Constant Mean Curvature Surfaces in a Half-Space 
of $\mathbb{R}^3$ With Boundary in the Boundary of the 
Half-Space

by

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The structure of the set of compact constant mean curvature surfaces whose boundary is a given Jordan curve in $\mathbb{R}^3$ seems far from being understood. We shall consider a simple, but interesting, situation concerning this problem: Assume that $M$ be an embedded compact $H$-surface in $\mathbb{R}^3_+ = \{x_3 \geq 0\}$ with $\partial M = \Gamma \subset P = \{x_3 = 0\}$. Then, there is little known about the geometry and topology of $M$ in terms of $\Gamma$. For example, if $\Gamma$ is convex, is $M$ of genus zero? When $\Gamma$ is a circle, it follows from Alexandrov [A] that $M$ is necessarily a spherical cap or the planar disk bounded by $\Gamma$.

We first show that if $\Gamma_n \subset P$ is a sequence of embedded (perhaps nonconnected) curves converging to a point $p$, and $M_n \subset \mathbb{R}^3_+$ is a sequence of 1-surfaces ($H = 1$), with $\partial M_n = \Gamma_n$, then some subsequence of $M_n$ converges either to $p$ or to the unit sphere tangent to $P$ at $p$ (the convergence being smooth in $\mathbb{R}^3 - p$). The same kind of result was obtained by Wente [W] when $\Gamma_n$ is an arbitrary Jordan curve in $\mathbb{R}^3$ converging to a point $p$ and $M_n$ is an immersed topological disc bounded by $\Gamma_n$ which minimizes area among disks bounding a fixed algebraic volume.

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Our second result gives a partial answer to the question above. Given a convex Jordan curve $\Gamma \subset P$ and an embedded compact $H$-surface $M \subset \mathbb{R}^3_+$ bounded by $\Gamma$, we shall show that, for $H$ sufficiently small (depending on $\Gamma$), $M$ is a topological disc. Moreover, we can give a rather complete description of the shape of $M$, even near its boundary. As a consequence we obtain an upper bound, which depends on $\Gamma$ but not on $H$, for the area of the surfaces above with positive genus.

Finally we remark that our assumptions about $M$ hold under different geometric restrictions. For general $\Gamma \subset P$, if we denote by $\Omega$ the compact domain in $P$ bounded by $\Gamma$, then the maximum principle shows that any constant mean curvature graph over $\Omega$ bounded by $\Gamma$, lies in $\mathbb{R}^3_+$ (or $\mathbb{R}^3_-\). When $\Gamma$ is convex, there are two other conditions for embedded compact $H$-surfaces $M$ with $\partial M = \Gamma$ which imply that $M$ lies in one of the half spaces bounded by $P$: this holds if either $M$ is transversal to $P$ along $\Gamma$, see Earp, Brito, Meeks and Rosenberg [E], or $\text{Area}(M)H^2 \leq 2\pi$, see Lopez and Montiel [L].

Preliminaries

Let $P(t)$ denote the horizontal plane \{${x_3 = t}$\}, $P = P(0)$. Vertical planes will be denoted by $Q$. Let $a = (0,0,1)$ and $B(p,r)$ (respectively $D(p,r)$) denote the closed Euclidean ball in $\mathbb{R}^3$ (respectively in $P$) centered at $p$, of radius $r$. If the ball is centered at the origin $p = 0$ we will write simply $B(r)$ (resp. $D(r)$). In this paper an $H$-surface $M$ will mean an embedded compact constant mean curvature (equal to $H$) surface in the half-space $\mathbb{R}^3_+$ with $\partial M \subset P$. If $M$ is minimal, then the maximum principle implies that $M$ is a domain in $P$. So we assume $H \neq 0$. 
We orient $H$-surfaces by their mean curvature vector $\vec{H}$; so if $n$ is the unit normal vector field along $M$, then $\langle n, \vec{H} \rangle = H > 0$. $\Gamma$ is oriented by the orientation of $M$ and bounds a compact region $\Omega$ in $P$ so that $M \cup \Omega$ is a 2-cycle of $\mathbb{R}^3$. The maximum principle for surfaces with nonnegative mean curvature implies that $\text{int}(M) \cap \text{int}(\Omega) = \emptyset$ and thus $M \cup \Omega$ is an embedded piecewise smooth surface without boundary. Let $W$ denote the closure of the bounded component of $\mathbb{R}^3 - (M \cup \Omega)$. Notice we do not assume $\Gamma$ is connected.

