Spectrum of Volterra type operators on weighted Banach spaces of entire functions

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Workshop on Banach Spaces, 60th Birthday Rafael Payá, Granada, 2015

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Project Prometeo II/2013/013
$H(\mathbb{C})$ and $\mathcal{P}$ denote the space of entire functions and the space of polynomials, respectively. The space $H(\mathbb{C})$ will be endowed with the compact open topology $\tau_{co}$.

The differentiation operator $Df(z) = f'(z)$ and the integration operator $Jf(z) = \int_0^z f(\zeta)d\zeta$ are continuous on $H(\mathbb{C})$.

Given an entire function $g \in H(\mathbb{C})$, the Volterra operator $V_g$ with symbol $g$ is defined on $H(\mathbb{C})$ by

$$V_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta \quad (z \in \mathbb{C}).$$

For $g(z) = z$ this reduces to the integration operator. Clearly $V_g$ defines a continuous operator on $H(\mathbb{C})$. 
Our purpose is to characterize boundedness and compactness of Volterra operators \( V_g \) acting between different weighted Banach spaces \( H^\infty_v(\mathbb{C}) \) of entire functions with sup-norms in terms of the symbol \( g \). We also investigate the spectrum of \( V_g \) acting on \( H^\infty_v(\mathbb{C}) \).

We complement recent work by Bassallote, Contreras, Hernández-Mancera, Martín and Paul in 2012 for spaces of holomorphic functions on the disc and by Constantin and Peláez in 2013 for reflexive weighted Fock spaces.

The case of functions on the disc was considered by Pommerenke, Aleman, Siskakis, Pau, Peláez and Rättyä, among others.

New results about weak compactness, the spectrum and about operators on the smaller spaces \( H^0_v(\mathbb{C}) \) are presented.
A weight $v$ is a continuous function $v : [0, \infty[ \to ]0, \infty[,$ which is non-increasing on $[0, \infty[ \text{ and satisfies } \lim_{r \to \infty} r^m v(r) = 0 \text{ for each } m \in \mathbb{N}.$ If necessary, we extend $v$ to $\mathbb{C}$ by $v(z) := v(|z|)$.

The weighted Banach spaces of entire functions are defined by

$$H^\infty_v(\mathbb{C}) := \{ f \in H(\mathbb{C}) \mid \|f\|_v := \sup_{z \in \mathbb{C}} v(|z|)|f(z)| < \infty \},$$

$$H^0_v(\mathbb{C}) := \{ f \in H(\mathbb{C}) \mid \lim_{|z| \to \infty} v(|z|)|f(z)| = 0 \},$$

and they are endowed with the weighted sup norm $\|\cdot\|_v.$
H_\infty^v(\mathbb{C}) coincides with the **weighted Fock space** F_\phi^\infty of order infinity when v(z) = \exp(-\phi(|z|)), and \phi : [0, \infty[ \to ]0, \infty[ is a twice continuously differentiable increasing function.

H_0^v(\mathbb{C}) is a closed subspace of H_\infty^v(\mathbb{C}).

The polynomials are contained and dense in H_0^v(\mathbb{C}) but the monomials do not in general form a Schauder basis (**Lusky**). The Cesàro means of the Taylor polynomials satisfy \|C_nf\|_v \leq \|f\|_v for each f \in H_\infty^v(\mathbb{C}) and the sequence (C_nf)_n is \|\cdot\|_v-convergent to f when f \in H_0^v(\mathbb{C})
For a weight \( v \), the associated weight \( \tilde{v} \) is defined by

\[
\tilde{v}(z) := \left( \sup \left\{ |f(z)| \mid f \in H^\infty_v(\mathbb{C}), \|f\|_v \leq 1 \right\} \right)^{-1} = (\|\delta_z\|_v)^{-1}, \quad z \in \mathbb{C},
\]

where \( \delta_z \) denotes the point evaluation of \( z \).

