# PROSPECTIVE TEACHERS SOLUTIONS TO A PROBABILITY PROBLEM IN A SAMPLING CONTEXT

Emilse Gómez-Torres[1], Carmen Díaz[2] and J. Miguel Contreras[3]

[1]Universidad Nacional de Colombia, [2]University of Huelva (Spain), [3]Universitty of Granada (Spain)

13th International Congress on Mathematical Education, Hamburg, 24-31 July 2016

*In this paper we analyse the knowledge of sampling showed by 157 prospective primary school teachers when solving a probability task. We observed correct intuitions about estimation, confusion between sample and population, and failures in proportional reasoning. Finally, we describe a didactical activity which served to improve these prospective teachers' knowledge.*

**Keywords**: *sampling, common mathematical knowledge, assessment, prospective teachers, primary education.*

## INTRODUCTION

Probability take part in the primary school mathematics curriculum in many countries and specifically in Spain (MEC, 2006; MECD, 2014). These proposals suggest a change in the teaching approach, from only a classical meaning of probability to a frequentist view, when children perform predictions about the behaviour of simple experiments, carry out experiments and simulations, collect and analyse data from these experiments and confront their data with their predictions. Since probability was only included in 2006 in the Spanish primary school curricula, many prospective primary school teachers have not received a specific education in probability. This situation is common in other countries, as suggested in several chapters from the ICMI and IASE Joint Study book (Batanero, Burrill, & Reading, 2011). Formative actions directed to empower these teachers in both their content and pedagogical content knowledge of probability should be based on previous assessment of the teachers' formative needs.

The research described in this paper was aimed to assess primary school prospective teachers' content knowledge of some ideas needed for teaching probability with a frequentist approach. More specifically, we consider the ideas of population, frequency, theoretical value of probability and estimating probability from frequencies in repeated trials. All these ideas are included in the curricular guidelines for primary school level, and, consequently should be considered part of prospective primary school teachers' CCK (Common Content Knowledge; Hill, Schilling & Ball, 2004; Hill, Ball, & Schilling, 2008). In the next sections we summarise the study background and methodology, and the responses by 157 prospective primary school teachers to an open probability problem in sampling context that served to assess their knowledge of these ideas. We finish with some conclusions and suggestions to better educate prospective primary school teachers.

## BACKGROUND

In our research we base on the construct "mathematical knowledge for teaching" (MKT) proposed by Ball and her colleagues (Ball, Lubienski, & Mewborn, 2001; Ball, Thames, & Phelps, 2008), which include *content knowledge* and *pedagogical content knowledg*e. The authors divide content knowledge in three different categories, the first of which is what is assessed in our research:

- *Common content knowledge (CCK)* is the mathematical knowledge shared with the students, which includes basic skills and general knowledge of the subject that teachers have to teach to their students.
- *Specialized content knowledge (SCK),* the particular way in which teachers master the subject matter; this type of knowledge supports the teacher activity in planning and handling classes and in assessing students' knowledge.
- *Knowledge in the mathematical horizon:* includes a more complete knowledge than the knowledge the teacher teaches; for example, the content the student need in the courses which follow.

Research related to understanding the idea of sampling started with the heuristics and biases programme, summarized in Kahneman, Slovic, and Tversky (1982), which made popular the idea of heuristics. According to this research, few people follow the normative mathematical rules to reason about sampling. Instead they commit systematic errors in decision making. For example, according to the *belief in the law of small number*s, people expect that small samples should exactly reflect all the characteristics in the population distribution.

A few of these studies were carried out with prospective teachers. Azcárate (1995) analysed the responses by 57 prospective teachers to two items related to variability in small and large samples and reported that 67% of participants showed the representativeness heuristics; 56% of them recognized the difference between probability and observed frequency, and 39% expected an exact convergence. Similar results were observed by Serrano (1996) in interviews to 20 prospective teachers. Mohamed (2012) applied a typical task in frequentist probability taken from Green (1983) to 283 prospective teachers, where participants were asked to predict the frequency in which 100 thumbtacks would landed up; only 31% gave the correct answer while 62% reasoned according to the equiprobability bias. A new version of the same task was proposed by Gómez, Batanero and Contreras (2014) to 157 prospective teaches; most of them provided accurate estimation is for the theoretical proportion; however, 40% assumed equiprobability of results, and only 45% understood the variability of results.