We say that $M$ is a small $H$-surface if $M \subset B(p, r)$ for some $p \in \mathbb{R}^3$ and $r \leq 1/H$. Otherwise we will say that $M$ is a large $H$-surface. It follows from the maximum principle that, for small $H$-surfaces $M$, one has $M \subset \bigcap\limits_{\alpha} B_{\alpha}$, where $B_{\alpha}$ denotes the family of balls $B(q, \rho)$ such that $q \in \mathbb{R}^3$, $\rho \leq 1/H$ and $\partial M \subset B(q, \rho)$.

One of our main tools will be the Alexandrov reflection technique. This idea was first introduced in [A] to show that the sphere is the unique compact $H$-surface (without boundary) embedded in $\mathbb{R}^3$. Now we sketch the main steps of this method, adapted to our situation. A detailed version of it can be found, for instance, in [S]. Let $Q$ be a vertical plane disjoint from $M$ and $b$ a nonzero vector normal to $Q$. Suppose that one parallel translates $Q$ in the direction of $b$ until the new plane $Q_1$ touches $M$ for the first time. Then when one translates $Q_1$ slightly further (always in the direction of $b$), to a position $Q_2$, the closure of the part of $M$ swept out, $M_2$, is a graph over a part of $Q_2$ and the reflection, $M_2^*$, of $M_2$ in the plane $Q_2$ is contained in $W$. This is clear if $Q_1$ touches $M$ only at the interior points. To see that the same is true in the case that some boundary point of $M$ lies in $Q_1$, we only need to observe that, thanks to the boundary maximum principle for $H$-surfaces, the angle (on the $W$ side) between $M$ and $\Omega$ along $\Gamma$ is everywhere positive. Denote by $\Gamma_2$ the part of $\Gamma$ left behind the plane $Q_2$ and by $\Gamma_2^*$ its symmetric image by $Q_2$. Thus $\Gamma_2^* \subset \Omega$. 
Now one continues parallel displacement of $Q_2$ until the last position $Q_3$ such that, following the notation above, the reflection $M_3^*$ of $M_3$ in the plane $Q_3$, is contained in $W$. Then $M_3$ is a graph over a domain in $Q_3$ and, using the (interior and boundary version of the) maximum principle for $H$-surfaces, one proves that $Q_3$ is also the last position such that $\Gamma_3^* \subset \Omega$. So $Q_3$ depends only on $\Gamma$. Moreover we obtain that, either $M$ is symmetric with respect to $Q_3$ or $\text{int}(M_3^*)$ does not meet $M$. Notice that $\Gamma_3$ is necessarily non empty and that, if $\Gamma$ is a convex curve, then the part of $M$ outside $\Omega \times [0, \infty[$ is a graph over a part of $\Gamma \times [0, \infty[$; in particular, each component has genus zero.

