A few results related to the Radon-Nikodým property

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Congrats Rafa. Happy birthday.
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## Contents:

1. Vector integration, RNP, dentability and fragmentability.
2. Two results related to RNP
   - Related with separable dual spaces: RNP and the Lindelöf property.
   - Related to small slices: RNP and norm attaining operators.
3. One last thing.
Vector integration, RNP, dentability and fragmentability.
To study RNP your start with vector integration:

- **Simple function.**
  \[ s = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \text{, where } \alpha_i \in E, A_i \in \Sigma, \text{ disjoints}. \]

- **Measurable function.**
  \[ \lim_{n} \| s_n(w) - f(w) \| = 0, \quad \mu \text{ a.e. } w \in \Omega. \]

- **Scalarly measurable function.**
  \[ x^* f \text{ is measurable for } x^* \in E^*. \]

- **Prove Pettis’s Theorem.**
- **Measurable \neq \text{scalarly measurable.}**
- **The integral of simple functions.**
- **Bochner integral.**
- **You carry on...**
Theorem 3 (Dominated Convergence Theorem). Let \((\Omega, \Sigma, \mu)\) be a finite measure space and \((f_n)\) be a sequence of Bochner integrable \(X\)-valued functions on \(\Omega\). If \(\lim_n f_n = f\) in \(\mu\)-measure, (i.e., \(\lim_n \mu\{\omega \in \Omega : \|f_n - f\| \geq \varepsilon\} = 0\) for every \(\varepsilon > 0\)) and if there exists a real-valued Lebesgue integrable function \(g\) on \(\Omega\) with \(\|f_n\| \leq g\) \(\mu\)-almost everywhere, then \(f\) is Bochner integrable and \(\lim_n \int_E f_n \, d\mu = \int_E f \, d\mu\) for each \(E \in \Sigma\). In fact, \(\lim_n \int \|f - f_n\| \, d\mu = 0\).
**Theorem 4.** If $f$ is a $\mu$-Bochner integrable function, then

(i) $\lim_{\mu(E) \to 0} \int_E f \ d\mu = 0$;

(ii) $\| \int_E f \ d\mu \| \leq \int_E \| f \| \ d\mu$, for all $E \in \Sigma$;

(iii) if $(E_n)$ is a sequence of pairwise disjoint members of $\Sigma$ and $E = \bigcup_{n=1}^{\infty} E_n$, then

$$\int_E f \ d\mu = \sum_{n=1}^{\infty} \int_{E_n} f \ d\mu,$$

where the sum on the right is absolutely convergent;

(iv) if $F(E) = \int_E f \ d\mu$, then $F$ is of bounded variation and

$$|F|(E) = \int_E \| f \| \ d\mu \quad \text{for all } E \in \Sigma.$$
Then, at Berkeley, California, Rieffel taught a real analysis course in which he opted to present the Bochner integral instead of the classical Lebesgue theory. As rumor has it, all went smoothly until he came to the Radon-Nikodým theorem and its attendant difficulties in infinite dimensional Banach spaces.
Example 1. The failure of the Radon-Nikodym theorem for a $c_0$-valued measure. Let $\Omega = [0, 1]$ and $\mu$ be Lebesgue measure on $\Sigma$, the $\sigma$-field of Lebesgue measurable subsets of $[0, 1]$. Define a measure $G: \Sigma \to c_0$ by

$$G(E) = \left( \int_E \sin(2^n \pi t) \, d\mu(t) \right).$$
Definition (Rieffel, 1967)

$D \subset E$ is dentable if for each $\varepsilon > 0$ there is a point $x \in D$ such that $x \not\in \overline{co}(D \setminus U_\varepsilon(x))$.
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Proposition (Small slices)

$D \subset E$ is dentable if, and only if, $D$ has slices of arbitrarily small diameter.

$$S = \{ y \in D : x^*(y) > \sup_D x^* - \alpha \} \quad \alpha > 0$$
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Theorem 7 (Rieffel-Maynard-Huff-Davis-Phelps). Any one of the following statements about a Banach space $X$ implies all the others.

(a) Every bounded subset of $X$ is dentable.
(b) Every bounded subset of $X$ is $\sigma$-dentable.
(c) The space $X$ has the Radon-Nikodým property.

Fragment from Phelps’ memorial article

“As I recall, there was a period of excitement involving Rieffel’s generalization of the classical Radon-Nikodým Theorem to Banach space valued functions provided the Banach space has the Radon-Nikodým Property (RNP) which is a geometric property. The RNP immediately attracted widespread attention, and Bob was not immune from the RNP-bug. He published in 1974–75 three papers\footnote{I. Namiola (August 2013).} (one jointly with Davis) on RNP. By 1975 it was shown that the RNP, which was shown to be necessary by Rieffel for his generalized Radon-Nikodým theorem, is also sufficient by the combined efforts of Chatterji, Davis, Huff, Maynard and Phelps.”

