the second condition (4.38) can be revisited to compute the relation between the radii of (A)dS,

\begin{equation}
\frac{\hat{L}_D^2}{\hat{L}_{D-N}^2} = \frac{D - 1}{D - N - 1} e^{2N\Omega_0}.
\end{equation}

Introducing these new relations in the Lovelock action, it yields

\begin{equation}
\hat{S}_D = \int d^{D-N} x \sqrt{|g_{D-N}|} \sum_{d=0}^{D-N} \frac{D/2}{d} \left(\frac{\left(-\epsilon\right)^{d+1}(D - 2d)!}{D(D-2)! \kappa_{D-N}} \left(\sqrt{\frac{D - 1}{D - N - 1} \hat{L}_{D-N}}\right)^{2(d-1)} \hat{\mathcal{R}}^d, \right.
\end{equation}

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and this is to be compared with

\[ S_{D-N} = \int d^{D-N} x \sqrt{|g_{D-N}|} \sum_{d=0}^{D-N} \left( \frac{D-N}{2} \right) \left( -\epsilon \right)^{d+1} \frac{(D-N-2d)! L_{D-N}^{2(d-1)}}{(D-N)(D-N-2)!} K_{D-N}^{d} \cdot (4.47) \]

A careful look in these expressions reveals that there are not any new combinations of the three-tuple \((D, d, N)\) that match the coupling constants of the two actions computed using both formalisms. This result means that regardless the dimension and regardless the curvature order, the coupling constants are not related using this two theories and therefore the uniqueness of the Lovelock Lagrangian is not guaranteed fixing the values through the gauge principle.
Chapter 5

Conclusions

The Lovelock Lagrangian has been presented, discussed the motivations and the reasons to consider this theory as a candidate to describe gravitation in higher dimensions. The problem of indetermination of coupling constants has been addressed.

In order to fix the value of the coupling constants, the gauge principle has been used. First, it has been showed that the Einstein-Hilbert Lagrangian can be reproduced when applying the gauge principle to the Poincaré group of transformations. This procedure does not include the cosmological constant so, in order to obtain it, a modification to the algebra of the group has been considered, the (anti-)de Sitter algebra. After obtaining the Einstein-Hilbert Lagrangian plus the cosmological constant, a natural extension of the theory has been constructed to include a generic number of dimensions and therefore, the Lovelock Lagrangian.

Once the values of the coupling constants have been determined, Kaluza-Klein dimensional reduction is applied to it with the goal of studying the uniqueness of the determination of the constants when applying the gauge principle.

First, a reduction over two dimensions, with the geometry of a two-torus, is performed and only the cosmological term and the Ricci scalar term, which coincides with the Einstein-Hilbert action, are compatible. This is rather natural because by construction, in order to find the relations between the Einstein’s constant and respectively the radii of the (A)dS, they were imposed to do so. However, for terms of higher curvature the coupling constants are different showing that uniqueness is not guaranteed when obtaining them using the gauge principle.

In an attempt to find a particular $N$ dimensional reduction that shows compatibility for any higher curvature term among the coupling constant using the geometry of a n-torus, the procedure has showed again non-matching constants, reinforcing the conclusion of arbitrariness when determining the coupling constant of the Lovelock series using the gauge principle.

Further questions arise at this point, because even though the dimensional reduction over torus does not agree the terms of the two actions, a reduction over different topologies may do so. This is left for future research.
Appendix A

Conventions

A.1 The Levi-Civita Symbol

The $D$-dimensional Levi-Civita symbol is a completely antisymmetric object, defined as

$$
\varepsilon_{\mu_1\ldots\mu_D} = \begin{cases} 
1 & \text{if } (\mu_1 \ldots \mu_D) \text{ is an even permutation of } (0 1 \ldots D-1), \\
-1 & \text{if } (\mu_1 \ldots \mu_D) \text{ is an odd permutation of } (0 1 \ldots D-1), \\
0 & \text{otherwise},
\end{cases} \tag{A.1}
$$

which is equivalent to

$$
\varepsilon_{\mu_1\ldots\mu_D} = \delta^1_{[\mu_1} \cdots \delta^D_{\mu_D]}. \tag{A.2}
$$