- \( \tilde{v} \) is continuous, radial, \( \tilde{v} \geq v > 0 \), and for each \( z \in \mathbb{D} \) we can find \( f_z \in H^\infty_v \), \( \|f_z\|_v = 1 \) with \( |f_z(z)|\tilde{v}(z) = 1 \).

- \( H^\infty_{\tilde{v}}(\mathbb{C}) \) coincides isometrically with \( H^\infty_v(\mathbb{C}) \), and \( H^0_{\tilde{v}}(\mathbb{C}) \) with \( H^0_v(\mathbb{C}) \).
The Volterra operator $V_g$ with symbol $g \in H(\mathbb{C})$ is defined on $H(\mathbb{C})$ by

$$V_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)\,d\zeta \quad (z \in \mathbb{C}).$$

**Question**

How it acts on $H^\infty_v(\mathbb{C})$ or $H^\infty_v(\mathbb{C})$?
Theorem

Let \( v \) be a weight and let \( w(r) := \exp(-\alpha r^p) \), where \( \alpha > 0, p > 0 \) are constants. The following conditions are equivalent for \( g \in H(\mathbb{C}) \):

1. \( V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C}) \) is continuous.
2. \( V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C}) \) is continuous.
3. There exists a constant \( C > 0 \) such that
   \[
   |g'(z)| \leq C |z|^{p-1} \exp(\alpha |z|^p) \tilde{v}(|z|) \]
   for all \( z \in \mathbb{C}, |z| \geq 1 \).

The following conditions are also equivalent for \( g \in H(\mathbb{C}) \):

1. \( V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C}) \) is compact.
2. \( V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C}) \) is compact.
3. \( |g'(z)| = o(|z|^{p-1} \exp(\alpha |z|^p) \tilde{v}(|z|)) \) as \(|z| \to \infty\).
Let $v$ and $w$ be weights. The following conditions are equivalent for an entire function $h \in H(\mathbb{C})$:

1. $M_h : H_v^\infty(\mathbb{C}) \rightarrow H_w^\infty(\mathbb{C})$ is continuous.
2. $M_h : H_v^0(\mathbb{C}) \rightarrow H_w^0(\mathbb{C})$ is continuous.
3. $\sup \frac{w(z)|h(z)|}{\tilde{v}(z)} < \infty$.
4. $\sup \frac{\tilde{w}(z)|h(z)|}{\tilde{v}(z)} < \infty$. 

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Proposition

Let $v$ and $w$ be weights. The following conditions are equivalent for an entire function $g \in H(\mathbb{C})$:

1. $M_h : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is compact.
2. $M_h : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is compact.
3. $\lim_{|z| \to \infty} \frac{w(z)|h(z)|}{\tilde{v}(z)} = 0.$
4. $\lim_{|z| \to \infty} \frac{\tilde{w}(z)|h(z)|}{\tilde{v}(z)} = 0.$
Let \( \phi : [0, \infty[ \rightarrow [0, \infty[ \) be a continuous non-decreasing function, \( C^1 \) on \([r_\phi, \infty[\) for some \( r_\phi \geq 0 \). Suppose that \( \phi' \) is non-decreasing in \([r_\phi, \infty[\), \( \phi'(r_\phi) > 0 \) and \( r^n = O(\phi'(r)) \) as \( r \rightarrow \infty \) for each \( n \in \mathbb{N} \). This implies \( r^n = O(\phi(r)) \) as \( r \rightarrow \infty \) for each \( n \in \mathbb{N} \), and that

\[
w_\phi(z) := 1/\phi(|z|), \quad z \in \mathbb{C},
\]

and

\[
u_\phi(z) := 1/\max\{\phi'(r_\phi), \phi'(|z|)\} = \begin{cases} 1/\phi'(r_\phi), & |z| \leq r_\phi \\ 1/\phi'(|z|), & |z| \geq r_\phi \end{cases}
\]

are weights.