Now consider a horizontal plane $P(t)$ and put $M(t) = M \cap \{x_3 \geq t\}$. If we assume $P(t)$ is very high then the above reflection argument works when one moves $P(t)$ down, if we change $W$ to $W' = W \cup (\Omega \times] - \infty, 0])$. In this case, if $P(t_1)$ is the highest plane touching $M$, then $M \cap P(t_1)$ contains only interior points of $M$. Let $P(t_3)$ be the lowest horizontal plane in $\mathbb{R}_+^3$ such that the reflected image $M(t_3)^*$ of $M(t_3)$ in $P(t_3)$ is contained in $W'$. Then $M(t_3)$ is a vertical graph over a part of $P$ and either $t_3 = 0$, in which case $M$ is a graph over $\Omega$, or $t_3$ occurs exactly at the first time that the reflected surface touches $\Gamma$. In the latter case, either $M$ is a spherical cap or $\text{int}(M(t_3)^*)$ meets $M$ only at points of $\Gamma$ and always transversally. Notice that, in particular, the highest point of $M$ must lie in the cylinder $\text{int}(\Omega) \times ]0, \infty[$. Moreover, it follows from the height estimate below that $t_3 < 1/H$ and, so the height of $M$ is at most $2/H$.

Finally note that, if $M \subset \Omega \times [0, \infty[$, or more generally, if $\langle n(p), a \rangle \leq 0$ for all $p \in \Gamma$, then doing Alexandrov reflexion with horizontal planes we get that $M$ is a graph over $\Omega$. 
Estimates of Height, Area and Curvature for $H$-surface graphs.

Serrin [Se] observed that one has height estimates for $H$-graphs with zero boundary values. Suppose $M \subset \mathbb{R}^3_+$ is an $H$-graph over a compact domain $\Omega \subset P$ with $\partial M = \partial \Omega$. Then we can see that the highest point of $M$ is below height $1/H$ as follows: let $\varphi = Hx_3 + n_3$. Then, if $K$ denotes the Gauss curvature of $M$,

$$\Delta \varphi = (H(2H) - (4H^2 - 2K))n_3 = -2(H^2 - K)n_3 \geq 0,$$

since our choice of $n$ makes $n_3 \leq 0$. Since $\varphi = n_3 \leq 0$ on $\partial M$, it follows that $\varphi \leq 0$ on $M$. So $Hx_3 \leq -n_3 \leq 1$, and $x_3 \leq 1/H$, as desired.

We claim that $M$ satisfies area and curvature estimates (depending only on $H$) on compact subsets of $\text{int}(\mathbb{R}^3_+)$. To see this, let $\varepsilon > 0$ and $M(\varepsilon)$ denote the part of $M$ above $P(\varepsilon)$. Assume $M(\varepsilon)$ is not empty. Since $\varphi$ is subharmonic and $\varphi = H\varepsilon/4 + n_3 \leq H\varepsilon/4$ on $\partial M(\frac{\varepsilon}{4})$, we have that, on $M(\frac{\varepsilon}{2})$,

$$n_3 \leq H\varepsilon/4 - Hx_3 \leq H\varepsilon/4 - H\varepsilon/2 \leq -H\varepsilon/4.$$

$M$ is the graph of a certain function $u \in C^\infty(\Omega)$. Consider the compact subdomain $\Omega(\varepsilon) = \{ x \in \Omega / u(x) \geq \varepsilon \}$. The above estimate of $n_3$ yields an estimate of the gradient of $u$ in $\Omega(\frac{\varepsilon}{2})$, which, in turn, we can use (in an obvious way) to show that the (euclidean) distance between $\Omega(\varepsilon)$ and $\partial \Omega(\frac{\varepsilon}{2})$ is bigger than some positive constant $\delta$ depending only on $H$ and $\varepsilon$. Thus, for each $p \in \Omega(\varepsilon)$ we have that $D(p, \delta) \subset \Omega(\frac{\varepsilon}{2})$ and we control $u$ and $|\nabla u|$ on $D(p, \delta)$. Then standard results in elliptic equations (see [G], theorems 13.1 and 6.2) show that we have $C^{2,\alpha}$ estimates for $u$ on $\Omega(\varepsilon)$ depending on $H$ and $\varepsilon$. Now the claim follows directly.
The main results

**Theorem 1.** Let $M_n \subset \mathbb{R}^3_+$ be $H$-surfaces with $H = 1$, and $\Gamma_n = \partial M_n \subset D(r_n) = \{(x_1, x_2, 0)/x_1^2 + x_2^2 \leq r_n^2\}$, with $r_n$ a sequence converging to zero. Then there is a subsequence of $M_n$ that either converges to the origin $0 \in \mathbb{R}^3$ or to the sphere $S \subset \mathbb{R}^3_+$ of radius one tangent to $P$ at $0$. In the first case the surfaces converge as subsets and in the second one the convergence is smooth (any $C^k$) on compact subsets of $\mathbb{R}^3 - 0$.

