Prospective Primary School Teachers' Perception of Randomness[1]

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ABSTRACT

Subjective perception of randomness has been researched by psychologists and mathematics educators, using a variety of tasks, resulting in a number of different descriptions for the biases that characterize people's performances. Analyzing prospective teachers' possible biases concerning randomness is highly relevant as new mathematics curricula for compulsory teaching levels are being proposed that incorporate increased study of random phenomena. In this chapter we present results of assessing perception of randomness in a sample of 208 prospective primary school teachers in Spain. We first compare three pairs of random variables deduced from a classical task in perception of randomness and deduce the mathematical properties these prospective teachers assign to sequences of random experiments. Then, the written reports, where prospective teachers analyse the same variables and explicitly conclude about their own intuitions are also studied. Results show a good perception of the expected value and poor conception of both independence and variation as well as some views of randomness that parallel some naïve conceptions on randomness held at different historic periods.

Keywords: Randomness, subjective perception, conceptions of randomness, prospective primary school teachers, assessment.
probabilistic reasoning, heuristics and biases, historical conceptions of randomness

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1. INTRODUCTION

Different reasons to teach probability have been highlighted over the past years (e.g., by Gal, 2005; Franklin et al., 2005, Jones, 2005; and Borovnick, 2011): the role of probability reasoning in decision making, the instrumental need of probability in other disciplines, and the relevance of stochastic knowledge in many professions. Moreover, students meet randomness not only in the mathematics classroom, but also in social activities (such as games or sports), and in meteorological, biological, economic, and political settings. Consequently, some of them may build incorrect conceptions of randomness in absence of adequate instruction (Borovcnick, 2012).

Consequently of this pervasive presence of chance in everyday life, probability has been included in schools since primary education in recent curricula that place more emphasis on the study of randomness and probability by very young children. For example, in the Spanish curriculum for compulsory primary education (MEC, 2006), we find the following contents in the first cycle (6-7 year-olds)*: "Random nature of some experiences. Difference between possible, impossible and that what is possible but not certain"*. Reference is made to using the chance language in everyday settings, in order to describe and quantify random situations. In the second cycle (8-9 year olds) the document suggests that children should evaluate the results of random experiences, and understand that there are more and less probable events, and that it is impossible to predict a specific result. In the last cycle (10-11 year olds) children are encouraged to recognize random phenomena in everyday life and estimate the probability for events in simple experiments.

This curriculum is not an exception, since the current tendency even for primary school levels is towards a data-orientated teaching of probability, where students are expected to perform experiments or simulations, formulate questions or predictions, collect and analyze data from these experiments, propose and justify conclusions and predictions that are based on data (e.g., Ministério da Educação, 1997; NCTM, 2000).

Changing the teaching of probability in schools will depend on the extent to which we can prepare adequately the teachers. Although teachers do not need high levels of mathematical knowledge, they do require a profound understanding of the basic mathematics they teach at school level, including a deep grasp of the interconnections and relationships among different aspects of this knowledge (Ma, 1999); for example, teachers need a sound understanding of the different meanings associated to randomness and probability. Unfortunately, several authors (e.g., Franklin & Mewborn, 2006;

Chadjipadelis, Meletiou, & Paparistodemou, 2010; Batanero & Díaz, 2012) suggest that many of the current programmes do not yet train teachers adequately for their task to teach statistics and probability. The situation is particularly challenging for primary teachers, few of whom have had suitable training in statistics, or in the related didactical knowledge (Jacobbe, 2010) and, consequently, they might share with their students a variety of probabilistic misconceptions (Stohl, 2005). Therefore it is important to assess teachers' probabilistic knowledge and find activities where teachers work with meaningful problems and are confronted to their own misconceptions in the topic (Batanero, Godino & Roa, 2004).

Understanding randomness is the base of understanding probability and conceptions of randomness are at the heart of people's probabilistic and statistical reasoning (Lecoutre et al., 2006); however, epistemological analysis of randomness, as well as psychological research have shown that there is no adequate perception of randomness in children or even in adults. There is an apparent contradiction in people's understanding of random processes and sequences, which is related to the psychological problems associated with the concept, namely that randomness implies that "anything possible might occur", but that, subjectively, however, many people believe that only the outcomes without visible patterns are "permissible" examples of randomness (Hawkins, Jolliffe, & Glickman 1991).

Despite the relevance of the topic in probability and statistics, little attention has been paid to prospective teachers' conceptions on randomness. To address this omission, in this chapter we analyse a research that was aimed at assessing prospective primary school teachers' perception of randomness using two different tools: (a) we first analyse some statistical variables deduced from a classical experiment related to perception of randomness that was carried out by the teachers; (b) we secondly analyze the written reports produced by the teachers, which were part of an activity, directed to confront them with their own misconceptions of randomness.

In the next sections we first analyse some different historical interpretation of randomness that can be parallel to some conceptions shown by prospective teachers in our research. We secondly analyze previous research on subjective perceptions of randomness in children and adults, and the scarce research dealing with teachers. Then we present the method, results and conclusions of our study. Finally some implications for teachers' education are provided.

2. RANDOMNESS: EMERGENCE AND PROGRESSIVE FORMALIZATION

In spite of being a basic idea in probability, randomness is not an easy concept. The term resists easy or precise definition, its emergence was slow and it has received various interpretations at different periods in history (Zabell, 1992; Bennet, 1998; Liu & Thompson, 2002, Batanero, Henry & Parzysz, 2005). Some of these interpretations are relevant to this research and may help understanding prospective teachers' difficulties in the theme.

2.1. Randomness and causality

Chance mechanisms such as cubic dice, or astragali have been used since antiquity to make decisions or predict the future. However, a scientific idea of randomness was absent in the first exploratory historical phase, which extended according to Bennet (1998), from antiquity until the beginning of the Middle Ages when randomness was related to causality and conceived as the opposite of something that had some known causes. According to Liu and Thompson (2002), conceptions of randomness and determinism ranged along an epistemological spectrum, where, on one extremun, random phenomena would not have an objective existence but would reflect human ignorance. This was the view, for example of Aristotle, who considered that chance results from the unexpected coincidence of two or more series of events, independent of each other and due to so many different factors that the eventual result is pure chance (Batanero, Henry, & Parzysz, 2005). It was also common in European Enlightenment where there was a common belief in universal determinism, as expressed for example by Laplace: "We ought then to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow" (Laplace, 1814/1995, p. vi). From this viewpoint, chance is seen as only the expression of our ignorance.

The other end of the spectrum consists in accepting the existence of "irreducible chance" and therefore that randomness is an inherent feature of nature (Liu & Thompson, 2002). As stated by Poincaré (1936), for the laws of Brownian motion, the regularity of macroscopic phenomena can be translated to deterministic laws, even when these phenomena are primarily random at the microscopic level. Moreover, ignorance of the laws governing certain natural phenomena does not necessarily involve a chance interpretation, because certain phenomena with unknown laws (such as death) are considered to be deterministic. Finally, among the phenomena for which the laws are unknown, Poincaré discriminated between random phenomena, for which probability calculus would give us some information, and those non-random phenomena, for which there is no possibility of prediction until we discover their laws. As Ayer (1974) stated, a phenomenon is only considered to be random if it behaves in accordance with probability calculus, and this definition will still hold even when we have found the rules for the phenomenon.