If \( \phi(r) = \exp(\alpha r^p), \) \( r \geq 0, \alpha > 0, p > 0, \) then
\( w_\phi(z) = \exp(-\alpha |z|^p), \) \( z \in \mathbb{C}, \) and \( u_\phi(z) = \alpha^{-1} p^{-1} |z|^{1-p} \exp(-\alpha |z|^p) \) for \( |z| \) large enough.
Proposition

The integration operators $J : H^\infty_{u\varphi}(\mathbb{C}) \rightarrow H^\infty_{w\varphi}(\mathbb{C})$ and $J : H^0_{u\varphi}(\mathbb{C}) \rightarrow H^0_{w\varphi}(\mathbb{C})$ are continuous.

Proposition

If the function $\varphi$ is of smoothness $C^2$ on $[r_\varphi, \infty[$ for some $r_\varphi > 0$ and it satisfies $\sup_{r \geq r_\varphi} \frac{\varphi''(r)\varphi(r)}{(\varphi'(r))^2} < \infty$ in addition to the general assumptions, then the differentiation operators $D : H^\infty_{w\varphi}(\mathbb{C}) \rightarrow H^\infty_{u\varphi}(\mathbb{C})$ and $D : H^0_{w\varphi}(\mathbb{C}) \rightarrow H^0_{u\varphi}(\mathbb{C})$, $Df := f'$, are continuous.
Under the assumptions on $\varphi$ in the last Proposition, $M(f, r) = O(\varphi(r))$, when $r \to \infty$ if and only if $M(f', r) = O(\varphi'(r))$ for $r \to \infty$. The argument can be traced back, at least, to Pavlovic (1999).

Examples of functions $\varphi$ that satisfy the assumptions of Proposition can be found in the work of Hardy. For example one can take

$$
\varphi(r) := r^a(\log r)^b \exp(cr^d + k(\log r)^m),
$$

for large $r$, where $c > 0$, $d > 0$ or $c = 0$, $k > 0$, $m > 1$. 
Theorem

Let $\varphi$ be of smoothness $C^2$ on $[r_\varphi, \infty[$ for some $r_\varphi > 0$ and let it satisfy
\[ \sup_{r \geq r_\varphi} \frac{\varphi''(r) \varphi(r)}{(\varphi'(r))^2} < \infty \]
in addition to the general assumptions.

The following conditions are equivalent for an entire function $g \in H(\mathbb{C})$:

1. $V_g : H_\varphi^\infty(\mathbb{C}) \to H_{w\varphi}^\infty(\mathbb{C})$ is continuous.
2. $V_g : H_\varphi^0(\mathbb{C}) \to H_{w\varphi}^0(\mathbb{C})$ is continuous.
3. $\sup_{|z| \geq r_\varphi} \frac{|g'(z)|}{\varphi'(|z|) \tilde{\nu}(z)} < \infty$. 

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Proof.

Assume that condition (1) holds. The differentiation operator $D : H^\infty_w(\mathbb{C}) \to H^\infty_u(\mathbb{C})$ is continuous. We can apply (1) and the identity $DV_g = M_{g'}$ to conclude that $M_{g'} : H^\infty_v(\mathbb{C}) \to H^\infty_u(\mathbb{C})$ is continuous. Now condition (3) follows from the characterization of bounded multiplication operators, since $u_\varphi(z) = 1/\varphi'(|z|), |z| \geq r_\varphi$.

Conversely, if condition (3) holds, the operator $M_{g'} : H^\infty_v(\mathbb{C}) \to H^\infty_u(\mathbb{C})$ is continuous by the characterization of bounded multiplication operators. We apply the boundedness of $J : H^\infty_u(\mathbb{C}) \to H^\infty_w(\mathbb{C})$ to get that $V_g = J \circ M_{g'} : H^\infty_v(\mathbb{C}) \to H^\infty_w(\mathbb{C})$ is continuous.
Theorem

Let $\varphi$ be of smoothness $C^2$ on $[r_{\varphi}, \infty[$ for some $r_{\varphi} > 0$ and let it satisfy
\[
\sup_{r \geq r_{\varphi}} \frac{\varphi''(r) \varphi(r)}{(\varphi'(r))^2} < \infty
\]
in addition to the general assumptions.