**Proof.** It follows from the height estimates and the Alexandrov reflection technique that all the $M_n$ are contained in a fixed compact of $\mathbb{R}^3$. Let $r > 0$ and $Q$ be a vertical plane outside this compact. For $n$ large, $\partial M_n \subset D(r)$ so, using the Alexandrov method, one can parallel translate $Q$ until $\partial D(r)$ and the part of $M_n$ swept out by $Q$ is a graph over a part of $Q$. Therefore one has uniform area and curvature estimates for this part of $M_n$. Alexandrov reflection with horizontal planes gives that the part of each $M_n$ above $P(1)$ is a vertical graph, so one has uniform area and curvature estimates for the $M_n(1 + \delta), \delta > 0$.

Standard compactness techniques yield a subsequence (which we also call) $M_n$ that converges on compact subsets of $\mathbb{R}^3 - I$, where $I = 0 \times [0, 1]$, see for instance [Wh]. The limit is either empty or a surface $M$ of mean curvature one, properly embedded in $\mathbb{R}^3 - I$ (embeddedness follows because the part of $M$ contained in each one of the half-spaces $\{\alpha x_1 + \beta x_2 > 0\}, \alpha^2 + \beta^2 = 1$, and $\{x_3 > 1\}$ is a graph). If the limit is empty then for $n$ large, $M_n$ is uniformly close to $I$. Thus $M_n$ is a small 1-surface and, as $\Gamma_n \subset B(r_n)$, it follows that $M_n \subset B(r_n)$. So, $M_n$ converges to 0.

Now we assume the $M_n$ converge to a surface $M$ properly embedded in $\mathbb{R}^3 - I$. If one does Alexandrov reflection, for each $r > 0$ vertical planes can be moved up to $\partial D(r)$ and the reflected images of $M$ by these planes lie in the domain enclosed
by $M$ (since this holds for $M_n$, $n$ large). So this works up till $r = 0$ by continuity, $M$ is a rotational surface about the vertical line through 0 and each component of $M$ has multiplicity one. $M$ has height at most two so it is neither a Delaunay surface nor a stack of spheres of radius one. So $M$ is the sphere $S$ of radius one passing through 0.

Finally we show the convergence is uniform on compact subsets of $\mathbb{R}^3 - 0$. Given $\varepsilon > 0$, there exists $r > 0$ so that for $n$ large,

$$M_n \cap (D(r) \times ]\frac{3}{2}, \infty[) = M_n \cap (D(r) \times ]2 - \varepsilon, 2 + \varepsilon[),$$

and this intersection is a graph above $D(r)$. Coming down with horizontal planes $P(t)$ from $t = 2$ to $t = 1$ we see that $M_n \cap (D(r) \times [\varepsilon, 2 - \varepsilon]) = \emptyset$. So we have uniform estimates for $M_n$ on compact subsets of $\mathbb{R}^3 - 0$, not just on compact subsets of $\mathbb{R}^3 - I$.

**Remark 1.** From theorem 1 we conclude that given a positive integer $k$ and $\varepsilon, \delta > 0$, there exists $r = r(k, \varepsilon, \delta) > 0$ such that any large $H$-surface $M \subset \mathbb{R}^3_+$ with $H = 1$ and $\partial M \subset D(r) \subset P$ satisfies that $M - B(\delta)$ is the graph (with respect to the normal lines of the sphere) of a function $u$, defined on a domain of $S$, with $\|u\|_{C^k} < \varepsilon$. In fact, if this statement were false we could construct a sequence $M_n$ which would contradict theorem 1.

