STABLE CONSTANT MEAN CURVATURE HYPERSURFACES ARE AREA MINIMIZING IN SMALL L^1 NEIGHBORHOODS

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ABSTRACT.- We prove that a strictly stable oriented constant-mean-curvature hypersurface in a smooth closed manifold of dimension less than or equal to 7 is uniquely homologically area minimizing for fixed volume in a small L^1 neighborhood, proving a conjecture of Choksi and Sternberg.

1. INTRODUCTION

By work of White [W] and Grosse-Brauckman [Gr], a strictly stable oriented constant-mean-curvature surface S_0 in a smooth ambient Riemannian manifold M is minimizing in a small Riemannian distance neighborhood U of S_0 among competitor hypersurfaces $S \subset U$ enclosing the same volume. Assuming M compact, we extend their results to a small L^1 neighborhood of S_0, i.e., to hypersurfaces S such that $S - S_0$ bounds a region with net volume 0 and small total volume.

Stable constant-mean-curvature hypersurfaces in M appear in particular as solutions of the isoperimetric problem; see for instance [R1, R2].

If the ambient space is a flat 3-torus T^3 there is a connection between the isoperimetric problem and the study of mesoscale phase separation phenomena. For example, in diblock copolymers, different pieces of large molecules repel each other and, in an attempt to minimize the interfaces between such pieces, create a periodic structure. See Choksi and Sternberg [CS]. One simple model postulates periodic surfaces separating regions of fixed volume fraction and minimizing interface energy or area. A more sophisticated model with diffuse interfaces replaces a function which is 1 on one region and -1 on the other with a general L^1 function u with fixed integral and minimizes the Cahn-Hilliard functional

$$E_\varepsilon(u) = \int_{T^3} \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) \right) dx,$$

where W is nonnegative with $W(\pm 1) = 0$. The W term encourages u to focus on the values ± 1, while the ∇u term would minimize the transitions. So-called Γ-convergence theory shows that as $\varepsilon \to 0$, some subsequence of global minimizers

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of the Cahn-Hilliard energy converges to a sharp interface limit with $u = \pm 1$ of solutions to the isoperimetric problem (see [B, Example 0.1 and Comments p. 112], [CS, Prop. 3.1]). Conversely, Kohn and Sternberg [KS, Prop. 2.1 and §3.3] and Choksi and Sternberg [CS, Prop. 3.2] show by compactness that if a hypersurface S_0 minimizes area uniquely (up to translations) for given volume in an L^1-neighborhood, then there are nearby L^1-local minimizers of the Cahn-Hilliard energy E_ε for small ε. Choksi and Sternberg [CS, p. 382 and Rmk. 7(ii)] conjecture that it suffices to assume S_0 has positive second variation. Our results prove this conjecture, providing local Cahn-Hilliard minimizers (Corollary 5). Pacard and Ritoré [PR, Thm. 4.2] use perturbation theory for PDEs to prove a similar result for nondegenerate critical points rather than local minima.

In flat 3-tori there are some beautiful minimal surfaces, the Schwarz P and D surfaces and the Gyroid G of A. Schoen, which are closely related to complex phases appearing in periodic phase separation. Ross [Ro] has proved that these surfaces are stable for fixed volume and there is a particular interest in providing a mathematical treatment of these complex phases by minimizing locally the Cahn-Hilliard energy or other more sophisticated models. Corollary 5 proves nearby Cahn-Hilliard diffuse-interface versions of these surfaces.

For background in geometric measure theory see Giusti [G] and Morgan [M1].

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2. The proof

Two homologous oriented hypersurfaces S and S' in M^n bound a region. Technically, this region is an n-dimensional integral current which is unique (up to multiples of M in the case that M is orientable). We define the L^1-distance $||S - S'||_{L^1}$ between them as the minimum of the masses of the regions they bound. For example in $\mathbb{R}^n$ if S and S' bound regions (of multiplicity +1) Ω and Ω', respectively, then $||S - S'||_L^1$ is just the volume of the symmetric difference between Ω and Ω'.