In summary, this research suggest that prospective teachers fund difficulties in concepts linked to the idea of sampling. In this paper we try to complement this previous research with a different task taken from a questionnaire given by Fischbein and Gazit (1984) to 285 10 -13 years old children, and in which they observed misunderstanding of inverse proportionality and extended use of additive strategies. This task was adapted to this research, as will be clarified in the next section.

## METHOD

The sample consisted in 157 prospective primary school teachers, following a mathematics education course at the Faculty of Education, University of Granada, Spain. All of them had studied simple and conditional probability in the previous academic year and along secondary school. The data were taken as part of a practical activity in the course in which participants solved and analysed different probability problems. Three sessions (one hour long each, with an interval of a week between sessions) were spent in the activity. In the first session, the prospective teachers solved individually the problem proposed and explained in detail their solutions in a response form. In the second session the teacher educator organised a debate where the different solutions to each problem were discussed until an agreement about the correct solution was reached. In the third session, students in pairs carried out a didactical analysis of the problems proposed. This analysis included the identification of the mathematical content of the problems and of possible difficulties of children when solving the problem, as well as some simulations activities.

To assess the prospective teachers' knowledge we used the methodology proposed by Godino (2009), where the researcher selects a mathematical task whose solution implies using the main topics of content, or competence to be evaluated. As we were interested in Common Content Knowledge, the task proposed to participants (Figure 1) was adapted from previous research by Fischbein and Gazit (1984) with primary school children; so that the task and the related knowledge is valid for primary school level.

The context of this item is estimation with capture-recapture method, used in moving populations and in sampling populations of animals. We are interested in the theoretical probability $p$ that a fish is marked; this probability (or proportion in the population) should be estimated from the frequency obtained in the sample of 250 fish; that is: $\hat{p} \approx \frac{Number\ of\ fish\ marked\ in\ the\ sample}{Number\ of\ fish\ in\ the\ sample} = \frac{25}{250}$ = 0.1. This estimate is reliable, given the large sample size.

On a farm there is a fishing pond. The owner wants to know how many fish there are in the pond. He took out 200 fish and marked each of them, with a coloured sign. He put the marked fish back in the pond and let them get mixed with the others. On the second day, he took out 250 fish in a random manner, and found that, among them, 25 were marked.

a. What is the approximate number of fish in the pond?

b. If owner takes 100 more fish, how many will be marked approximately?

Figure 1: Task proposed to participants

The total number of fish in part a) can be estimated by taking into account that $p$ can also be interpreted as the population proportion and that it should be close to the observed proportion $\hat{p}$ in the sample. Since the total number of marked fish in the pond is known: $p = \frac{200}{N}$, it is possible to obtain $N = \frac{200}{p}$ , and if we substitute $p$ by its estimation, then $N \approx \frac{200}{0{,}1} = 2000$. To reply to part b) the prospective teacher have to understand that the relative frequency of marked fish in a new sample of 100 fish might be close to $p$; $f_r = \frac{y}{100}$, where y is the number of marked fish in the new sample; fr ≈ p ≈ 0.1, then y ≈ 0.1×100=10.

The analysis of the task solution suggests that the participants should remember and apply some concepts, properties and procedures linked to frequentist approach of probability, such as those of frequency (absolute and relative), non-equiprobable events, probability, estimation, variability of estimates, population and sample, proportion and inverse proportion. Most of these concepts are studied in Spain in the last year of primary school or in the first years of secondary school.

## RESULTS

Once the written responses to the task were collected, a quantitative and qualitative analysis of responses was carried out in order to identify categories of responses. The quantitative analysis consisted in displaying the distribution of the participants' estimates in parts a) and b) of the task (Figures 2 and 3). In part a) the correct response (2000) was given by only 20.4 % participants (see Figure 2). The most common estimations of the number of fish in the pond was 425; these participants added the 250 fish in the sample with the additional number of marked fish (200-25). Other isolate results were 225 (confusing sample and population, 2 participants), 450 (adding the fish in the sample with the 200 marked fish; 5 cases), and some non-sensical values (e.g., 300, 400, 500, 1000, 2500).