I. Namiola (August 2013).
A remarkable result

Let $E$ be a Banach space. Then the following conditions are equivalent:

1. $E$ is an Asplund space, i.e., whenever $f$ is a convex continuous function defined on an open convex subset $U$ of $E$, the set of all points of $U$ where $f$ is Fréchet differentiable is a dense $G_δ$-subset of $U$.

2. every $w^*$-compact subset of $(E^*, w^*)$ is fragmented by the norm;

3. each separable subspace of $E$ has separable dual;

4. $E^*$ has the Radon-Nikodým property.
Two results
RNP and the Lindelöf property
Theorem (Namioka-Orihuela-Cascales, 2003)

$K$ compact subset of $M^D$, $(M, \rho)$ metric space.

T.F.A.E:

(a) The space $(K, \tau_p)$ is fragmented by $d$.

(b) For each $A \in \mathcal{C}$, the pseudo-metric space $(K, d_A)$ is separable.

(c) $(K, \gamma(D))$ is Lindelöf.

(d) $(K, \gamma(D))^\mathbb{N}$ is Lindelöf.

Corollary

$E^*$ has the RNP if, and only if, $(E^*, \gamma(B_X))$ is Lindelöf.

Theorem (Solution to a problem by Corson)

If $(E^*, w)$ is Lindelöf, then $(E^*, w)^2$ is Lindelöf
RNP and norm attaining operators
The Bishop-Phelps property for operators

Definition

An operator $T : X \to Y$ is **norm attaining** if there exists $x_0 \in X$, $\|x_0\| = 1$, such that $\|T(x_0)\| = \|T\|$.

Definition (Lindenstrauss)

$(X, Y)$ has the Bishop-Phelps Property (BPp) if every operator $T : X \to Y$ can be uniformly approximated by norm attaining operators.

1. $(X, K)$ has BPp for every $X$ Bishop-Phelps (1961);
2. $\{T \in L(X; Y) : T^{**} \in NA(X^{**}; Y^{**})\} = L(X; Y)$ for every pair of Banach spaces $X$ and $Y$, Lindenstrauss (1963);
3. $X$ with RNP, then $(X, Y)$ has BPp for every $Y$, Bourgain (1977);
4. there are spaces $X$, $Y$ and $Z$ such that $(X, C([0, 1]))$, $(Y, \ell^p)$ ($1 < p < \infty$) and $(Z, L^1([0, 1]))$ fail BPp, Schachermayer (1983), Gowers (1990) and Acosta (1999);
5. $(C(K), C(S))$ has BPp for all compact spaces $K, S$, Johnson and Wolfe, (1979).
6. $(L^1([0, 1]), L^\infty([0, 1]))$ has BPp, Finet-Payá (1998),
Theorem (Guirao-Kadets-Cascales, 2013)

Let $\mathcal{A} \subset C(K)$ be a uniform algebra and $T : X \to \mathcal{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\tilde{T} \in S_{L(X,\mathcal{A})}$ satisfying that

$$\|\tilde{T}u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon \quad \text{and} \quad \|T - \tilde{T}\| < 2\varepsilon.$$ 

This gives

- for $C(K)$ an example of the BPBp for $c_0$ as domain and an infinite dimensional Banach space as range (answer a question by Acosta-Aron-García-Maestre, 2008);
- new cases, in particular, disk algebra as range.
Co-authors

R. M. Aron, B. Cascales and O. Kozhushkina,
*The Bishop-Phelps-Bollobás theorem and Asplund operators*,

B. Cascales, A. J. Guirao and V. Kadets,
*A Bishop-Phelps-Bollobás type theorem for uniform algebras*,
Advances in Mathematics 240 (2013) 370-382
A Urysohn type lemma for uniform algebras

**Proposition 2.8.** Let $A \subset C(K)$ be a unital uniform algebra, $\Omega \subset \mathbb{C}$ a bounded simply connected region such that all points in its boundary $\partial \Omega$ are simple. Let us fix two different points $a$ and $b$ with $b \in \partial \Omega$, $a \in \overline{\Omega}$ and a neighborhood $V_a \subset \overline{\Omega}$ of $a$. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \emptyset$ and for every $t_0 \in U \cap \Gamma_0$, there exists $f \in A$ such that

(i) $f(K) \subset \overline{\Omega}$;
(ii) $f(t_0) = b$;
(iii) $f(K \setminus U) \subset V_a$.

1. **Fragmentability** gives an open set $U \cap \Gamma_0 \neq \emptyset$, $y^* \in S_{X^*}$ & $\rho < 2\varepsilon$ with $1 = |y^*(u_0)| = \|u_0\|$ and $\|x_0 - u_0\| < \varepsilon$ & $\|T^*(\delta_t) - y^*\| < \rho \ \forall t \in U$.

2. Uryshon’s lemma that applied to an arbitrary $t_0 \in U \cap \Gamma_0$ produces a function $f \in \mathcal{A}$ satisfying $f(t_0) = \|f\|_\infty = 1$, $f(K) \subset St\varepsilon'$ and $f$ small in $K \setminus U$.

explicitly defined by

$$\tilde{T}(x)(t) = f(t) \cdot y^*(x) + (1 - \varepsilon')(1 - f(t)) \cdot T(x)(t)$$

suitability of $U$ is used to prove that $\|T - \tilde{T}\| < 2\varepsilon$. 