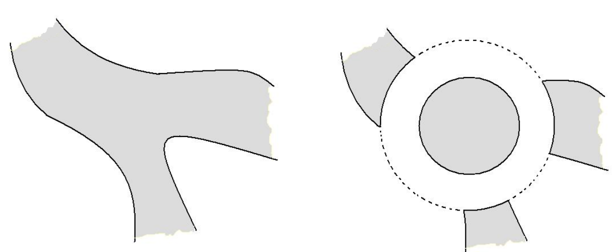

Figure 1. Proof of the area growth estimate: S at the left. At the right we have a competitor hypersurface which differs from S inside a ball of radius r and encloses the same volume as S in this ball.

We will need the following isoperimetric version of the classical result after Fleming [F, Sect. 5] that for $n \leq 7$, area-minimizing hypersurfaces in the $\mathbb{R}^n$ are hyperplanes. For $n = 3$ da Silveira ([dS], see also [LR]) proved the result under the weaker hypothesis that S be stable for fixed volume, i.e., in competition with surfaces which together with S bound net oriented volume 0.

Proposition 1. *Let S be an oriented hypersurface (multiplicity one) without boundary in $\mathbb{R}^n$, $n \leq 7$, area-minimizing for fixed volume under changes of compact support. Then S is either a hypersphere or a hyperplane.*

Proof. If S is compact, S is a hypersphere by the standard isoperimetric inequality. Assuming S is not compact, the hypothesis of the proposition implies that S has constant mean curvature and is stable for fixed volume. Given $r > 0$, inside the ball $B(r)$ around a fixed point of S, replace the region bounded by S by a ball $B(\rho)$, $0 \leq \rho \leq r$, of the same volume as in Figure 1. The resulting area inside $B(r)$ is at most twice the area of the hypersphere $\partial B(r)$. Hence the original area inside the ball of the minimizer S is at most Cr^{n-1}, for some C. Since this holds for all $r > 0$, by Cheung [C] S has mean curvature 0. By monotonicity of the mass ratio [A, Cor. 5.1(3) p. 446], the area divided by $\alpha_n r^{n-1}$, where α_n is the volume of the unit ball in $\mathbb{R}^{n-1}$, is nondecreasing in r, varying from 1 as r approaches 0 to a limit C_0, as r approaches infinity. Therefore homothetic contractions, restricted to balls about the origin, have area bounded below and above, so that by compactness [M1, 5.5 and remark page 88], a subsequence converges to a nonzero limit, which has constant area ratio C_0 and is therefore a cone [A, Cor. 5.1(2)]. Since the cone minimizes area for given volume and $n \leq 7$, by regularity [M2] the cone must be a hyperplane (with multiplicity 1 because it is the boundary of a region) and $C_0 = 1$. Hence likewise S has constant mass ratio 1 and must be a hyperplane. □

Now we prove our main result.

Theorem 2. *In a smooth closed Riemannian manifold of dimension $n \leq 7$, let S_0 be a smooth oriented constant-mean-curvature hypersurface, possibly with boundary, with positive second variation for fixed volume and boundary. Then S_0 is uniquely homologically area minimizing for fixed volume among oriented hypersurfaces in a small L^1 neighborhood.*

In particular, if S_0 bounds a region, then it minimizes area among hypersurfaces S enclosing the same volume with $||S - S_0||_{L^1}$ small. It is not necessary to assume that S_0 is a boundary. Our proof gives that S_0 minimizes among competitors S such that $\partial S = \partial S_0$ and $S - S_0$ bounds net oriented volume 0.

Proof. Denote area, volume, and mean curvature by A, V, and H. The subscript 0 refers to S_0. Our hypersurface S_0 has positive second variation under smooth variations which fix volume (or equivalently under smooth variations which fix volume to first order). By Grosse-Brauckmann [Gr, Lemma 5], for some $C > 0$, S_0 has positive smooth second variation for the energy

$$F = A + H_0 V + (C/2)(V - V_0)^2$$

under general smooth variations. As Grosse-Brauckman [Gr, last paragraph] points out, White [W, Thm. 3] applies to show that S_0 uniquely minimizes F in a neighborhood. To see this, let ω be a smooth differential form which over homologous surfaces gives the volume enclosed with S_0, such that $C\omega$ is small in a neighborhood of S_0 [W, end of Intro.]. To apply [W, Thm. 3], take F to be the area integrand, $F_1 = F + C\omega$, $F_2 = F$, and $\phi(x, y) = (x-y)^2/2C$. By [W, Thm. 3], S_0 uniquely minimizes F in a small neighborhood U of its support. In particular, for fixed volume, S_0 uniquely minimizes A in U.