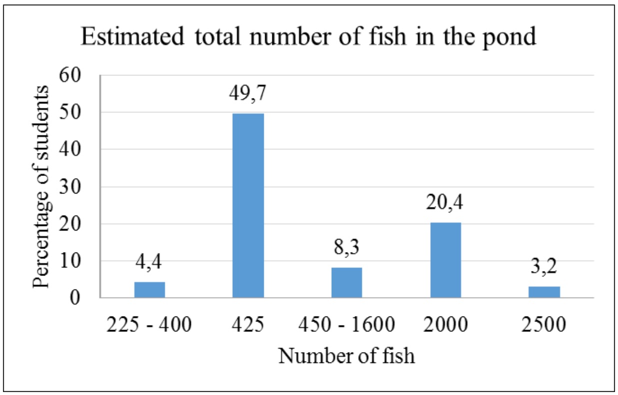


Figure 2. Estimated number of fish in the pond

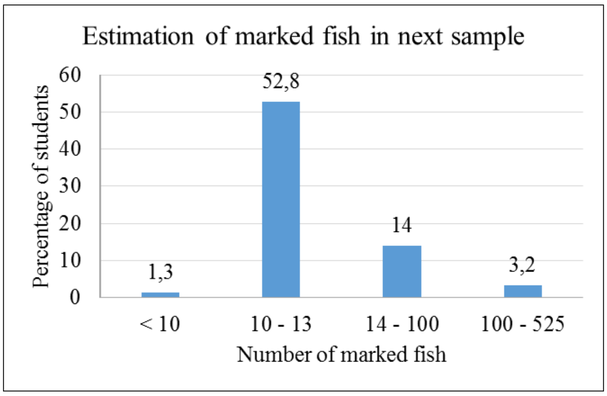


Figure 3. Estimated number of fish in the second sample

These results are summarised in Table 1, where we observe 14% of non-responses. These results were worse than those observed by Fischbein and Gazit (1984) with 12 year olds children who attended experimental classes of probability (12 lessons), 44% of whom provided correct responses to question a). We explain this result because of lack of previous instruction on the idea of sampling and lack of proportional reasoning by these prospective teachers.

| Response | Percentage of the sample |
|---|---|
| Correct (2000) | 20.4 |
| Incorrect (425) | 49.7 |
| Other numerical values | 15.9 |
| No response | 14.0 |

Table 1: Percentages of responses to Part a.

In part b) 54.1% of participants gave a correct estimation (10 or 35), and many others provided values around 10; however some estimation were extremely high (Figure 3). There were a great variety of other responses; while a few provided values close to 10 (11-13), which could be admissible; other participants provided too small values (e.g. 1 or 4) and some estimated a number of marked fish in the sample as big as 150 or provided values higher than the sample size (e.g., 325 or 525; in this last case the participant added the second sample size (100) with the number of marked fish in the pond (200) and the 225 non-marked fish in the first sample. These results are summarised in Table 2, where 28.7% of participants provided no response.

| Response | b. Number of marked fish in a second sample: 10 fish |
|---|---|
| 1-4 | 1.3 |
| Correct (10-13 or 35[1]) | 56.0 |
| 14-100 (except 35) | 10.8 |
| 100-525 | 3.2 |
| Do not reply | 28.7 |

Table 2: Percentages of responses to question b.

The percentage of correct responses to part b) was higher than in part a); however about half the sample was unable to solve the problem correctly. This second question does not appear in Fischbein and Gazit's research. Finally, only 17% of participants replied correctly both questions; these poor results suggest the need to reinforce the training of these teachers as regards elementary sampling ideas.

We additionally performed a qualitative analysis of participants' strategies to solve the problem, which were classified in the following way:

- *Proportional reasoning*. These strategies involve multiplicative operations and comparing / performing operations with proportions. Participants who solved both questions using these strategies, in general, obtained correct results. In some cases, participants combined proportional reasoning with informal reasoning, or used a rote algorithm such as cross multiplication, instead of working directly with proportions. In the following example, the subject combines deductive reasoning with proportional reasoning. In Part a) he first establishes the ratio between marked and non-marked fish; then he computed the number of non-marked fish in the pond, using this ratio, and, finally, he added the number of marked and unmarked fish in the pond to estimate the population size. In part b) he directly applied a cross product to estimate the number of marked fish in the sample. He confused the size of the first sample (250) and instead uses the number of unmarked fish in that sample (225) and then obtain a mistaken estimation.