2.2. Randomness and probability

With the pioneer developments of probability theory, randomness was related to equi-probability (for example in the Liber of Ludo Aleae by Cardano), because this development was closely linked to games of chance, where the number of possibilities is finite and the principle of equal probabilities for the elementary events of the sample space in a simple experiment is reasonable.

Nowadays, we sometimes find randomness explained in terms of probability, although such an explanation would depend on the underlying understanding of probability. If we adopt a Laplacian's view of probability, we would consider that an object is chosen at random out of a given class (sample space), if the conditions in this selection allow us to give the same probability for any other member of this class (Lahanier- Reuter, 1999). This definition of randomness may be valid for random games based on dice, coins, etc., but Kyburg (1974) suggested that it imposes excessive restriction to randomness and it would be difficult to find applications of the same. For example, it only can be applied to finite sample spaces; if the sample space is infinite, then the probability associated to each event is always null, and therefore still identical, even when the selection method is biased. Furthermore, this interpretation precludes any consideration of randomness applied to elementary events that are not equiprobable.

When we transfer the applications of probability to the physical or natural world, for example studying the blood type of a new-born baby or any other hereditary characteristic, we cannot rely on the equiprobability principle. In this new situation, we may consider an object as a random member of a class if we can select it using a method providing a given 'a priori' relative frequency to each member of this class in the long run. Thus, we use the frequentist view of probability, which is most appropriate when we have data from enough cases. However, we are left with the theoretical problem of deciding how many experiments it is necessary to consider in order to being sure that we have sufficiently proven the random nature of the object (Batanero, Henry, & Parzysz, 2005).

Within either of these two frameworks, randomness is an 'objective' property assigned to the event or element of a class. Kyburg (1974) criticizes this view and proposes a subjevtive interpretation of randomness composed of the following four terms:

- The object that is supposed to be a random member of a class;
- The set of which the object is a random member (population or collective);
- The property with respect to which the object is a random member of the given class;
- The knowledge of the person giving the judgment of randomness.

Whether an object is considered to be a random member of a class or not, depends, under this interpretation, on our knowledge. Consequetnly, this view is coherent with the subjective conception of probability, and adequate when we have some information affecting our judgment about the randomness of an event (Fine, 1973).

2.3. Formalization of randomness

By the end of the XIXth century, theoretical developments of statistical inference and publication of tables of pseudo-random numbers, produced concern about how to ensure the 'quality' of those numbers. According to Zabell (1992), an important development was the distinction between a random process and a random sequence of outcomes: Although randomness is a property of a process, rather than of the outcomes of that process, it is only by observing outcomes that we can judge whether the process is random or not (Johston-Wilder & Pratt, 2007). The possibility of obtaining pseudorandom digits with deterministic algorithms and relate debates led to the formalization of the concept of randomness (Fine, 1973).

Von Mises (1928/1952) based his study of this topic on the intuitive idea that a sequence is considered to be random if we are convinced of the impossibility of finding a method that lets us win in a game of chance where winning depends on forecasting that sequence. This definition of randomness is the basis for statistical tests that are used for checking random number tables before presenting them to the scientific community. However, since in all statistical tests there is the possibility of error, we can never be totally certain that a given sequence, in spite of having passed all the tests, does not have some unnoticed pattern within it. Thus, we cannot be absolutely sure about the randomness of a particular finite sequence. We only take a decision about its randomness with reference to the outcomes of test techniques and instruments. This explains why a computer-generated random sequence (which is not random in an absolute sense) can still be random in a relative sense (Harten & Steinbring, 1983).

Another attempt to define the randomness of a sequence was based on its *computational complexity*. Kolmogorov's interpretation of randomness reflected the difficulty of describing it (or storing it in a computer) using a code that allows us to reconstruct it afterwards (Zabell, 1992). In this approach, a sequence would be random if it cannot be codified in a more parsimonious way, and the absence of patterns is its essential characteristic. The minimum number of signs necessary to code a particular sequence provides a scale for measuring its complexity, so this definition allows for a hierarchy in the degrees of randomness for different sequences. It is important to remark that in both Von Mises' and Kolmogorov's approaches perfect randomness would only apply to sequences of infinite outcomes and therefore, randomness would only be a theoretical concept (Fine, 1973).

3. PERCEPTION OF RANDOMNESS

3.1. Adult's perception of randomness

There has been a considerable amount of research into adults' subjective perception of randomness (e.g., Wagenaar, 1972; Falk, 1981; Bar-Hillel & Wagenaar, 1991; Engel & Sedlmeier, 2005). Psychologists have used a variety of stimulus tasks, which were classified in a review by Falk and Konold (1997) into two main types: (a) In *generation tasks* subjects generate random sequences under standard instructions to simulate a series of outcomes from a typical random process, such as tossing a coin; (b) In *recognition tasks*, which were termed as *comparative likelihood task* by Chernoff (2009a), people are asked to select the most random of several sequences of results that might have been produced by a random device or to decide whether some given sequences were produced by a random mechanism. Similar types of research have also been performed using two-dimensional random distributions, which essentially consist of random distributions of points on a squared grill.

One main conclusion of all these studies is that humans are not good at either producing or perceiving randomness (Falk, 1981; Falk & Konold, 1997; Nickerson, 2002) and numerous examples demonstrate that adults have severe difficulties when dealing appropriately with aspects of randomness, so that systematic biases have consistently been found. One such bias is known as the *gambler's fallacy,* or belief that the probability of an event is decreased when the event has occurred recently, even though the probability is objectively known to be independent across trials (Tversky & Kahneman, 1982). Related to this is the tendency of people in sequence generation tasks to include too many alternations of different results (such as heads and tails in the flipping of a coin) in comparison to what would theoretically be expected in a random process. Similarly, in perception tasks people tend to reject sequences with long runs of the same result (such as a long sequence of heads) and consider sequences with an excess of alternation of different results to be random (Falk, 1981; Falk & Konold, 1997). Comparable results are found in people's performance with two-dimensional tasks, in which clusters of points seem to prevent a distribution from being perceived as random. These results support the findings by Tversky and Kahneman (1982) that suggest people do not follow the principles of probability theory in judging the likelihood of uncertain events and apply several heuristics, such as local representativeness, where subjects evaluate the probability of an event by the degree of likeness to the properties of its parent population or the degree it reflects the features of the process by which it is generated. Other authors (e.g. Falk, 1981; and Falk & Konold, 1997) believe that individual consistency in people's performance with diverse tasks suggests underlying misconceptions about randomness.

3.2. Children's perception of randomness

From a didactic point of view, a crucial question is whether these biases and misconceptions are spontaneously acquired or whether they are a consequence of poor instruction in probability. Below, we outline a number of key research studies looking at children's and adolescents' conceptions of randomness, and their performance when faced with tasks requiring the generation or recognition of sequences of random results.