**Remark 2.** Theorem 1 remains true if one assumes that the surfaces $M_n \subset \mathbb{R}^3_+$ are compact, embedded, have constant mean curvature $H = 1$ and non necessarily planar boundary $\partial M_n \subset B(r_n)$, with $r_n \to 0$ as $n \to \infty$. To prove that, instead of using vertical planes, one does Alexandrov reflection with planes that are $\varepsilon$-tilted from the vertical, i.e. planes $Q$ whose unit normal vector $b(\varepsilon)$ satisfies $\langle b(\varepsilon), a \rangle = \varepsilon$. Given $\varepsilon, r > 0$, one can choose $r_n$ small enough so that Alexandrov
reflection works with \( \varepsilon \)-tilted planes \( Q + tb(\varepsilon) \), \( t \) coming from \( -\infty \), up till the plane reaches \( B(r) \). Taking \( \varepsilon \to 0 \) we get the assertion in Theorem 1, see [K] for more details in a related situation.

**Theorem 2.** Let \( \Gamma \subset P \) be a strictly convex curve. There is an \( H(\Gamma) > 0 \), depending only on the extreme values of the curvature of \( \Gamma \), such that whenever \( M \subset \mathbb{R}^3_+ \) is an \( H \)-surface bounded by \( \Gamma \), with \( 0 < H < H(\Gamma) \), then \( M \) is topologically a disk and either \( M \) is a graph over the domain \( \Omega \) bounded by \( \Gamma \) or \( N = M \cap (\Omega \times ]0, \infty[) \) is a graph over \( \Omega \) and \( M - N \) is a graph over a subannulus of \( \Gamma \times ]0, \infty[ \), with respect to the lines normal to \( \Gamma \times ]0, \infty[ \). In the latter case (i.e. when \( M \) is not a graph over \( \Omega \)), given any \( \theta, 0 < \theta < \pi/2 \), and \( H \)-surface \( M \) as above, one can also ensure that the angle between \( n \) and \( a \) is less than \( \theta \) along \( \Gamma \) (so \( H(\Gamma) \) will also depend on \( \theta \)).

Before proving Theorem 2, we state a lemma whose proof we will give later. We remark that the radius \( r \) of the lemma is independent of the value of \( H \).

**Lemma 2.1.** Let \( \Gamma \subset P \) be strictly convex. There is an \( r > 0 \), depending only upon the extreme values of the curvature of \( \Gamma \), such that whenever \( M \subset \mathbb{R}^3_+ \) is an \( H \)-surface with boundary \( \Gamma \), there is a point \( p \in \Omega \) (\( p \) depends on \( M \)) such that \( D(p, r) \subset int(\Omega) \) and \( M \cap (D(p, r) \times ]0, \infty[) \) is a graph over \( D(p, r) \).

**Proof of theorem 2.** Let \( M \) be an \( H \)-surface as in Theorem 2 and let us first suppose that \( M \) is small. In this case, if \( H \) is smaller than the curvature values of \( \Gamma \), then \( M \subset \cap B_\alpha \), where \( B_\alpha \) is the family of closed euclidean balls of radius \( 1/H \) centered at points of \( P \) with \( \Gamma \subset B_\alpha \). It is clear, from the relation between the curvatures of \( \Gamma \) and \( B_\alpha \), that each point of \( \Gamma \) lies in the boundary of some \( B_\alpha \). In particular, \( \cap B_\alpha \) is contained in the solid cylinder \( \Omega \times ]0, \infty[ \) and, thus, \( M \) is a graph over \( \Omega \).
Suppose now that $M$ is a large $H$-surface. Let $r > 0$ and $p \in \Omega$ be given by Lemma 2.1. Let $\Sigma'$ be the unique vertical catenoid meeting $P$ in the circle $C_0 = \partial D(p, \rho)$ where $\rho < r$ and $\rho$ is smaller than the smallest radius of curvature of $\Gamma$ (the latter condition allows us to translate $C_0$ horizontally in $\Omega$ so as to touch every point of $\Gamma$), and such that the angle between $\Sigma'$ and $P$ along $C_0$ is $\theta$. Here the angle $\theta$ is the angle between $\Sigma'$ and the non compact component of $P - C_0$; so $\Sigma'$ is a graph over this non compact component. Let $\Sigma = \Sigma' \cap \{0 \leq x_3 \leq 1\}$ and $C_1$ be the circle of $\Sigma$ at height one. Let $V = \{v \in P/C_0 + v \subset \Omega\}$ and let $D(R)$ be a sufficiently large disk in $P$ so that the translated disc $D(r) + a \subset \{x_3 = 1\}$ contains the circle $C_1 + v$ for all $v \in V$. Clearly $V$ is compact and convex.