The Levi-Civita symbol is a fundamental symbol: it has these values in all coordinate systems. In order for this to be true, the Levi-Civita symbol has to transform as a pseudo-tensor density of weight $w = +1$ under general coordinate transformations

$$
\varepsilon_{\alpha_1\ldots\alpha_D} = \text{sgn} \left( \frac{\partial y}{\partial x} \right) \left| \frac{\partial y}{\partial x} \right|^D \prod_{i=1}^D \frac{\partial x^{\mu_i}}{\partial y^{\alpha_i}} \varepsilon_{\mu_1\ldots\mu_D}. \tag{A.3}
$$

Using these transformation rules the invariant volume element is given by

$$
\sqrt{|g|} d^D x = \sqrt{|g|} \varepsilon_{\mu_1\ldots\mu_D} \prod_{i=1}^D dx^{\mu_i}. \tag{A.4}
$$

An alternating symbol with upper indices can also be defined as

$$
\varepsilon^{\nu_1\ldots\nu_D} = \left( \prod_{i=1}^D g^{\mu_i\nu_i} \right) \varepsilon_{\mu_1\ldots\mu_D}, \tag{A.5}
$$

such that

$$
\varepsilon^{01\ldots D-1} = (-1)^{D-1} |g|^{-1}. \tag{A.6}
$$

The contraction of two Levi-Civita symbols is given by

$$
\varepsilon_{\mu_1\ldots\mu_d \nu_1\ldots\nu_{D-d}} \varepsilon^{\mu_1\ldots\mu_D} = (-1)^{D-1} |g|^{-1} d! (D-d)! \delta^{\nu_1}_{[\mu_1} \cdots \delta^{\nu_{D-d}}_{\nu_{D-d]}}, \tag{A.7}
$$

from which yields two interesting cases, namely

$$
\varepsilon_{\mu_1\ldots\mu_D} \varepsilon^{\mu_1\ldots\mu_D} = (-1)^{D-1} D! |g|^{-1}, \tag{A.8a}
$$

$$
\varepsilon_{\mu_1\ldots\mu_D} \varepsilon^{\nu_1\ldots\nu_D} = (-1)^{D-1} D! |g|^{-1} \delta^{\nu_1}_{[\mu_1} \cdots \delta^{\nu_D}_{\mu_D]}, \tag{A.8b}
$$

$$
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$$
Appendix B

Geometry from the Tangent Space

B.1 Vielbein

Let $\mathcal{M}^D$ be a $D$-dimensional manifold. A property of the differential manifold is that each point $P$ can be locally described by its tangent $T_P(\mathcal{M}^D)$. The coordinates $x^\mu$ of $\mathcal{M}^D$ define at every $P$ a basis $\{|e_\mu\rangle\} = \partial_\mu \in T_P(\mathcal{M}^D)$. This basis is called coordinate basis and it is related to an arbitrary basis $\{|e_a\rangle\}$ through

$$|e_\mu\rangle = e^a_\mu |e_a\rangle, \quad |e_a\rangle = e^\mu_a |e_\mu\rangle. \quad (B.1a)$$

The coefficients $e^a_\mu$ are the components of the vector $|e_\mu\rangle$ in the new basis $\{|e_a\rangle\}$. The matrix $e^a_\mu$ is called Vielbein\(^1\) whereas $e^\mu_a$ is the inverse Vielbein, verifying

$$e^\mu_a e^a_\nu = \delta^\mu_\nu, \quad (B.2a)$$
$$e^\mu_a e^b_\mu = \delta^b_a. \quad (B.2b)$$

In a particular tangent space $T_P(\mathcal{M}^D)$, the Vielbein allows to transform the components of a tensor of type $(1,1)$ from a basis to another as

$$T^\mu_\nu = e^a_\mu e^b_\nu T^a_b, \quad (B.3a)$$
$$T^a_b = e^a_\mu e^\nu_b T^{\mu}\nu, \quad (B.3b)$$

leaving a trivial generalisation to the tensors of type $(p,q)$. Using this transformation property, the metric tensor $g_{\mu\nu}$, which is curved in general, can be locally expressed with Cartesian coordinates,

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}, \quad (B.4a)$$
$$\eta_{ab} = e^\mu_a e^\nu_b g_{\mu\nu}. \quad (B.4b)$$