According to Piaget and Inhelder (1951), chance is due to the interference of a series of independent causes, and the 'non presence' of all the possible outcomes when there are only a few repetitions of an experiment. Each isolated case is indeterminate or unpredictable, but the set of possibilities may be found using combinatorial reasoning, thereby making the outcome predictable. The authors' notion of probability is based on the ratio between the number of possible ways for a particular case to occur and the number of all possible outcomes. This theory would suggest that chance and probability cannot be totally understood until combinatorial and proportional reasoning are developed what, for Piaget and Inhelder, does not happen until a child reaches the formal operations stage (12-14 years).

Piaget and Inhelder (1951) investigated children’s understanding of patterns in two-dimensional random distributions. They designed a piece of apparatus to simulate rain drops falling on paving stones. The desire for regularity appeared to dominate the young children’s predictions. When they were asked where the following rain drop would fall, children at stage 1 (6 to 9 years) allocated the rain drops in approximately equal numbers on each pavement square, thereby producing a uniform distribution. With older children, proportional reasoning begins to develop, and Piaget and Inhelder reported that such children tolerate more irregularity in the distribution. The authors believed that children understood the law of the large numbers, which explains the global regularity and the particular variability of each experiment simultaneously.

Fischbein and Gazit (1984) and Fischbein et al. (1991), have also documented children's difficulties in differentiating random and deterministic aspects, and their beliefs in the possibility of controlling random experiments. In contrast to the Piagetian view, these authors have suggested that even very young children display important intuitions and precursor concepts of randomness and consequently argue that it is not didactically sound to delay exploiting and building on these subjective intuitions until the formal operations stage is reached.

Moreover, Green’s (1983) findings, also contradicted Piaget and Inhelder’ theory. His investigations with 2930 children aged 11-16, using paper and pencil versions of piagetian tasks, showed that the percentage of children recognizing random or semi-random distributions actually decreased with age. In a second study with 1600 pupils aged 7 to 11 and 225 pupils aged 13 to 14 (Green, 1989, 1991), Green gave the children generation and recognition tasks related to a random sequence of heads and tails representing the results of flipping a fair coin. The study demonstrated that children were able to describe what was meant by equi-probable. However, they did not appear to understand the independence of the trials, and tended to produce series in which runs of the same result were too short compared to those that we would expect in a random process. In both studies, children based their decisions on the following properties of the sequences: results pattern, number of runs of the same result, frequencies of results, and unpredictability of random events. However, these properties were not always correctly associated to randomness or determinism.

Toohey (1995) repeated some of Green's studies with 75 11-15 year-old students and concluded that some children have only a *local perspective* of randomness, while that of other children is entirely global. The local perspective of randomness emphasized the spatial arrangement of the outcomes within each square, while the *global perspective* concentrates on the frequency distribution of outcomes.

Batanero and Serrano (1999) analysed the written responses of 277 secondary students in two different age groups (14 and 17 year-olds) to some test items taken from Green (1989, 1991) concerning the perception of randomness in sequences and two dimensional distributions. The authors also asked the students to justify their answers. Batanero and Serrano's results suggest that students' subjective meaning of randomness could parallel some interpretations that randomness has received throughout history. For example, when a student associated the lack of pattern to randomness, Batanero and Serrano suggested this view was consistent with the *complexity approach* to randomness described before. Other students showed a conception compatible with, the *classical, frequentist or subjective approach* to probability and randomness.

More recently, researchers have used computers to simulate random processes in order to discover children's understanding of randomness. Pratt (2000) analysed 10-11 year-old children's ideas of randomness as they worked in a computer environment and suggested these children showed the local and global perspective of randomness described by Toohey (1995). While the local perspective children's attention is mainly paid to the uncertainty of the next outcome and the ephemeral patterns in short sequences, in the global view the children were aware of the long term predictability of either the empirical distribution of outcomes (frequency of observed outcomes) or the theoretical distribution (expressing beliefs about the behaviour of the random generator, such as equally likelihood of different outcomes) (see also Johston-Wilder & Pratt, 2007).

3.3. Teachers' perception of randomness

In this section we summarise the scarce research focussed on teachers' perception of randomness, which suggests the need to develop specific training where teachers can increase their probabilistic knowledge for teaching.

Azcárate, Cardeñoso and Porlánd (1998) analysed the responses of 57 primary-school teachers to Konold et al.' (1991) questionnaire in order to analyse these teachers' conception of randomness. They also asked participants to list examples or random and non random phenomena and to describe the features they assigned to random phenomena. In general, participants showed a partial conception of randomness, which reflected, in most cases, causal argumentations and poor perception of random processes in everyday settings (beyond games of chance). Many participants considered a phenomena to be deterministic if they could identify some causes that could influence the phenomena apart pure chance (e.g. in meteorology). Other criteria to judge randomness included multiple possibilities or unpredictability of results.

The most relevant study with teachers is that by Chernoff (2009a & b). After a pilot study with 56 prospective teachers, Chernoff (2009a) analysed the responses and justifications given by 239 prospective mathematics teachers (163 elementary school teachers and 76 secondary school teachers) to a questionnaire consisting in comparative likelihood tasks. The questionnaire included several sequences of 5 trials of flipping a fair coin, in which the author fixed the ratio of heads to tails and varied the arrangement of outcomes. In order to show that responses that were assumed as incorrect in previous research could be derived from participants' subjective probabilistic thinking, Chernoff (2009b) analysed the justifications of 19 prospective teachers that apparently had incorrect perception of randomness. The result of his analysis suggested that these prospective teachers may be reasoning from three different interpretations of the sample space: (a) taking into account the switches from head to tail; (b) considering the longest run; and (c) considering the switches and longest run together. Consequently, their reasoning as regards randomness and their judgement of whether a sequence was random or not could be consistent with these views of the sample space; therefore their apparent incorrect responses were not due to lack of probabilistic reasoning, but to use of personal subjective probabilities.

In summary, research carried out with prospective teachers is scarce and suggest a poor perception of randomness as well as use of subjective probabilities. Below we summarise our own research in which we explore the teachers' capability to judge their own intuitions when analysing the data collected by themselves in a generation task and the possibility that part of the teachers show some naïve conceptions on randomness that parallel those held at different historic periods.

4. METHOD

Participants in our study were 208 prospective primary school teachers in the Faculty of

Education, University of Granada, Spain; in total 6 different groups (35-40 prospective teachers by group) took part in this research. All of these prospective teachers (in their second year of University) were following the same mathematics education course, using the same materials and doing the same practical activities. They all had followed a mathematics course, which included descriptive statistics and elementary probability the previous year.

The data were collected as a part of a formative activity, which is discussed in depth in Godino, Batanero, Roa and Wilhelmi (2008) as consisted of two sessions (90 minutes long each). The two main goals of the formative activity were: (a) assessing prospective teachers' conceptions of randomness; (b) confronting prospective teachers with their possible misconceptions on this concept.