  Part a: If for each 25 marked fish, there are 225 non- marked fish, then 225x8=1800 without marked; more marked fish, that are 200 $\rightarrow$ 1800+200=2000

  Part b: If 225 –25 / $100 - x$ $\quad x = \frac{100\times 25}{225} = 11'11 \cong 11$ marked fish (G012)

- *Additive strategies*. These strategies, described by Fischbein and Gazit (1984) involve a confusion between intersection and union of events; consequently, participants estimate the values by adding or subs tracing data, instead of performing multiplicative operations. The concept of sampling is not applied by many of participants using these strategies, as the problem is interpreted in a deterministic way, even if they knew, the practice was part of a

[1] We accepted 35 as a correct response, since some participants added the 10 fish in the new sample to the 25 fish in the first sample

lesson of probability. In the first example below, the total number of fish in the pond was supposed to be composed of the first sample of fish (250 fish) and the fish put initially in the pond (200 marked fish); consequently, the number of fish in the intersection was subtracted, getting 425 fish (200 marked fish +250-25 non-marked fish in the first sample). In the second example, some participants added the number of marked fish put in the pond and the total number of fish in the first sample; they, assumed these sets were disjoint, and then they obtained 450 fish (200+250).

250-25 =225 are not marked. If before he marked 200 fish, then should be there 425 (G017)
200+250=450 (C035)

- *Other strategies*. Other different strategies included: a) Reasoning according to the equiprobability bias (Lecoutre, 1992), that is, assuming that all the events had the same probability: *"25 fish marked in the first ample + 50 more = 75 possible marked fish. When we take 100 fish, there is 50% chance to be marked or unmarked"* (A033, part b); b) Providing non-numerical answers based on deterministic reasoning: *"There might be any number of marked fish because you never know; it is impossible to count all the fish at once"* (G049, part a); c) Assuming unpredictability: *"we only know that there is a minimum of 200+ (250-25) = 425 fish, but do not know exactly how many there are"* (G046, part a); and d) Misinterpreting the question: *"We do not know how many fish there are, then we cannot know the proportion; we do not know how the fish is collected, it seems random"* (G046, part b).

| | a. Estimated total: 2000 fish | | b. Estimated marked fish in second sample: 10 fish | |
|---|---|---|---|---|
| Strategy | Correct | Incorrect/NA | Correct | Incorrect/NA |
| Proportional reasoning | 75.0 | 1.6 | 64.7 | 19.3 |
| Additive strategies | | 35.2 | | |
| The procedure used was not explained (only write number) | 25.0 | 48.8 | 35.3 | 36.1 |
| Other strategies | | 1.6 | | 6.9 |
| No response | | 11.2 | | 37.5 |

Table 3: Percentage of different strategies in each question.

In Table 3 we cross the participants' responses and strategies in the problem. We observe an important percentage of prospective teachers who were unable to describe their strategies to solve the problem. This is a matter of concern, since a teacher will need to explain their students the response to the problems. The majority of correct responses were associated to proportional reasoning; while in incorrect responses, participants mostly used additive strategies or did not explain their procedure. In our sample the proportion of proportional strategies in the correct responses was higher than that in Fischbein and Gazit's research, whose children basically used additive strategies (only 5% of older children who provided correct response used proportional strategies).

## DISCUSSION AND SUGGESTIONS FOR TRAINING THE TEACHERS

Although previous research have analyzed the probability knowledge in prospective teachers, very few focused on sampling and none of them analyzed the task proposed in this paper. This task, taken from Fischbein and Gazit (1984) has been enriched with a question related to sampling variability and replicability of trials.

Our results suggest a low level of probability knowledge in this sampling context by the participant prospective teachers, which may be also common in other prospective teachers in Spain. Only one out of every five participants obtained a correct estimation of the population size in a finite population using the information in a sample. Our participants' performance improved in the second part of the task, where more than 50% gave correct estimations for the expected frequency of a given result in a new sample. Although the majority of these prospective teachers used proportional reasoning, many of them also restricted to additive strategies, that are typical of pre-formal operational stages in Piaget and Inhelder (1951) theory; other were unable to explain their strategies or applied rote algorithms. These results suggest that these prospective teachers need extensive training in probability and sampling, in agreement with previous research by Azcárate (1995) and Serrano (1996).

