THE ISOPERIMETRIC PROBLEM

ANTONIO ROS

Contents

1. Presentation

The isoperimetric problem is an active field of research in several areas: in differential geometry, discrete and convex geometry, probability, Banach spaces theory, PDE,. . .

In this section we will consider some situations where the problem has been completely solved and some others where it remains open. In both cases I have chosen the simplest examples. We will start with the classical version: the isoperimetric problem in a region, for instance, the whole Euclidean space, the ball or the cube. Even these concrete examples profit from a broader context. In these notes we

Research partially supported by a MCYT-FEDER Grant No. BFM2001-3318.

will study the isoperimetric problem not only for a region but also for a measure, especially the Gaussian measure, which is an important and fine tool in isoperimetry.

Concerning the literature on the isoperimetric problem we mention the classic texts by Hadwiger [33], Osserman [57] and Burago & Zalgaller [18]. We will not treat here the special features of the 2-dimensional situation, see [43] for more information about this case.

I would like to thank F. Morgan, J. Pérez and M. Ritoré for several helpful discussions and comments on these notes.

1.1. **The isoperimetric problem for a region.** Let M^n be a Riemaniann manifold of dimension n with or without boundary. In this paper *volumes* and *areas* in M will mean n-dimensional and $(n-1)$-dimensional Riemannian measures, respectively. The n-dimensional sphere (resp. closed Euclidean ball) of radius r will be denoted by $\mathbf{S}^n(r)$ (resp. $B^n(r)$). When $r=1$ we will simply write $\mathbf{S}^n$ (resp. B^n).

Given a positive number $t < V(M)$, where $V(M)$ denotes the volume of M, the *isoperimetric problem* consists in studying, among the compact hypersurfaces $\Sigma \subset M$ enclosing a region Ω of volume $V(\Omega) = t$, those which minimize the area $A(\Sigma)$, see Figure 1. Note that we are not counting the area coming from the boundary of M. If the manifold M has smooth boundary or it is the product of manifolds with smooth boundary (like the cube), there are no differences between the isoperimetric problems for M and for the interior of M. So we will take one manifold or the other at our convenience, depending on the arguments we are going to use.

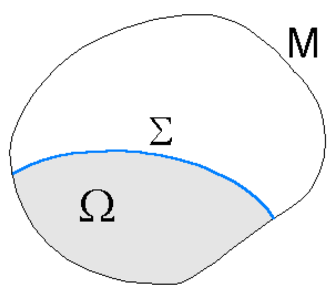

FIGURE 1. Isoperimetric problem in a region

By combination of the results of Almgren [2], Grüter [32], Gonzalez, Massari, Tamanini [26] we obtain the following fundamental existence and regularity theorem (see also Morgan [55]).

Theorem 1. *If M^n is compact, then, for any t, $0 < t < V(M)$, there exists a compact region $\Omega \subset M$ whose boundary $\Sigma = \partial\Omega$ minimizes area among regions of volume t. Moreover, except for a closed singular set of Hausdorff dimension at most $n-8$, the boundary Σ of any minimizing region is a smooth embedded hypersurface with constant mean curvature and, if $\partial M \cap \Sigma \neq \emptyset$, then ∂M and Σ meet orthogonally.*

In particular, if $n \leq 7$, then Σ is smooth. The region Ω and its boundary Σ are called an *isoperimetric region* and an *isoperimetric hypersurface* respectively. The hypersurface Σ has an area-minimizing tangent cone at each point. If a tangent cone at $p \in \Sigma$ is an hyperplane, then p is a regular point of Σ. The theorem also applies to the case when M itself is noncompact but M/G is compact, G being the isometry group of M.

The *isoperimetric profile* of M is the function $I = I_M :]0, V(M)[\to \mathbb{R}$ defined by $I(t) = \min\{A(\partial\Omega) \mid \Omega \subset M \,\text{region with}\, V(\Omega) = t\}$. General properties of this function will be considered in section §2.4. Explicit lower bounds for the profile I are very important in applications and are called *isoperimetric inequalities.* For results of this type see Chavel [19, 20], Gallot [25] and sections §2 and §3 below.

Another important aspect of the problem (which, in fact, is one of the main matters of this paper) is the study of the geometry and the topology of the solutions and the explicit description of the isoperimetric regions when the ambient space is simple enough. If M is the Euclidean space, the sphere or the hyperbolic space, then isoperimetric domains are metric balls. However, this question remain open in many interesting cases. Concerning that point, there are in the isoperimetry field a certain number of *natural conjectures* which may be useful, at this moment, to fix our ideas about the problem. Let's consider, for instance, the following ones:

1) The isoperimetric regions on a flat torus $T^n = \mathbf{S}^1 \times \cdots \times \mathbf{S}^1$ are of the type (ball in T^m) $\times T^{n-m}$.