In the first session (90 minutes long), the prospective teachers were given the statistical project "Check your intuitions about chance" (Godino et al, 2008) in which they were encouraged to carry out an experiment to decide whether the group had good intuitions on randomness or not. The experiment consisted of trying to write down apparent random results of flipping a coin 20 times (without really throwing the coin, just inventing the results) in such a way that other people would think the coin was flipped at random (simulated sequence). This is a classical generation task was similar to that used in Engel and Sedlmeier (2005)'s research.

Participants recorded the simulated sequences on a recording sheet. Afterwards they were asked to flip a fair coin 20 times and write the results on the same recording sheet (real sequence). At the end of the session, in order to confront these future teachers with their misconceptions, participants were given the data collected in their classroom. These data contained six statistical variables: number of heads, number of runs and length of the longest run for each of real and simulated sequences from each student. Results in the experiments are presented in Figures 1 to 3. Sample size for the data analyzed by the prospective teachers in each group were smaller (30-40 experiments per group), although the shape of the distribution and summaries for each variable were very close to those presented in Figures 1 to 3.

Teachers were asked to compare the variables collected from the real and simulated sequences, finish the analysis at home and write a report with a complete discussion of the project, including all the statistical graphs and procedures they used and their conclusions regarding the group's intuitions about randomness. Participants were given freedom to build other graphs or summaries in order to complete their reports. In a second session the reports were collected and the different solutions to the project given by the prospective teachers were collectively discussed in the classroom. In addition, a didactical analysis was carried out in order to reflect on the statistical knowledge needed to solve the project and the pedagogical content knowledge involved in teaching statistics in primary school through project work.

5. RESULTS AND DISCUSSION

In order to assess prospective teachers' conceptions of randomness we first analysed the number of heads, number of runs and longest run in each of the simulated and real sequences in the data collected by the prospective teachers in their experiments. In the following, the data collected by the six groups taking part in the study (n=208) will be analysed together, although similar results were found in each of the six subsamples.

5.1. Perception of the Binomial distribution

The theoretical distribution for the number of heads in 20 trials can be modelled by the Binomial distribution $B(n, p)$, where n=20 and p= 0.5. The expectation and variance for this distribution is $\mu = np = 10$; $Var = npq = 5$. The empirical distributions for the number of heads in the real and simulated sequences in the experiments carried out by the participants in the study are presented in Figure 1 and the summary statistics in Table 1.

We can observe that participants produced "heads" and "tails" in about equal numbers, and the majority of the students were close to the theoretically correct expected value of 10. There was no significant preference for "heads" over "tails" or vice versa, in agreement with previous research (e.g. Wagenaar, 1972, Green, 1991; Falk & Konold, 1997; Engel & Sedlmeier, 2005).

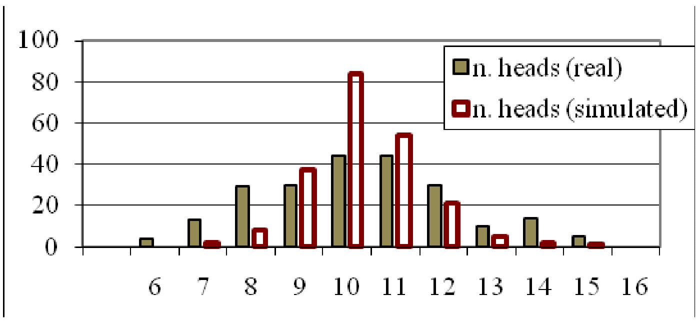

Figure 1: Distribution for Number of heads.

Table 1: Summary statistics for Number of heads, longest run and number of runs

	Number of heads		Longest run		Number of runs	
	Real	Simulated	Real	Simulated	Real	Simulated
Mean	10.45	10.29	4.35	3.32	10.10	10.78
Mode	10; 11	10	4	3	10; 11	12
Median	10	10	4	3	10	12
Std. Deviation	2.05	1.22	1.6	1.12	2.9	2.8
Range	11	8	10	11	14	13

Results show a good perception of the expected number, median and mode of the binomial distribution (number of heads in 20 flipping of a coin) as it is shown in the mode, median and average number of heads in the simulated sequences, which are close to the theoretical value $np=10$ and in the non significant difference in the t- test of difference on averages between real and simulated sequences (t = -1.00; p=0.31). The standard deviation in the simulated sequences was, however, almost half the theoretical value and the differences were statistically significant in the F- test (F=2.83, p=0.001).

5.2. Perception of independence

Even if the concept of independence is very easy to define mathematically by the property that a joint probability of two events is the product of the two single event probabilities, its application is difficult for many people. Perception of independence was poor in our sample, as prospective teachers produced in average shorter runs and higher number of runs than expected in a random process (according to Engel and Sedlmeier, 2005, the expected number of runs in a coin flipping sequence of length 20 is 10.5, and the expected longest run 4.33). This is consistent with previous research, in which, a majority of participants of different ages -examined under various methods- identify randomness with an excess of alternations between different outcomes (e.g., Wagenaar, 1972; Green, 1991, Engel & Sedlmeier, 2005) and whereas the expected probability of alternation in a random binary sequence is .5, people’s average subjective preference is a probability of .6 or .7 in generation tasks (Falk & Konold, 1997). Moreover, according Tversky and Kahneman’s (1982) representativeness heuristic, people judge that a sequence is random if it is representative of its parent population; therefore the sequence comprise about equal numbers of heads and tails and display an irregular order, not only globally, but also locally. This result is visible in Figures 2 and 3 and in the t- tests of differences in averages that was statistically significant for both variables (t = -7.76; p = 0.001 for the longest run; t=2.48; p= 0.01 for the number of runs).

Some teachers recognized this difference in their reports and consequently they correctly considered this was an indication of an incomplete perception of randomness: "In the simulated sequence we tend to produce many short runs" (AB). On the contrary, misconceptions of independence, was also observed in some written reports by other participants, who rejected the sequence as random because some runs were longer than what they believed should be expected in a random sequence: "Some students cheated and invented their sequences, since they produced too many successive heads or tails to be random" (EA).

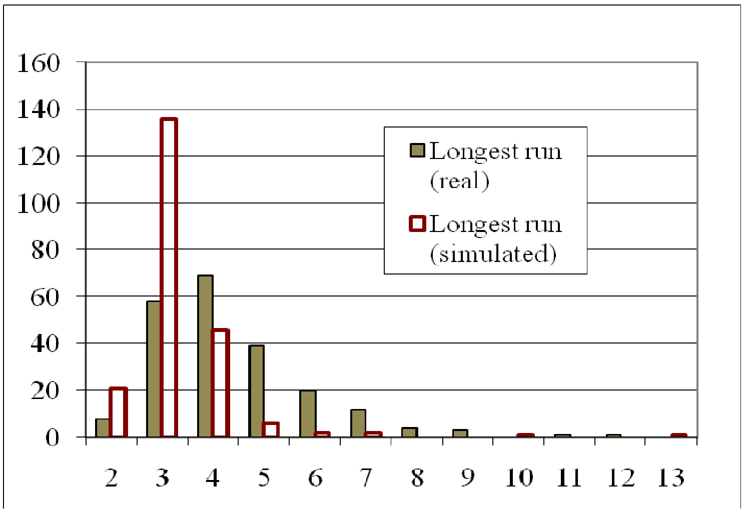

Figure 2: Distribution and summary statistics for the longest run.