Now consider Alexandrov reflection of $M$ with horizontal planes coming down from above $M$. A rescaled version of Theorem 1 will imply that if $H$ is small enough, the part of the reflected surface which is in $D(R) \times [0, \infty[ \text{ is uniformly near } D(R)$; it is a graph above $D(R)$ of height less than $1/2$, and $M \cap (D(R) \times [\frac{1}{2}, \infty[)$ is a graph above $D(R)$ of height bigger than $1/H$. We make this precise in the next paragraph.

We know the highest point $q$ of $M$ is in $\Omega \times ]0, \infty[$, hence $q$ is in $D(R) \times [0, \infty[$. From Remark 1, if $H$ is small enough, the image of $M(\frac{1}{2H})$ by the homothety of ratio $H$ centered at the origin is arbitrarily near the unit upper half-sphere $S \cap \{x_3 \geq 1\}$, with respect to the $C^k$-distance, $k \geq 1$. Thus the component $G$ of $M \cap (D(R) \times [0, \infty[)$, that contains $q$ is almost flat when $H$ is small. In particular, since $M$ is horizontal at $q$, $G$ is a graph above $D(R)$, $G \subset \{x_3 > 1/H\}$ and the total vertical oscillation of $G$ is less than one half for $H$ small enough (depending on $R$). Then when one does Alexandrov reflection with horizontal planes, the last position of the reflected image $G^*$ of $G$ is a graph above $D(R)$ of height less than one half (remember that when one does reflection coming down from $q$, the
last position occurs just the first times that a point of $M$ reflects to a point of $\partial M$). Since the oscillation of $G$ is less than one half, $G^*$ lies below $D(R) \times \frac{1}{2}$.

Moreover, as another consequence of the Alexandrov technique, we obtain that $M \cap (D(R) \times [\frac{1}{2}, \frac{1}{4}]) = \emptyset$.

Therefore $\Sigma + a$ and $C_1 + ta$, $0 \leq t \leq 1$, are contained in $\text{int}(W)$ (recall $W$ is the compact component bounded by $M \cup \Omega$) and, as Lemma 2.1 gives $C_0 + ta \subset \text{int}(W)$ for $0 < t \leq 1$, it follows that $\Sigma \subset W$. Otherwise when one translates $\Sigma + a$ down to $\Sigma$, there would be a first point of contact of $\Sigma + ta$ with $M$. This contact point occurs on the $W$ side of $M$, the side to which the mean curvature vector of $M$ points. This is impossible since the point of contact is an interior point of both $M$ and $\Sigma + ta$ and $\Sigma + ta$ is a minimal surface.

We know that the boundary component of $\Sigma + v$, for $v \in V$, at height one, is contained in $\text{int}(W)$. Hence $\Sigma + v \subset W$ for each $v \in V$ by similar reasoning as above: the family $\Sigma + tv$, $0 \leq t \leq 1$, can have no first point of interior contact with $M$ as $t$ goes from 0 to 1.