This procedure can be generalised to the whole tangent bundle of the manifold by making the Vielbein functions depending on the coordinates $x^\nu$ of the manifold,

$$e^a_\mu = e^a_\mu(x^\nu). \quad (B.5)$$

\(^1\)Vierbein if $D = 4$, Fünfbein when $D = 5$ and subsequently.
Due to the dependency of the Vielbein with the coordinates of the manifold, a feature arises, the non-zero commutativity of partial derivatives acting on an arbitrary field \( \chi \), this is
\[
[\partial_a, \partial_b] \chi = -\Omega_{ab}^c \partial_c \chi,
\]  
where \( \Omega_{ab}^c \) are called the anholonomy coefficients,
\[
\Omega_{ab}^c = e^\mu_a e^\nu_b \left( \partial_\mu e^c_\nu - \partial_\nu e^c_\mu \right).
\]  
If another base \( \{e'_a\} \) is considered in \( TP(M^D) \), it will be related to the coordinate basis through
\[
|e_\mu\rangle = e^a_\mu |e'_a\rangle,
\]  
\[
|e'_a\rangle = |e_\mu\rangle e^a_\mu.
\]  
Since the relation between the basis \( |e_a\rangle \) and \( |e'_a\rangle \) are the Lorentz transformations \( (\Lambda^{-1})_a^b \), then the Vielbein verifies that
\[
e'^a_\mu = \Lambda^a_b e^b_\mu,
\]  
\[
e'^\mu_b = (\Lambda^{-1})_a^b e^a_\mu.
\]  
From this, a straight-forward relation between any tensor of type \((1,1)\) is found using expression (B.3),
\[
T'^a_b = e'^a_\mu e'^\nu_b T^\mu_\nu = e^\mu_a e^\nu_b \Lambda^a_\mu \Lambda^b_\nu (\Lambda^{-1})_d^c T^\mu_\nu = \Lambda^a_c (\Lambda^{-1})^d_b T^c_d.
\]  
Although the Lorentz transformations are global in a tangent flat \( TP(M^D) \), they are local in the tangent bundle \( T(M^D) \), namely \( \Lambda^a_b = \Lambda^a_b(x) \). This is the reason of being called local Lorentz transformation. Once again this means troubles with the partial derivative acting on a tensor, the result shall not be a tensor,
\[
\partial_\mu T^a_b = \partial_\mu \left[ \Lambda^a_i c_i (\Lambda^{-1})^d_j b_j \right] T^c_d + \Lambda^a_c (\Lambda^{-1})^d_b \partial_\mu T^c_d.
\]  
In order to solve this situation, and following the same procedure of Riemannian geometry, a derivative that acts properly under the local Lorentz transformations is defined,
\[
D_\mu T^a_b = \partial_\mu T^a_b + \omega^a_{\mu c} T^c_b - \omega^a_{\mu b} T^a_c,
\]  
where \( \omega^a_{\mu b} \) is called the spin connection and transforms as
\[
\omega'_a c_{\mu b} = (\Lambda^{-1})_a^c \Lambda^b_\mu \omega^d_c + \Lambda^b_\mu \partial_\nu (\Lambda^{-1})^c_d a.
\]  
Notice that, although the spin connection does not transform like a tensor due to the last term, the difference of two spin connections actually does. An aside before continuing is that the infinitesimal variation of this connection requires the infinitesimal Lorentz transformation,
\[
\Lambda^a_b = \delta^a_b - \frac{1}{2} \theta_{cd} \left( S^{cd} \right)_a^b,
\]  
where the set of matrix operators \( (S^{cd})_a^b \) is a representation of the Lorentz algebra defined as
\[
\left( S^{cd} \right)_a^b = i \eta^{ca} \delta^d_b - i \eta^{da} \delta^c_b.
\]
Thus, the variation of the spin connection yields
\[
\delta \omega_{\mu ab} = \frac{i}{2} \theta_{cd} S^{cd}_{\ a} \omega_{\mu eb} + \frac{i}{2} \theta_{cd} S^{cd}_{\ b} \omega_{\mu ea} + \partial_{\mu} \theta_{cd} S^{cd}_{\ a} \delta_{eb}.
\]  
(B.16)