5.3. *Perception of variation*

Variability is omnipresent throughout the statistical enquiry cycle and is fundamental to statistical thinking (Wild & Pfannkuch, 1999). However, the prospective teachers in our study produced sequences with little variation in the length of runs (a wide majority produced a longest run with only 3 similar outcomes). This is visible in Figure 2 and in the F test, that was statistically significant (F=2.06; p=0.0001). However, perception of variation was good as regards the number of runs (F=1.07; p=0.6; no significant). This result is reasonable, since, participants were not committed, as a group, to reproduce the sampling distribution of the proportions of heads. Consequently, their deviation from the expected variability in the distribution of all the random sequences is a result of in each individual's matching with the expected proportion in his/her sequence produced

(Falk, Falk, & Ayton, 2009). However, in the next sections we can show how the perception of variation in these prospective teachers was poor, since only a few of them made reference to variation in their reports as an important feature of random processes. Anyway, part of the teachers perceived variation in the data, and even more, justified randomness basing on this variation "There is more variety in the random sequence. This is pretty logical, since these results were obtained by a random experiment that involved chance" (NC); " In the number of heads there is a difference, … since when we made up the data (in the simulated sequence) results were more even, but in the real sequence these results are more uneven, since they are due to chance" (IE).

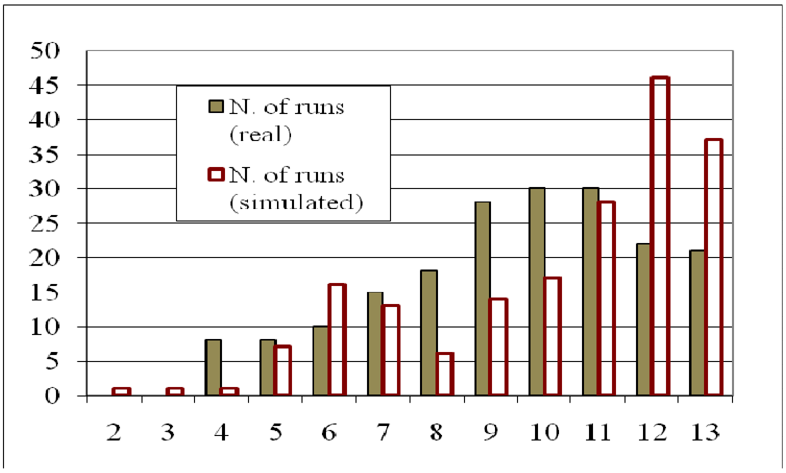

Figure 3: Distribution and summary statistics for Number of runs

5.4. Teachers' analyses of their own intuitions

All the above results reproduced those obtained by Green (1991) and Batanero and Serrano (1999) with secondary school students, which is reasonable, because the statistics training that Spanish prospective primary teachers receive is reduced to their study of statistics along secondary education.

As we have previously explained, after performing the experiment, participants in the study were given a data sheet with the data recorded in the classroom and were asked to analyse these data and conclude on their own intuitions in a written report. While the experiment proposed to the prospective teachers was a *generation task*, the production of the report includes a *recognition task*, since the subjects were asked to recognise the features of randomness in the data. According to Wagenaar (1972) and Falk and Konold (1997) these recognition tasks are more appropriate for assessing subjective perception of randomness since a person could have a good perception of randomness in spite of being unable to reproduce it.

Most participants represented the data using different graphs that varied in complexity and that have been analyzed in a previous paper (Batanero, Arteaga & Ruiz, 2010). Many of them also computed averages (mean, median or modes) and variation parameters (range or standard deviation). Although the graphs and summaries produced were generally correct and many of them used Excel to produce a variety of statistical analysis, few of them got a complete correct conclusion about the group intuitions (Table 2).

Table 2: Frequency (percent) of prospective teachers' conclusions.

Conclusion about the intuitions (n=200)		Correct	Incorrect	No conclusion
Perceiving the expected value	Number of heads	32 (16.0)	114 (57.0)	54 (27.0)
Perceiving independence	Expected longest run	6 (3.0)	87 (43.5)	107(53.5)
	Expected number of runs	9 (4.5)	100 (50.0)	91 (45.5)
Perceiving variation	Number of heads	25 (12.5)	121 (60.5)	54 (27.0)
	Longest run	8 (4.0)	85 (42.5)	107 (53.5)
	Number of runs	11 (5.5)	98 (49.0)	91 (45.5)

Only a small number of prospective teachers explicitly were able to get a correct conclusion for the class' perception of expected values and variation in the different variables. Those who succeeded completed an informal inference process and were able to relate the empirical data to the problem posed in the project (the intuitions in the group), completing a modelling cycle (posing a problem; collecting data; using mathematical models, building with the model and interpreting the results as regards the problem posed). They concluded that the group had good intuitions as regards the average number of heads and, at the same time, that the perception of variability in this random variable was poor. An example is given below (similar responses were obtained for the other variables):

> *As regards the number of heads, the intuitions in the classroom were very close to what happen in reality; but not complete. The means in the real and simulate sequences are very close; the medians and modes are identical; however the standard deviations suggest the spread in both distributions is different (CG).*

Other participants reached a partial conclusion, being able only to conclude about the central tendency or about the spread in the data. For example, the following teacher was able to perceive the similarity of means, but did not realize that the variation in the two sequences was quite different:

Observing the table, I think that my colleagues have good intuition; since the most frequent values for the number of heads in the simulated sequence coincide with those in the real sequence; 10 and 11 are the most frequent values in both cases. The means are close to 10 in both sequences; therefore the intuitions are good (TG).

In the next example, the student concluded about the difference in spread, but was unable to relate this result to the students' intuitions. He argued that the different students had similar intuitions but did not relate these intuitions to the results from the generation task experiment and did not analyse the central tendency:

Intuitions are very similar in the different students; there is no much irregularity. But when we compare to the real sequences, we realise the graph is more irregular. In the real sequence the maximum number of heads is 16 and the minimum 7; however in the simulate graph the maximum is 13 and the minimum 8; the range is smaller than that of real flipping of the coin (MM).

The remaining students either were unable to conclude or reached an incorrect conclusion. Part of them could not connect the results of their statistical analyses to the students' intuitions; that is, they did not see the implications of the results provided by the mathematical model to the solution of the problem posed (assessing the students' intuitions). An example is given below:

When I compare the data I realise that many students coincided in their results. In spite of this, I still think there is mere chance; since in the simulated sequences we invented the results (EL).

Other students connected the mathematical work to the problem situation, but they failed in their conclusions because they made an incorrect interpretation of the question posed by the lecturer. They assumed a good intuition would mean getting exactly the same results in the simulated and real sequences. In the next example, the prospective teacher shows a correct conception of randomness (randomness means lack of prediction in the short term) mixed with an incorrect conception: Instead of comparing the two distributions, he compared the students one by one and tried to assess the number of coincidences between the number of heads in the real and simulated sequences for each student:

Studying the graphs the prediction of the group was not too bad. A game of chance is unpredictable; but when we add the number of students who guessed the result, the number of guesses is higher than the number of failures (students who were very far from the real result) (LG).