Our choice of $C_0$ guarantees that for each $q \in \Gamma$, there is a $v \in V$ such that $C_0 + v$ is tangent to $\Gamma$ at $q$. Hence $\theta$ is strictly bigger than the outer angle that $W$ makes with $P$ at $q$. So this holds along $\Gamma$.

Since the horizontal translations of $\Sigma$, $\Sigma + tv$, $v \in V$, $0 \leq t < 1$ are all in $W$ and $D(r) \times [0, 1] \subset \text{int}(W)$ by Lemma 2.1, we know that $\Omega \times [0, 1] \subset \text{int}(W)$. Also $M$ meets $D(R) \times [\frac{1}{2}, \infty]$ in a graph above $D(R)$ of height bigger than $1/H$, so $M \cap (\Omega \times [0, \infty])$ is a graph above $\Omega$ of height bigger than $1/H$. The part of $M$ outside $\Omega \times [0, \infty]$ is a graph over a subannulus of $\Gamma \times [0, \infty]$, so $M$ is topologically a disk and Theorem 2 is established.

Proof of Lemma 2.1. Consider doing Alexandrov reflection with horizontal planes, coming down from above $M$. If we can come down to $P$, then $M$ is a graph above
Ω and 2.1 is clear. Otherwise there is a height $t > 0$ where the reflected surface touches $\Gamma$ for the first time at a point $q \in \Gamma$. So $q \times [0, \infty]$ intersects $M$ exactly once and $q \times [0, 2t] \subset \text{int}(W)$. Also the part of $M$ above height $t$ is a vertical graph.

Now consider Alexandrov reflection with vertical planes $Q$, $v$ normal to $Q$, $|v| = 1$. Suppose one can do Alexandrov reflection of $M$, moving the plane $Q$ slightly beyond $q$, and denote by $J(v)$ the open segment in $\text{int}(\Omega)$ joining $q$ to its reflected image by this plane. Clearly the vertical rectangle $J(v) \times [0, 2t] \subset \text{int}(W)$. Suppose we could repeat this reasoning for a family of directions $F \subset \{v \in P; |v| = 1\}$, such that, for some $p \in \Omega$ and $r > 0$, we have $D(p, r) \subset \cup_{v \in F} J(v)$. Then we would have

$$D(p, r) \times [0, 2t] \subset \cup_{v \in F} J(v) \times [0, 2t] \subset \text{int}(W).$$

Hence $D(p, r) \times [0, \infty]$ would intersect $M$ only at points above $P(2t)$, so this intersection would be a graph above $D(p, r)$ as desired. So we have to understand the horizontal directions $v$ for which Alexandrov reflection goes beyond a given point $q \in \Gamma$.

First recall, that for horizontal directions $v$, one can always do Alexandrov reflection up till $\Gamma$. Let $k$ be the minimum curvature of $\Gamma$ and let $C \subset P$ be a circle of curvature $k$. So if $C$ is tangent to $\Gamma$ at $q$, then $\Gamma$ is inside $C$. Take $\rho > 0$ smaller than the smallest radius of curvature of $\Gamma$. Thus the tubular neighbourhood of $\Gamma$ in $P$ of radius $\rho$ is an embedded annulus. Then for each horizontal $v$, $|v| = 1$, one can do Alexandrov reflection with vertical planes orthogonal to $v$ at least a distance $\rho/2$ beyond the first time the plane meets $\Gamma$ and, so, at least a distance $\rho/2$ beyond the first time the vertical plane meets the circle $C$.

Now consider those horizontal vectors $v$ such that Alexandrov reflection with planes orthogonal to $v$ left behind $q$ (this will hold for those directions in some neighborhood $F \subset \{v \in P/|v| = 1\}$ of the inward pointing normal to $C$ at $q$). It
is clear from the geometry of the circle, that $\cup_{v \in F} J(v)$ contains a disk $D(p, r)$, $r > 0$, where $r$ depends on $p$ and $C$ (but not on $q \in \Gamma$). This completes the proof of Lemma 2.1.