The following conditions are equivalent for an entire function $g \in H(\mathbb{C})$:

1. $V_g : H_v^\infty(\mathbb{C}) \to H_{w\varphi}^\infty(\mathbb{C})$ is compact.
2. $V_g : H_v^0(\mathbb{C}) \to H_{w\varphi}^0(\mathbb{C})$ is compact.
3. $\lim_{|z| \to \infty} \frac{|g'(z)|}{\varphi'(|z|) \hat{v}(z)} = 0.$
We have already proved

Theorem

Let $v$ be a weight and let $w(r) := \exp(-\alpha r^p)$, where $\alpha > 0, p > 0$ are constants. The following conditions are equivalent for $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is continuous.

(2) $V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is continuous.

(3) There exists a constant $C > 0$ such that

$$|g'(z)| \leq C|z|^{p-1} \exp(\alpha|z|^p)\tilde{v}(|z|)$$

for all $z \in \mathbb{C}, |z| \geq 1$.

The following conditions are also equivalent for $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is compact.

(2) $V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is compact.

(3) $|g'(z)| = o(|z|^{p-1} \exp(\alpha|z|^p)\tilde{v}(|z|))$ as $|z| \to \infty$. 

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Main consequence

Proposition

Assume that \( v(r) = \exp(-\alpha r^p), \alpha > 0, \ p > 0. \)

(i) \( V_g : H^\infty_v(\mathbb{C}) \to H^\infty_v(\mathbb{C}) \) is continuous if and only if \( g \) is a polynomial of degree less than or equal to the integer part of \( p \).

(ii) \( V_g : H^\infty_v(\mathbb{C}) \to H^\infty_v(\mathbb{C}) \) is compact if and only if \( g \) is a polynomial of degree less than or equal to the integer part of \( p - 1 \).
Now we investigate the **spectrum of the Volterra operator** when it acts continuously on a weighted Banach space of entire functions $H^\infty_v(\mathbb{C})$.

**Aleman and Constantin** in 2009 and **Aleman and Peláez** in 2012 investigated the spectra of Volterra operators on several spaces of holomorphic functions on the disc. **Constantin** started in 2012 the study of the spectrum of Volterra operator on spaces of entire functions, more precisely on the classical Fock spaces.

We assume that $g \in H(\mathbb{C})$ be a non-constant entire function such that $g(0) = 0$ and $V_g$ is the Volterra operator.
$X$ is a Hausdorff locally convex space (lcs).

$L(X)$ is the space of all continuous linear operators on $X$.

The **resolvent set** $\rho(T, X)$ of $T$ consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $L(X)$.

The **spectrum** of $T$ is the set $\sigma(T, X) := \mathbb{C} \setminus \rho(T, X)$. The **point spectrum** is the set $\sigma_{pt}(T, X)$ of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective.
Preliminary results

Proposition

The operator \( V_g - \lambda I : H(\mathbb{C}) \to H(\mathbb{C}) \) is injective for each \( \lambda \in \mathbb{C} \). In particular \( \sigma_{pt}(V_g, H(\mathbb{C})) = \emptyset \). Moreover, \( 0 \in \sigma(V_g, H(\mathbb{C})) \).

Lemma

Given \( \lambda \in \mathbb{C} \), \( \lambda \neq 0 \), and \( h \in H(\mathbb{C}) \), the equation \( f - (1/\lambda)V_g f = h \) has a unique solution given by

\[
\begin{align*}
f(z) &= R_{\lambda,g} h(z) = h(0) e^{g(z)/\lambda} + e^{g(z)/\lambda} \int_0^z e^{-g(\zeta)/\lambda} h'(\zeta) d\zeta, \quad z \in \mathbb{C}.
\end{align*}
\]
Proposition

Let $g \in H(\mathbb{C})$ be a non-constant entire function such that $g(0) = 0$. The Volterra operator $V_g$ satisfies $\sigma(V_g, H(\mathbb{C})) = \{0\}$ and $\sigma_{pt}(V_g, H(\mathbb{C})) = \emptyset$.