Although the binomial distribution (number of heads) was more intuitive for the prospective teachers, still the number of correct conclusions as regards the perception of the binomial distribution was very scarce. These results suggest that these prospective teachers did not only hold some misconceptions of randomness, but they also were unconscious of their misconceptions and were unable to recognise these misconceptions when confronted with the statistical data collected in the experiments.

5.5. Further analysis of pre-service teachers' conceptions

Another interesting point is that some of these teachers justified their wrong conclusions as regards some of the variables in the project by making explicit their own views of randomness that reproduced some of the conceptions described by Batanero and Serrano (1999) in secondary school students. Below we present examples of these conceptions, many of which are partly correct, but are incomplete and parallel some view of randomness that were described in Section 2.

Randomness and causality: The principle of cause and effect is deeply rooted in human experience and there is a tendency to relate randomness to causality. A possible way to relate randomness and cause is to think that what appears as randomness to a person's limited mind could well be explained by a extremely complex causal system that is unknown to the person, who is incapable of perceive the causes for the phenomena. Another view is considering that causality is an illusion and that all events are actually random. Some participants assumed "Chance" as the cause of random phenomena, as was apparent in some participants' responses:

> *"We define random experiments as a consequence of chance"* (SG).
>
> *"Some people think that the number of possibilities of getting a tail is 50%; but, although in this experiment results were very close to what we expected, this result was due to chance or good luck, because we cannot deduce this result, since it depends on chance"* (NG).

Randomness as unpredictability: A common feature in different conceptions of randomness is unpredictability: the fact that we cannot predict a future event based on a past outcome (Bennet, 1998). Some participants expressed this idea in their responses, since they assumed they could not reach a conclusion about the differences in distribution for the number of runs, number of heads or longest run because anything might happen in a random process. In these responses the "outcome approach" (Konold, 1989) that is the interpretation of probability questions in a non probabilistic way may also operate:

> *"I want note that it is impossible to make a prediction of results since in this type of experiment any result is unpredictable" (AA).*
>
> *"Results of random experiments cannot be predicted until they happen" (SG).*

"My final conclusion is that in experiences related to chance, there are more or less likely events, but it is impossible to predict the exact result (MN).

Randomness as equiprobability. A few subjects connected randomness as equiprobability (in the classical approach to this concept) and stated that any result was possible, since the experiment was random and consequently there was equal probability for each result. This view was parallel to the classical conception of probability where an event as random only in case there is the same probability for this event and for any other possible event in the experiment.

"The probability for head and tails is the same, and therefore, in 20 throwing there is the same probability to obtain 20 heads, 20 tails or any possible combination of heads and tails" (EB).

"You know that there is 50% possibility to get head and another 50% possibility to get tail because there are only two different results. Consequently each student got a different result" (CG).

Lack of pattern or lack of order. Some participants associated randomness to lack of model or lack of pattern, a view close to Von Mises's (1952/1928) modelling of randomness, where a sequence of outcomes is random whenever it is impossible to get an algorithm that serves to produce the sequence. In particular some students rejected the idea that a random sequence could appear as ordered. In spite that this view is partly correct, in fact in the analysis of the project data a variety of models appeared, such as the Binomial distribution, the distribution of runs, or the geometrical distribution. These models arise in any random sequence and consequently, randomness should also been interpreted as multiplicity of models.

"You cannot find a pattern, as it is random" (BS).

"We do not think it is possible to get a sequence so well ordered as [C,C,C,C,C,+,+,+,+,+], since our intuition led us to alternate between head and tails and to produce unordered sequences, such as, for example, [C,C,+,C,+,+,C,+,C,+,C,+]" (RE).

"It is not random, it is too ordered" (SG).

Randomess and control. A few prospective teachers described randomness as something that cannot be controlled, a vision common until the Middle Ages according Bennett (1998):

"Despite our inability to control randomness, we got equal number of heads and tails" (AG).

"These predictions are really accurate. Although the simulation and reality are very close we should take into account that randomness can never be controlled 100%, even if you have much knowledge of the situation" (EC).

On the contrary, it was also observed the illusion of control (Langer, 1975), defined as an expectancy of a personal success probability inappropriately higher than the objective probability would warrant. Consequently of this belief some participants believed they could predict or control the result of the experiment. For example, one participant classified all his classmates according their capacity to predict the results:

> *"Only 21.7% students guessed the number of heads in the experiment; 13% were very close because they had an error of (± 1); the remaining students failed in their prediction"* (LG).

6. DISCUSSION AND IMPLICATIONS FOR TRAINING TEACHERS

As stated by Bar-Hillel and Wagenaar (1991), randomness is a concept which somehow eludes satisfactory definition; although theoretically randomness is a property of a generating (random) process, in practice we can only infer indirectly randomness from properties of the generator's outcomes. In addition, although expressions such as 'random experiments', 'random number', 'random variable', 'random event', 'randomness' frequently appear in daily language as well as in school textbooks, the meaning of randomness in not clarified in school textbooks, thus increasing the likelihood of students having difficulties at this point (Batanero, Serrano, & Green, 1998). It is not then surprising that, given such complexity, the prospective teachers in our sample showed different misconceptions of randomness, in both the sequences they produced and in their reports when analysing their data.

However, understanding randomness is an essential step in learning probability and therefore it is essential that prospective teachers acquire a sound understanding of this concept along their initial education if we want them succeed in their future teaching of probability. As stated by Ball Lubienski, & Mewborn (2001, p. 453), some of the activities in which teachers regularly engage, such as "figuring out what students know; choosing and managing representations of mathematical ideas; selecting and modifying textbooks; deciding among alternative courses of action" involve mathematical reasoning and thinking. Consequently, teachers' instructional decisions as regards the teaching of probability are dependent on the teacher's probabilistic knowledge.

Prospective teachers in our sample showed a mixture of correct and incorrect beliefs concerning randomness. On a hand, their perception of averages in the binomial distribution was good, since as stated by Falk, Falk, & Ayton (2009) one major characteristics of a sequence of coin tosses is the equiprobability of the two outcomes and equal proportions of the two symbol types are more likely to be obtained by chance than any other result.

However, at the same time, misconceptions related to variation, and independence, as well associating randomness with ignorance, or trying to control randomness, also appear in the sample. This is cause for concern when paired with evidence that prospective mathematics teachers in our sample may have a weak understanding of randomness and present different biases that could be transmitted to their future students.

These prospective teachers in our research were asked to solve a problem and complete a modelling cycle. According to Chaput, Girard & Henry (2011), modelling consists of describing an extra- mathematical problem in usual language and building up an experimental protocol in order to carry out an experiment (in this research the problem consisted in checking the intuitions on randomness and a particular experiment was chosen to get data on the teachers' intuitions). This description leads to setting some hypotheses which are intended to simplify the situation (in the example, the length of the sequences to be produced was fixed and the equiprobability of heads and tails in the coin was assumed).