**Corollary 2.2.** Let $\Gamma \subset P$ be a strictly convex curve. Then there exists $V(\Gamma)$, $A(\Gamma) > 0$, depending only on the extreme values of the curvature of $\Gamma$, such that any $H$-surface $M \subset R^3_+$ bounded by $\Gamma$ which either encloses a volume $\text{Vol}(W) > V(\Gamma)$ or verifies $\text{Area}(M) > A(\Gamma)$, is a topological disk.

**Proof.** From the height estimate we have that $M \subset B(r + \frac{2}{H})$ where $r > 0$ is chosen such that $\Gamma \subset D(r)$. So $W$ is contained in the same ball and

$$\text{Vol}(W) \leq c(r + 2/H)^3,$$

for a certain positive constant $c$. Thus if $\text{Vol}(W)$ is big enough, we will have $H < H(\Gamma)$ and the result follows from Theorem 2.

Computing the Laplacian of $|X|^2$, $X$ being the position vector, in $R^3$ and its restriction to $M$, we conclude, using the divergence theorem twice, the standard formulae:

$$3 \text{Vol}(W) + \int_M \langle X, n \rangle = 0,$$

$$\text{Area}(M) + H \int_M \langle X, n \rangle = \frac{1}{2} \int_{\partial M} \langle X, \nu \rangle,$$

where $\nu$ is the outward pointing conormal vector along $\partial M$. Thus

$$2\text{Area}(M) = 6H \text{Vol}(M) + \int_{\partial M} \langle X, \nu \rangle.$$

As $H$ is bounded above by $1/r'$, where $r' > 0$ is chosen so that $D(p, r') \subset \text{int}(\Omega)$ (note that some of the balls $B(p, r') + ta$, $t$ coming from $-\infty$, will have a first
contact with the positively curved side of $\text{int}(M)$ and thus the maximum principle gives $H \leq 1/r'$ it follows that when $\text{Area}(M)$ is big then $\text{Vol}(W)$ is big too. So we conclude our argument using the step above.

**Remark 3.** The results of this paper extend to compact hypersurfaces with constant mean curvature embedded in the Euclidean halfspace $\mathbb{R}^{n+1}_+$ and with boundary in $P = \partial \mathbb{R}^{n+1}_+$.

To see that theorem 1 extends, we only need to note that both, the Alexandrov reflection technique and the curvature estimates for $H$-graphs work for any dimension.

Concerning theorem 2, $\Gamma \subset P = \mathbb{R}^n$ will be a strictly convex compact hypersurface of $P$ and $H(\Gamma)$ will depend on the maximum and the minimum principal curvatures of $\Gamma$. The only change in our arguments is that, as the height of the higher dimensional vertical Catenoid $\Sigma'$ is bounded, now the piece $\Sigma \subset \Sigma'$ we use in our proof will be, not $\Sigma' \cap \{0 \leq x_{n+1} \leq 1\}$, but $\Sigma = \Sigma' \cap \{0 \leq x_{n+1} \leq \varepsilon(\theta)\}$, where $\varepsilon(\theta) > 0$ is small enough to assure the compactness of $\Sigma$.

**Remark 4.** An interesting problem is to understand the topology and geometry of solutions to the isoperimetric problem for a convex curve $\Gamma \subset P$. More precisely, given $V \geq 0$, we know from geometric measure theory, that there exists an embedded constant mean curvature surface $M$ with $\partial M = \Gamma$, which (together with the planar domain $\Omega$ enclosed by $\Gamma$) bounds a volume $V$, and minimizes area among such surfaces. What is the nature of $M$ as $V$ goes from 0 to infinity? When does $M$ traverse the plane $P$?

Another problem we wish to mention: Let $\Gamma_1$ and $\Gamma_2$ be convex curves in parallel planes. Is there a constant mean curvature surface $M$ with boundary $\Gamma_1 \cup \Gamma_2$, $M$ topologically an annulus?
Bibliography


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