Back with the topic, the constructed covariant derivatives \(D_\mu\) do not commute when acting on 
vectors in the same way as covariant derivatives \(\nabla_\mu\) of Riemannian geometry,
\[
[D_\mu, D_\nu] V^b = R_{\mu\nu a} V^a,  \tag{B.17a}
\]
\[
[D_\mu, D_\nu] V^a = R_{\mu\nu a} V^b,  \tag{B.17b}
\]
where \(R_{\mu\nu a}\) are the Riemann curvature tensors, defined as
\[
R_{\mu\nu a} = \partial_{\mu} \omega_{\nu a} - \partial_{\nu} \omega_{\mu a} - \omega_{\rho a} \omega_{\mu \nu}^{ \rho} + \omega_{\mu a} \omega_{\nu \rho}^{ \rho}.  \tag{B.18}
\]

At this point, it turns out that the manifold \(\mathcal{M}^D\) can be described in terms of the curved metric \(g_{\mu\nu}\) and the affine connection \(\Gamma_{\mu \nu}^{\rho}\) as well as in terms of the Vielbein and the spin connection \(\omega_{\mu a}^{\ b}\).
A priori, both descriptions are independent in spite of being related through (B.4), however, they can be related using the Vielbein postulates.

### B.2 Vielbein Postulates

In general, the constructed derivatives \(D_\mu\) are not covariant with respect to general changes of coordinate systems, therefore a totally covariant derivative \(\mathcal{D}_\mu\) (respect to curved indices and flat indices) must be defined. For a \((2, 2)\) type tensor with two covariant and contravariant flat indices and with the other two curved indices, the aforementioned covariant derivative is defined as
\[
\mathcal{D}_\mu T^{a \alpha}_{\ b \beta} = \partial_\mu T^{a \alpha}_{\ b \beta} + \omega_{\mu c} T^{a \alpha}_{\ c b \beta} - \omega_{a \mu c} T^{a \alpha}_{\ c b \beta} + \Gamma_{\mu \gamma}^{\alpha} T^{a \gamma}_{\ b \beta} - \Gamma_{\mu \beta}^{\gamma} T^{a \alpha}_{\ c \gamma}.  \tag{B.19}
\]

This derivative is finally covariant with respect to general changes of coordinate systems as well as with respect to Lorentz transformations in the tangent space. It is used in the formulation of the first Vielbein postulate, which reads
\[
\mathcal{D}_\mu e^a_{\ \beta} = 0.  \tag{B.20}
\]

According to the previous expression, this means that
\[
\mathcal{D}_\mu e^a_{\ \beta} = \partial_\mu e^a_{\ \beta} + \omega_{\mu c} e^c_{\ \beta} - \Gamma_{\mu \beta}^{\gamma} e^a_{\ \gamma} = 0,  \tag{B.21}
\]
which yields a relation between the spin connection and the affine connection,
\[
\omega_{\mu c} = \Gamma_{\mu \beta}^{\gamma} e^a_{\ \gamma} e^c_{\ \beta} - e^c_{\ \beta} \partial_\mu e^a_{\ \beta},  \tag{B.22}
\]
and this allows to establish the relation between curvature tensors with mixed curved and flat indices,
\[
R_{\mu \rho \sigma} = e^a_{\ \rho} e^\sigma_{\ b} R_{\mu \nu a}^{\ b}.  \tag{B.23}
\]

It is clear that both tensors carry the same information about the geometrical properties of the manifold and therefore both formalisms are completely equivalent.
The second Vielbein postulate is the connection compatibility condition with the metric \( \nabla_{\rho} g_{\mu\nu} = 0 \), using expression (B.4) it can be written in terms of the Vielbein as

\[
\nabla_{\rho} \left( e^{a}_{\mu} e^{b}_{\nu} \eta_{ab} \right) = 0, \tag{B.24}
\]

which implies together with the first postulate (B.20) the compatibility of the spin connection \( \omega_{\mu a}^{\ b} \) with the Minkowskian metric,

\[
D_{\mu} \eta_{ab} = 0. \tag{B.25}
\]

This compatibility condition implies a relation of antisymmetry on the last two indices of the curvature tensor \( R_{\mu \nu a}^{\ b} \) showing up when computing the commutator,

\[
[D_{\mu}, D_{\nu}] \eta_{ab} = R_{\mu \nu a}^{\ b} \eta_{cb} + R_{\mu \nu b}^{\ a} \eta_{ca} = R_{\mu \nu a}^{\ b} + R_{\mu \nu b}^{\ a} = 0. \tag{B.26}
\]