To obtain a contradiction, suppose that there is a sequence of surfaces S_i of no more area than S_0 converging in L^1 to S_0 and enclosing net signed volume 0 with S_0. Because the ambient manifold is compact, we may assume that S_i minimizes area for fixed $||S_i - S_0||_{L^1} = \varepsilon_i \to 0$. On the complement of S_0, S_i has two parts: where the region Ω_i bounded by $S_0 - S_i$ has positive or negative orientation. (If S_i and hence Ω_i has multiplicity, both decompose into pieces of multiplicity 1 [M1, Fig. 10.1.1].) Each part minimizes area for fixed volume; therefore S_i is a smooth constant mean curvature surface [M2, Cor. 3.7] (although the constants on the two parts need not be equal; we assert no regularity at points of S_0). By the first paragraph of this proof, each S_i strays outside U. By replacing S_i by a subsequence, we may assume that each S_i strays outside of U always with the same part or always with both parts. Hence by monotonicity, for a relevant part of S_i, the curvature of the sequence S_i is not bounded in $M - U$. Indeed, if the mean curvature were bounded, then by monotonicity of the mass ratio [A, Cor. 5.1(3) p. 446 and Rmk. 4.4], the area of S_i outside a smaller neighborhood U' would be bounded below by some positive constant δ, and then

$$A(S_0) \leq \liminf A(S_i) - \delta \leq A(S_0) - \delta,$$

the desired contradiction.

Choose a point outside of U on a relevant part of S_i of maximum $|II|^2$ (the sum of the squares of the principal curvatures) and scale the picture to make $|II|^2 = 1$. A limit is minimizing for fixed volume in $\mathbf{R}^n$ and hence must be a round sphere by Proposition 1. Hence for some large i, S_i includes a small, nearly round sphere partly outside U. We may assume that there are no other points of that part of S_i outside U, since otherwise we could repeat the argument on S_i minus the first sphere and obtain a second such sphere, while replacing them with one sphere would do better. Hence in each part of S_i, there is at most one such sphere partly outside U. For a constant c_n depending only on the dimension n, the total area and volume of such spheres satisfy $a > c_n v^{(n-1)/n}$.

Let T_i be S_i minus such spheres, so that T_i lies in the neighborhood U of S_0. Now

$$F(T_i) < A(S_i) - c_n v^{(n-1)/n} + |H_0|v + (C/2)v^2 < A(S_i)$$

for small v and hence for large i. Then

$$F(T_i) < A(S_i) < A(S_0) = F(S_0),$$

a contradiction of the fact that S_0 minimizes F in U. □

Remark 3. For minimal surfaces, the result also holds without volume constraints. The same proof holds, with simplifications.

Remark 4. When the ambient manifold M has nontrivial isometries, it suffices to assume that S_0 has positive second variation orthogonal to the isometries, for fixed volume. Our same proof applies because White [W, Thm. 3] immediately generalizes. White's proof observes that a sequence of other minimizers in shrinking physical neighborhoods of S are *almost minimizing* and hence Hölder differentiable manifolds that converge Hölder differentiably to S, contradicting the positive second variation of S. In the presence of isometries, one may translate the nearby minimizers to be graphs of functions orthogonal to the isometries to obtain the same contradiction.

As a direct consequence of Choksi and Sternberg [CS, Prop. 3.2], Theorem 2, and Remark 4, we have the following:

Corollary 5. *In a flat torus T^n of dimension $n \leq 7$, let S_0 be a smooth oriented constant-mean-curvature hypersurface, possibly with boundary, with positive second variation orthogonal to any isometries of M for fixed volume and boundary. Then for some $\varepsilon_0 > 0$, for $0 < \varepsilon < \varepsilon_0$, there is a family u_ε of L^1-local minimizers of the Cahn-Hilliard energy E_ε converging in L^1 to S_0.*

In particular, as Ross [Ro] proved that the P, D, and G minimal surfaces in T^3 have positive second variation orthogonal to the isometries of T^3, it follows that there are L^1-nearby diffuse-interface local minimizers of the Cahn-Hilliard energy.

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