**Figure**: Illustration of the Urysohn type lemma for uniform algebras.
A short comment about indexes related to RNP
With the help of the lifting theorem, it is possible to prove that if $(\Omega, \Sigma, \mu)$ is a finite measure space and $T: L_1(\mu) \rightarrow X^*$ is a bounded linear operator, then there exists a function $g: \Omega \rightarrow X^*$ that is weak*-measurable and such that for each $x \in X$ and $f \in L_1(\mu)$, one has $(Tf)(x) = \int_{\Omega} f(\omega)g(\omega)(x) \, d\mu(\omega)$. On the surface, this is a vast generalization of Theorem 3.1 which it includes. Unfortunately, this generalization is mostly an esthetic generalization because the measurability properties of the kernel $g$ are not, in general, strong enough to exhibit structural properties of the operator under representation.
A bit of notation

- \( m : \Sigma \rightarrow E^* \) vector measure c. a. and \( \mu \)-continuous;
- For every \( B \in \Sigma^+ \) the average range of \( m|_B \) is denoted by
  \[
  \Gamma_B := \left\{ \frac{m(C)}{\mu(C)} : C \in \Sigma_B^+ \right\}.
  \]
- Fix \( \rho : \Sigma \rightarrow \Sigma \) a lifting, and write:
  - \( \mathcal{F} := \rho(\Sigma) \setminus \{ \emptyset \} \).
  - \( (\mathcal{U}, \succ) \) is the directed set of all finite partitions of \( \Omega \) into elements of \( \mathcal{F} \) ordered by refinement \( \succ \).

Lemma (Folklore + a little thing)

The net

\[
\left\{ \sum_{A \in \pi} \frac{m(A)}{\mu(A)} \chi_A : \pi \in \mathcal{U} \right\}
\]

converges pointwise \( \mu \)-almost everywhere in the \( \omega^* \)-topology to a function \( \psi : \Omega \rightarrow E^* \) which is \( \omega^* \)-measurable and that satisfies the following properties for every \( A \in \Sigma \):

(I) \( \langle x, m(A) \rangle = \int_A \langle x, \psi(t) \rangle \, d\mu \)

(II) \( \psi(t) \in \overline{\Gamma_A}^{\omega^*} \) for \( \mu \)-almost every \( t \in A \).
Index of representability vs. dentability

**Definition**

Given a $\mu$-continuous vector measure of bounded variation $m : \Sigma \rightarrow E$, the *index of representability* $R(m)$ of $m$ is defined as

$$R(m) := \inf \{ \varepsilon > 0 : \forall A \in \Sigma^+, \exists B \in \Sigma^+_A \text{ with } \text{rad}(\Gamma_B) < \varepsilon \}.$$ 

- $D \subset E$, $\text{rad}(D) = \inf\{\delta > 0 : \exists x \in E \text{ such that } D \subseteq B(x, \delta)\}.$

**Theorem**

Let $m : \Sigma \rightarrow E \hookrightarrow E^{**}$ be a $\mu$-continuous measure of bounded variation and $\psi : \Omega \rightarrow E^{**}$ a Gelfand derivative of $m$. Then

$$R(m) \leq \text{meas}(\psi) \leq 2R(m)$$

and there exists a $\mu$-null set $D$ such that

$$\hat{d}(\psi(\Omega \setminus D), E) \leq R(m).$$
Let $(\Omega, \Sigma, \mu)$ be a finite measure space and $T : L^1(\mu) \to E$ a continuous linear operator. Then

$$d(T, \mathcal{L}_{\text{rep}}(L^1(\mu), E)) \leq 2 \gamma(T(B_{L^1(\mu)})) .$$

B. Cascales, A. Pérez and M. Raja, 
The Gelfand Integral for Multi-Valued Functions

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SUBDIFFERENTIALS OF NONCONVEX INTEGRAL FUNCTIONALS IN BANACH SPACES WITH APPLICATIONS TO STOCHASTIC DYNAMIC PROGRAMMING

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Dedicated to Nino Maugeri with great respect

Abstract. The paper concerns the investigation of nonconvex and nondifferentiable integral functionals on general Banach spaces, which may not be reflexive and/or separable. Considering two major subdifferentials of variational analysis, we derive nonsmooth versions of the Leibniz rule on sub differentiation under the integral sign, where the integral of the subdifferential set-valued mappings generated by Lipschitzian integrands is understood in the Gelfand sense. Besides examining integration over complete measure spaces and also over those with nonatomic measures, our special attention is drawn to a stronger version of measure nonatomicity, known as saturation, to invoke the recent results of the Lyapunov convexity theorem type for the Gelfand integral of the subdifferential mappings. The main results are applied to the subdifferential study of the optimal value functions and deriving the corresponding necessary optimality conditions in nonconvex problems of stochastic dynamic programming with discrete time on the infinite horizon.
THANKS