Next, the second step of the modelling process is translating the problem and the working hypotheses into a mathematical model in such a way that working with the model produces some possible solution to the initial problem. The teachers translated the question (what conceptions they had on randomness) and the working hypotheses to statistical terms (they compared three pairs of distributions: the number of heads, number of runs and length of the longest run in both sequences and for the whole classroom). Consequently participants in our sample built and worked with different statistical models (each student chose and produced particular graphs, tables or statistical summaries to compare these pairs of distributions).

The third and final step consists of interpreting the mathematical results and relate these results to reality, in such a way that they produce some answers to the original problem. Although the majority of participants in our research correctly completed steps 1 and 2 in the modelling cycle, few of them were capable to translate the statistical results they got to a response about what the intuitions of the classroom on randomness were like. That is, few of them could understand what the statistical results indicated about the intuitions in the group and therefore, these prospective teachers failed to complete the last part of the modelling process.

Dantal (1997) suggests that in our classroom, we concentrate in step 2 ("the real mathematics) in the modelling cycle), since this is the easiest part to teach to our students. However all the steps are equally relevant for modelling and in learning mathematics, if we want our students understand and appreciate the usefulness of mathematics. It is therefore very important that teachers' educators develop the prospective teachers' ability to model in probability and the capacity to learn from data if we want succeed in implementing statistics education at school level.

Considering the importance and difficulty of the topic, the question remains about how to make randomness understandable by prospective teachers. Possibly this understanding should develop gradually, by experiments and simulations, starting with concrete materials and moving later to computer simulations and observation of randomness in demographic or social phenomena. As stated by Batanero, Serrano, and Green (1998), it is important that prospective teachers understand that, in randomness apparent disorder, a multitude of global regularities can be discovered. These regularities allow us to study random phenomena using the theory of probability.

Finally we suggest the usefulness of working with activities similar to the one described in this report to help prospective teachers make the conceptions about randomness and probability explicit. In order to overcome possible misconceptions, after working with these activities, it is important to continue the formative cycle with a didactical analysis of the situation. In our experience, in the second session the correct and incorrect solutions to the project "checking your intuitions on randomness" were debated and the different conceptions of randomness explicit in the teachers' responses were discussed.

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REFERENCES

Ayer, A. J. (1974). El azar [Chance]. In Kline, M. (Ed.), *Matemáticas en el mundo moderno* (pp. 172-171). Barcelona: Blume.

Azcárate, P., Cardeñoso, J. M., & Porlánd, R. (1998). Concepciones de futuros profesores de educación primaria sobre la noción de aleatoriedad [Prospective primary school teachers' conceptions on the notion of randomness]. *Enseñanza de las Ciencias, 16*(1), 85-97.

Ball, D.L., Lubienski, S.T., & Mewborn, D.S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V.

Richardson (Ed.), *Handbook of research on teaching* (pp. 433-456). Washington, DC: American Educational Research Association.

Bar-Hillel, M., & Wagenaar, W. A. (1991). The perception of randomness. *Advances in Applied Mathematics 12*, 428-454.

Batanero, C., Arteaga, P., & Ruiz, B. (2010). Statistical graphs produced by prospective teachers in comparing two distributions. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth European Conference of Research in Mathematics Education, CERME-6 Conference*. Lyon, France: ERME. Online: www.inrp.fr/editions/editions-electroniques/cerme6/.

Batanero, C. & Díaz, C. (2012). Training teachers to teach probability: Reflections and challenges. *Chilean Journal of Statistics 3*(1), 3-13.

Batanero, C., Godino, J. D., & Roa, R. (2004). Training teachers to teach probability. *Journal of Statistics Education* 12. On line: www.amstat.org/publications/jse/.

Batanero, C., Henry, M., & Parzysz, B. (2005). The nature of chance and probability. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 15-37). New York: Springer.

Batanero, C., & Serrano, L. (1999).The meaning of randomness for secondary school students. *Journal for Research in Mathematics Education 30*(5), 558-567.

Batanero, C., Serrano, L., & Green, D. R. (1998). Randomness, its meanings and implications for teaching probability. *International Journal of Mathematics Education in Science and Technology, 29* (1), 113-123.

Bennett, D. J. (1998). *Randomness.* Cambridge, MA: Harvard University Press.

Borovcnick, M. (2011). Strengthening the role of probability within statistics curricula. In C. Batanero, G. Burrill, & C. Reading (Eds.) (2011). *Teaching Statistics in School Mathematics- Challenges for Teaching and Teacher Education. A Joint ICMI/IASE Study* (pp. 71-83). New York: Springer.

Borovcnick, M. (2012). Multiple perspectives on the concept of conditional probability. *AIEM. Avances de Investigación en Educación Matemática 2*, 5 – 27.

Chadjipadelis, T., Meletiou, M., & Paparistodemou, E. (2010). Statistics teacher of the new era: another specialized mathematician or a totally different person? In C. Reading (Ed.), *Proceedings of the Eight International Conference on Teaching Statistics.* Lubjana, Slovenia: International Association for Statistical Education. Online: www.stat.auckland.ac.nz/~iase/publications/icots8.

Chaput, B., Girard, J. C., & Henry, M. (2011). Modeling and simulations in statistics education. In C. Batanero, G. Burrill, & C. Reading (Eds.) (2011). *Teaching Statistics in School Mathematics- Challenges for Teaching and Teacher Education. A Joint ICMI/IASE Study* (pp. 85-95-83). New York: Springer.

Chernoff, E. (2009a). *Subjective probabilities derived from the perceived randomness of sequences of outcomes.* Unpublished Ph. D. Simon Fraser University.

Chernoff, E. (2009b). Sample space partitions: An investigative lens. *Journal of Mathematical Behavior, 28,* 19–29.

Dantal, B. (1997). Les enjeux de la modélisation en probabilité [The betting of modelling in probability]. In M. Henry (Coord.), *Enseigner les probabilités au lycée* (pp. 57-59). Reims: Commission Inter-IREM.

Engel, J. & Sedlmeier, P. (2005). On middle-school students' comprehension of randomness and chance variability in data. *Zentralblatt fur Didaltik der Mathematik, 37*(3), 168-177.

Falk, R. (1981). The perception of randomness. In C. Laborde (Ed.*), Proceedings of the Fifth International Conference for the Psychology of Mathematics Education* (pp. 222-229). University of Grenoble.

Falk, R., Falk, R. & Ayton, P. (2009). Subjective patterns of randomness and choice: some consequences of collective responses. *Journal of Experimental Psychology: Human Perception and Performance, 35*(1), 203–224.

Falk, R., & Konold, C. (1997). Making sense of randomness: Implicit encoding as a basis for judgment. *Psychological Review*, *104*, 301-318.

Fine (1973). *Theories of probability. An examination of foundations*. London: Academic Press.

Fischbein, E., & Gazit, A. (1984). Does the teaching of probability improve probabilistic intuitions? *Educational Studies in Mathematics, 15 (1),* 1-24.