Proposition

Let $X \subset H(\mathbb{C})$ be a locally convex space that contains the constants and such that the inclusion $X \subset H(\mathbb{C})$ is continuous. Assume that $V_g : X \to X$ is continuous for some non-constant entire function $g$ such that $g(0) = 0$. Then

$$\{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} \mid e^{\frac{g}{\lambda}} \notin X\} \subset \sigma(V_g, X).$$

If $X$ is a Banach space, then

$$\{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} \mid e^{\frac{g}{\lambda}} \notin X\} \subset \sigma(V_g, X).$$
Lemma

Let $X \subset H(\mathbb{C})$ be a locally convex space that contains the constants and such that the inclusion $X \rightarrow H(\mathbb{C})$ is continuous. Assume that $V_g : X \rightarrow X$ is continuous for some non-constant entire function $g$ such that $g(0) = 0$. The following conditions are equivalent:

(i) $\lambda \in \rho(V_g, X)$.

(ii) $R_{\lambda, g} : X \rightarrow X$ is continuous.

(iii) (a) $e^\frac{g}{\lambda} \in X$, and

(b) $S_{\lambda, g} : X_0 \rightarrow X_0$, $S_{\lambda, g} h(z) := e^\frac{g(z)}{\lambda} \int_0^z h'(\zeta)e^{-\frac{g(\zeta)}{\lambda}} d\zeta$, $z \in \mathbb{C}$, is continuous on the subspace $X_0$ of $X$ of all the functions $h \in X$ with $h(0) = 0$. 

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Lemma

Let $X \subset H(\mathbb{C})$ be a locally convex space that contains the constants and such that the inclusion $X \rightarrow H(\mathbb{C})$ is continuous. Let $X_0$ be the subspace of $X$ of all the functions $h \in X$ with $h(0) = 0$. The following conditions are equivalent for $\lambda \in \mathbb{C} \setminus \{0\}$.

(i) $S_{\lambda,g} : X_0 \rightarrow X_0$, $S_{\lambda,g} h(z) := e^{\frac{g(z)}{\lambda}} \int_0^z h'(\zeta)e^{-\frac{g(\zeta)}{\lambda}} d\zeta$, $z \in \mathbb{C}$, is continuous.

(ii) $T : X_0 \rightarrow X_0$, $Th(z) := e^{\frac{g(z)}{\lambda}} \int_0^z h(\zeta)g'(\zeta)e^{-\frac{g(\zeta)}{\lambda}} d\zeta$, $z \in \mathbb{C}$, is continuous.

The proof is obtained integrating by parts.
We deal with the Volterra operator acting on the Banach space \( H^\infty_v(\mathbb{C}) \), with \( v(r) = \exp(-\alpha r^p) \), where \( \alpha, p > 0 \). Recall:

**Proposition**

Assume that \( v(r) = \exp(-\alpha r^p) \), \( \alpha > 0 \), \( p > 0 \).

(i) \( V_g : H^\infty_v(\mathbb{C}) \to H^\infty_v(\mathbb{C}) \) is continuous if and only if \( g \) is a polynomial of degree less than or equal to the integer part of \( p \).

(ii) \( V_g : H^\infty_v(\mathbb{C}) \to H^\infty_v(\mathbb{C}) \) is compact if and only if \( g \) is a polynomial of degree less than or equal to the integer part of \( p - 1 \).
Lemma

Let $\nu$ be a weight such that $\nu(r)e^{\alpha r^n}$ is non-increasing on $[r_0, \infty[$ for some $r_0 > 0$, $\alpha > 0$ and $n \in \mathbb{N}$. The operator $T_\gamma : H_\nu^\infty(\mathbb{C}) \to H_\nu^\infty(\mathbb{C})$ defined by

$$T_\gamma h(z) := e^{\gamma z^n} \int_0^z \zeta^{n-1} h(\zeta) e^{-\gamma \zeta^n} d\zeta, \quad z \in \mathbb{C},$$

is continuous if $|\gamma| < \alpha$. 
Theorem

Assume that \( v(r) = \exp(-\alpha r^p) \), \( \alpha > 0 \), \( p > 0 \). Let \( g \) be a polynomial of degree \( n \) less than or equal to the integer part of \( p \) with \( g(0) = 0 \).