In our study, the assessment was followed by some formative activities. A second session was spent in a debate, where the prospective teachers presented their solutions to the problem and their justifications. This discussion helped revealing potential conflicts in their understanding of sampling and in distinguishing the correct and incorrect responses. After each participant supported its view, in order to improve the participants' intuitions on sampling, a third session was devoted to didactic analysis. First, some simulation activities were performed, using a freely available applet: (http://science.jburroughs.org/mbahe/BioA/starranimations/chapter40/videos_animations/capture_recapture.html), which simulated capture-recapture sampling. Other sampling problems, adapted from Green (1983), were also simulated using different Internet applets and manipulative materials: a) number of new-born babies in small and big hospitals; b) identifying which girl was cheating when recording 150 results of flipping a fair coin; and c) estimating the frequency of repeated trials when 100 pins are thrown on a table were solved.

As a part of the discussion, participants were also asked to list the concepts and properties they used to solve each problem and to identify the children's difficulties when solving each of the problems. Some of the possible errors identified were: a) adding number of marked fish to the population size and getting 2200 fish as population size; b) considering the population as union of the first sample and the marked fish (450 fish); c) considering that only marked fish remain in the population after selecting the first sample (425); and d) considering that it is impossible to predict the total number, due to randomness.

After the activity these prospective teachers were asked to write a summary of all the activities in the sessions, including correct and incorrect answers (with underlying biases) for each problem, summary of simulation activities and of the didactic analysis. This practice helped to develop the participants' content and pedagogical content knowledge in probability and sampling and may be useful in other courses directed to teachers.

## Acknowledgement

Project EDU2013-41141-P (MEC) and group FQM126 (Junta de Andalucía).

## References

Azcárate, P. (1995). *El conocimiento profesional de los profesores sobre las nociones de aleatoriedad y probabilidad* (Teachers' professional knowledge of randomness and probability). Unpublished Ph.D. University of Cádiz, Spain.

Batanero, C., Burrill, G., & Reading, C. (Eds.) (2011). *Teaching statistics in school mathematics. Challenges for teaching and teacher education. A joint ICMI and IASE study*. New York: Springer.

Fischbein, E., & Gazit, A. (1984). Does the teaching of probability improve probabilistic intuitions? *Educational Studies in Mathematics, 15* (1), 1-24.

Godino, J. D. (2009). Categorías de análisis de los conocimientos del profesor de matemáticas (Categories in the analysis of teacher's kowledge). *UNIÓN, Revista Iberoamericana de Educación Matemática, 20*, 13-31.

Gómez, E., Batanero, C., & Contreras, J. M. (2014). Conocimiento matemático de futuros profesores para la enseñanza de la probabilidad desde el enfoque frecuencial (Prospective teachers' mathematical knowledge to teach probability from the frequentist approach). *Bolema, 28* (48), 209-229.

Green, D. R. (1983). From thumbtacks to inference. *School Science and Mathematics*, 83 (7), 541-551.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal, 105*(1), 11-30.

Hill, H. C., Ball, D. L., & Schilling, S. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39* (4), 372-400.

Kahneman, D., Slovic, P., & Tversky, A. (Eds.) (1982). *Judgment under uncertainty: Heuristic and biases*. New York: Cambridge University Press.

Lecoutre, M. P. (1992). Cognitive models and problem spaces in "purely random" situations. *Educational Studies in Mathematics*, 23, 557-568.

Ministerio de Educación y Ciencia, MEC (2006). *Real Decreto 1513/2006, de 7 de diciembre, por el que se establecen las enseñanzas mínimas de la Educación Primaria* (Royal Decree establishing the minimum content for Primary Education). Madrid: Author.

Ministerio de Educación, Cultura y Deporte, MECD (2014). *Real Decreto 126/2014, de 28 de febrero, por el que se establece el currículo básico de la Educación Primaria* (Royal Decree establishing the basic curriculum for Primary Education). Madrid: Author.

Mohamed, N. (2012). *Evaluación del conocimiento de los futuros profesores de educación primaria sobre probabilidad* (Assessing prospective primary school teacher knowledge of probability). Unpublished Ph.D. University of Granada, Spain.

Piaget, J. & Inhelder, B. (1951). *La genése de l'idée de hasard chez l'enfant*. París: Presses Universitaires de France.

Serrano, L. (1996). *Significados institucionales y personales de objetos matemáticos ligados a la aproximación frecuencial de la enseñanza de la probabilidad* (Institutional and personal meaning of mathematical objects linked to the frequentist approach in teaching probability). Unpublished Ph.D. University of Granada, Spain.