Likewise, the compatibility (B.25) forces an antisymmetry relation on the two last indices of the spin connection, \( \omega_{\mu ab} = -\omega_{\mu ba} \), because

\[
D_{\mu} \eta_{ab} = \partial_{\mu} \eta_{ab} - \omega_{\mu a}^{\ c} \eta_{cb} - \omega_{\mu b}^{\ c} \eta_{ca} = -\omega_{\mu ab} - \omega_{\mu ba} = 0. \tag{B.27}
\]

Since in the present work only manifolds equipped with the Levi-Civita connection are considered\(^2\), expression (B.22) can be written with all its indices flat as

\[
\omega_{ab}^{\ c} = e_{\mu}^{a} \omega_{\mu b}^{\ c} = \frac{1}{2} \left( \Omega_{ad}^{\ e} \eta_{dc} + \Omega_{bd}^{\ e} \eta_{ac} - \Omega_{ab}^{\ e} \right), \tag{B.28}
\]

where the definition of anholonomy coefficient (B.7) has been used to shorten the expression. Furthermore, an interesting relation when subtracting two spin connections with its first pair of indices exchanged is found,

\[
\omega_{ab}^{\ c} - \omega_{ba}^{\ c} = \Omega_{ab}^{\ c}, \tag{B.29}
\]

the operation yields the anholonomy coefficients, this fact will be used afterwards to avoid some tedious calculations.

### B.3 Curvature Tensors from the Tangent Space

Previously, in expression (B.23), a relation between the Riemann tensor and the curvature tensor written in terms of the spin connection is found. Now, the objective is to find the relations between the contractions of \( R_{\mu \nu a}^{\ b} \) and its relation to the Ricci tensor \( R_{\mu \nu} \) and the Ricci scalar \( R \). The first contraction is

\[
R_{\mu a} = e^{\nu}_{b} R_{\mu a}^{\ b} = \partial_{a} \omega_{cb}^{\ c} - \partial_{c} \omega_{ca}^{\ b} - \omega_{\mu a}^{\ c} \omega_{bc}^{\ b} + e_{\nu}^{\ c} \Gamma_{\mu \nu}^{\ c} \omega_{\mu a}^{\ b}, \tag{B.30}
\]

which transforms properly under general change of coordinate systems and Lorentz transformations. The contraction with two flat indices can also be constructed as

\[
R_{ab} = e^{\nu}_{a} R_{\mu b}^{\ b} = \partial_{b} \omega_{ab}^{\ c} - \partial_{a} \omega_{ab}^{\ c} - \omega_{ab}^{\ c} \omega_{cd}^{\ d} + \omega_{ca}^{\ d} \omega_{db}^{\ c}, \tag{B.31}
\]

and this transforms as a rank two tensor under Lorentz transformation and as a scalar under general change of coordinate systems. This tensor is the flat version of the Ricci tensor,

\[
R_{ab} = e^{\nu}_{a} e^{\nu}_{b} R_{\mu \nu}, \tag{B.32}
\]

\(^2\)Torsion equal to zero.
which is also obtained from the completely flat curvature tensor

\[
R_{abc}^d = e^\mu_a e^\nu_b R_{\mu\nu c}^d
\]

\[
= \partial^d \omega_{bc}^d - \partial^d \omega_{ac}^d - \omega_{ac}^d \omega_{bc}^d + \omega_{bc}^e \omega_{ae}^d - \omega_{ab}^d \omega_{ec}^d + \omega_{bd}^e \omega_{ec}^d
\]

\[
= \partial^d \omega_{bc}^d - \partial^d \omega_{ac}^d - \omega_{ac}^d \omega_{be}^d + \omega_{be}^e \omega_{ae}^d - \omega_{ab}^d \omega_{ae}^d - \Omega_{ab}^e \omega_{ec}^d ;
\]

contracting the second and fourth indices

\[
R_{ac} = R_{abc}^b = \eta^{bd} R_{abcd}.
\]

Notice that in equation (B.33), the relation (B.29) has been used. The contraction of the Ricci tensor gives the Ricci scalar,

\[
R = \eta^{ab} R_{ab} = g^{\mu\nu} R_{\mu\nu},
\]

and it coincides when using the flat indices and the curved indices. It is obvious that the symmetry and antisymmetry relations hold in both formalisms.

With all the geometric objects expressed with curved and flat indices, the gravitational actions can be constructed with any of the both formalisms carrying the same information.
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Bibliography