(i) If the degree \( n \) of \( g \) satisfies \( n < p \), then \( \sigma(V_g, H^\infty_v(\mathbb{C})) = \{0\} \).

(ii) If \( p = n \in \mathbb{N} \) and \( g(z) = \beta z^n + k(z) \), \( k \) a polynomial of degree strictly less than \( n \), then \( \sigma(V_g, H^\infty_v(\mathbb{C})) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq |\beta|/\alpha\} \).

Moreover, we have
\[ \sigma(V_g, H^\infty_v(\mathbb{C})) = \{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} \mid e^{\frac{\lambda}{\alpha}} \notin H^\infty_v(\mathbb{C})\} . \]

In this case we also have \( \sigma(V_g, H^0_v(\mathbb{C})) = \sigma(V_g, H^\infty_v(\mathbb{C})) \).
Spectra of Volterra operators on \( H_v^\infty(\mathbb{C}) \)

Idea of the proof.

(i) If \( n \) is less than or equal to the integer part of \( p - 1 \), then \( V_g : H_v^\infty(\mathbb{C}) \rightarrow H_v^\infty(\mathbb{C}) \) is compact.

Assume now that \( p - 1 < n < p \). For each \( \lambda \neq 0 \), \( e^{\frac{r}{\lambda}} \in H_v^\infty(\mathbb{C}) \).

Suppose first that \( g(z) = \beta z^n \) for some \( \beta \neq 0 \). For \( \lambda \neq 0 \), take \( \gamma > |\beta| / |\lambda| \). Clearly \( v(r)e^{\gamma r^n} \) is non-increasing on \([r_0, \infty[\) for some \( r_0 > 0 \). Our Lemmas above imply \( \lambda \in \rho(V_g, H_v^\infty(\mathbb{C})) \).

Suppose now that \( g(z) = \beta z^n + k(z) \) for some \( \beta \neq 0 \) and some polynomial \( k \) of degree strictly less than \( n \). Setting \( g_1(z) := \beta z^n \), we have \( V_g = V_{g_1} + V_k \), and \( V_k \) is a compact injective operator on \( H_v^\infty(\mathbb{C}) \). If \( \lambda \neq 0 \), we have \( V_g - \lambda I = (V_{g_1} - \lambda I) + V_k \). A classical result on operator theory yields \( \sigma(V_g, H_v^\infty(\mathbb{C})) = \sigma(V_{g_1}, H_v^\infty(\mathbb{C})) = \{0\} \).
Idea of the proof continued.

(ii) We suppose now that $v(r) = \exp(-\alpha r^n)$, $\alpha > 0$, and that $g$ is a polynomial of degree exactly $n$.

Consider first the case $g(z) = \beta z^n$. For $\lambda \in \mathbb{C} \setminus \{0\}$, we have $e^{\frac{g}{\lambda}} \in H^\infty_v(\mathbb{C})$ if and only if $|\beta|/|\lambda| \leq \alpha$. Therefore, 
\[
\{\lambda \mid |\lambda| \leq |\beta|/\alpha\} \subset \sigma(V_g, H^\infty_v(\mathbb{C})).
\]

Now take $\lambda \in \mathbb{C}$ with $|\lambda| > |\beta|/\alpha$. Since $v(r) \exp(\alpha r^n) = 1$, our Lemmas above imply $\sigma(V_g, H^\infty_v(\mathbb{C})) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq |\beta|/\alpha\}$ in the present case.

In the general case $g(z) = \beta z^n + k(z), \beta \neq 0$ and some polynomial $k$ of degree strictly less than $n$, we proceed as in the proof of part (i).