Fischbein, E., Nello, M. S., & Marino, M. S. (1991). Factors affecting probabilistic judgements in children and adolescents. *Educational Studies in Mathematics*, *22*, 523-549.

Franklin, C., Kader, G., Mewborn, D.S., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2005). *A curriculum framework for k-12 statistics education. GAISE report.* American Statistical Association. On line: www.amstat.org/education/gaise/.

Franklin, C., & Mewborn, D. (2006). The statistical education of PreK-12 teachers: A shared responsibility. In G. Burrill (Ed.), *NCTM 2006 yearbook: thinking and* *reasoning with data and chance* (pp. 335-344). Reston, VA: NCTM.

Gal, I. (2005). Towards “probability literacy” for all citizens: building blocks and instructional dilemas. In G. Jones (Ed.), *Exploring probability in schools: challenges for teaching and learning* (pp. 39-63). New York: Springer.

Godino, J. D., Batanero, C., Roa, R., & Wilhelmi, M. R. (2008). Assessing and developing pedagogical content and statistical knowledge of primary school teachers through project work. In C. Batanero, G. Burrill, C. Reading & A. Rossman (2008). *Joint ICMI/IASE Study: Teaching Statistics in School Mathematics. Challenges for Teaching and Teacher Education. Proceedings of the ICMI Study 18 and 2008 IASE Round Table Conference*. Monterrey: ICMI and IASE. Online: www.stat.auckland.ac.nz/~iase/publications.

Green, D. R. (1983). A survey of probabilistic concepts in 3000 students aged 11-16 years. In D.R. Grey et al. (Eds.), *Proceedings of the First International Conference on Teaching Statistics* (v.2, pp. 766-783). University of Sheffield.

Green, D. R. (1989). Schools students' understanding of randomness. In R. Morris (Ed.), *Studies in Mathematics Education.* v.7*: The Teaching of Statistics* (pp. 27-39). Paris: Unesco.

Green, D. R. (1991). *A longitudinal study of pupil's probability concepts.* Technical Report ME90/91. Loughborough University of Technology.

Hawkins, A., Jolliffe, F., & Glickman, L. (1991). *Teaching statistical concepts.* London: Longman.

Harten, G., & Steinbring, H. (1983). Randomness and stochastic independence. On the relationship between intuitive and mathematical definition. In R. W. Scholz (Ed.), *Decision making under uncertainty* (pp. 363- 373). Amsterdan: Reidel.

Jacobbe, T. (2010). Preparing elementary school teachers to teach statistics: an international dilemma. In C. Reading (Ed.), *Proceedings of the Eight International Conference on Teaching Statistics.* Lubjana, Slovenia: International Association for Statistical Education. Online: www.stat.auckland.ac.nz/~iase/publications/icots8.

Jones, G. (2005). Introduction. In G. Jones (Ed.), *Exploring probability in school: challenges for teaching and learning* (pp. 1-12). New York: Springer.

Johnston-Wilder, P., & Pratt, D. (2007). The relationship between local and global perspectives on randomness. In D. Pitta & G. Philippou (Eds.). *Proceedings of the Fifth European Conference on Research in Mathematics Education.* Lamarca, Cyprus: ERME. Online: www.erme.unito.it/CERME5b/

Konold, C. (1989). Informal conceptions of probability. *Cognition and Instruction, 6*, 59-98.

Konold, C., Lohmeier, J., Pollatsek, A., Well, A.D., Falk, R., & Lipson, A. (1991). Novice Views on Randomness. In R.G. Underhill (Ed.), *Proceedings of the Thirteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 167-173) Blacksburg: Virginia Polytechnic Institute and State University.

Kyburg, H. E. (1974). *The logical foundations of statistical inference.* Boston: Reidel.

Lahanier- Reuter, D. (1999). *Conceptions du hasard et enseignement des probabilités statistiques* [Conceptions of chance and teaching of statistics and probability]. París: Presses Universitaires de France.

Liu, Y., & Thompson, P. (2002). Randomness: Rethinking the foundation of probability. In D. Mewborn, P. Sztajn, E. White, H. Wiegel, R. Bryant, & K. Nooney (Eds.), *Proceedings of the Twenty Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Athens, GA* (Vol. 3, pp. 1331–1334). Columbus, OH: Eric Clearinghouse.

Langer, E. J. (1975). The illusion of control. *Journal of Personality and Social Psychology, 32,* 311-328.

Laplace P. S. (1995). *Théorie analytique des probabilités* (Analytical theory of probabilities). Paris: Jacques Gabay. (Original work published 1814).

Lecoutre, M. P., Rovira K., Lecoutre, B., & Poitevineau, J. (2006). People's intuitions about randomness and probability: an empirical study. *Statistics Education Research Journal*, 5, 20–35. Online: www.stat.auckland.ac.nz/~iase/serj/.

Ma, L. P. (1999). Knowing and teaching elementary mathematics. Mahwah, NJ: Lawrence Erlbaum.

MEC (2006). *Real Decreto 1513/2006, de 7 de diciembre, por el que se establecen las enseñanzas mínimas correspondientes a la Educación primaria.* Madrid: Author.

Ministério da Educação. (1997). *Parâmetros curriculares nacionais: Matemática* [National curricular parameters: Mathematics]. Brasilia, Brazil: Author.

Mises, R. Von (1952). Probability, statistics and truth. London: William Hodge and company [Original work published in 1928].

NCTM (2000). Principles and standards for school mathematics. Reston, VA; NCTM. Online: standards.nctm.org/ .

Nickerson, R. S. (2002). The production and perception of randomness. *Psychological Review*, *109,* 330-357.

Piaget, J., & Inhelder, B. (1951). *La genése de l'idée de hasard chez l'enfant.* [The origin of the idea of chance in children]. Paris: Presses Universitaires de France.

Stohl, H. (2005). Probability in teacher education and development. In G. Jones (Ed), *Exploring probability in schools: Challenges for teaching and learning* (pp. 345-366). New York: Springer.

Poincaré, H. (1936). Chance. *Journal of the American Statistical Association,* 31, 10-30.

Pratt, D. (2000). Making sense of the total of two dice. *Journal for Research in Mathematics Education, 31*(5), 602-626.

Toohey, P. G. (1995). *Adolescent perceptions of the concept of randomness.* Unpublished Master Thesis. University of Adelaide.

Tversky, A. y Kahneman, D. (1982). Judgement of and by representativeness. In D. Kahneman, P. Slovic y A. Tversky (Eds.), *Judgement under uncertainty: Heuristics and biases* (pp. 84 – 98). Cambridge, MA: Cambridge University Press.

Wagenaar, W. A. (1972). Generation of random sequences by human subjects: A critical survey of literature. *Psychological Bulletin, 77*, 65-72.

Wild, C. J., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review,* 67(3), 223-265.

Zabell. S. L. (1992). The quest for randomness and statistical applications. In F. Gordon and S. Gordon (Eds.), *Statistics for the XXI Century* (pp. 139-150). The Mathematical Association of America.