2) Isoperimetric regions in the projective space $\mathbf{P}^n = \mathbf{S}^n/\pm$ are tubular neighborhood of linear subvarieties.

3) Isoperimetric regions in the complex hyperbolic space are geodesics balls.

In high dimension many of these conjectures probably fail. In fact, the natural conjecture is already wrong for a very simple space: in the slab $\mathbb{R}^n \times [0, 1]$ with $n \geq 9$, unduloids are sometimes better than spheres and cylinders; see Theorem 4. However, in the 3-dimensional case these kinds of conjectures are more accurate: 2) is known to be true (see Theorem 14) and partial results in §1.5 and §2.1 support 1).

1.2. **Soap bubbles.** Soap bubbles experiments are also useful to improve our intuition about the behavior of the solutions of the isoperimetric problem.

From the theoretical point of view, a soap film is a membrane which is also a homogeneous fluid. To deduce how a soap film in equilibrium pushes the space near it, we cut a small piece S in the membrane, see Figure 2. If we want S to remain in equilibrium, we must reproduce in some way the action of the rest of the membrane over the piece. Because of the fluidity assumption this action is given by a distribution of forces along the boundary of S. Moreover these forces follow the direction of the conormal vector ν: they are tangent to the membrane and normal to the boundary of S. Finally the homogeneity implies that the length of the vectors in this distribution is constant, say of length one, and thus S pushes the space around

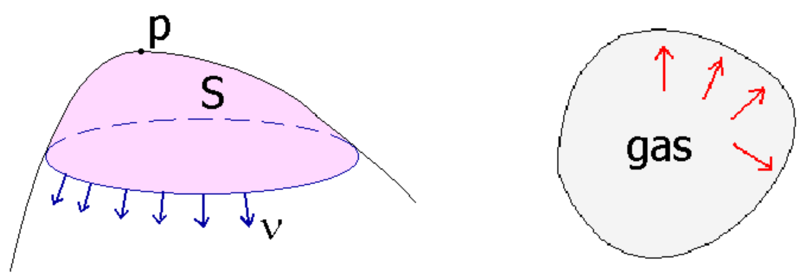

FIGURE 2. Soap bubbles and isoperimetric problem

it with a force given by

$$F(S) = \int_{\partial S} \nu \, ds.$$

Then, by a simple computation we can obtain the value of *the pressure* at a point p in the film (which is defined as the limit of the mean value of the forces $F(S)$ when S converges to p),

$$\lim_{S \to p} \frac{F(S)}{A(S)} = 2H(p)N(p),$$

where N is the unit normal vector field of S and H is its mean curvature.

Consider now a soap bubble enclosing a volume of gas like in Figure 2. The gas inside pushes the bubble in a homogeneous and isotropic way: it induces on the surface of the bubble a distribution of forces which is normal to the surface and has the same intensity at any point of the bubble. As the membrane is in equilibrium, the pressure density on the surface must be opposite to this distribution and, thus, the mean curvature of the bubble must be constant.

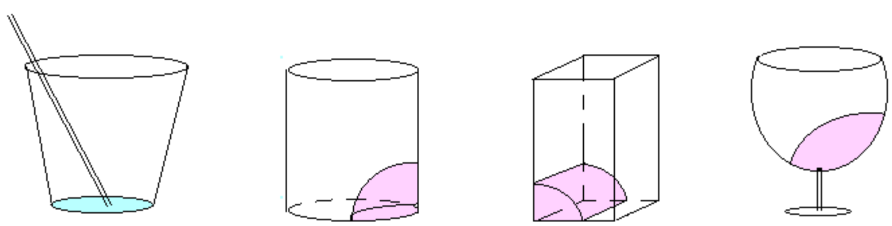

FIGURE 3. Soap bubble experiment

We can also reason that the (part of the) energy (coming from the surface) of the bubble is proportional to its area and, so, an equilibrium soap bubble minimizes area (at least locally) under a volume constraint. Therefore we can visualize easily (local) solutions of the isoperimetric problem for different kind of regions in the Euclidean 3-space by reproducing the soap film experiment in Figure 3. The case of a cubical vessel is particularly enlightening: it is possible to obtain the five bubbles which appear in Figure 9.

1.3. **Euclidean space, slabs and balls.** The symmetry group of the ambient space M can be used to obtain symmetry properties of its isoperimetric regions. These kind of arguments are called *symmetrizations* and were already used by Schwarz [70] and Steiner [74] to solve the isoperimetric problem in the Euclidean space (see §3.2 for a generalization of the ideas of these authors).

In this section we will see a different symmetrization argument, depending on the existence and regularity of isoperimetric regions in Theorem 1, which was first introduced by Hsiang [40, 42]. We will show how the idea works in the Euclidean case.

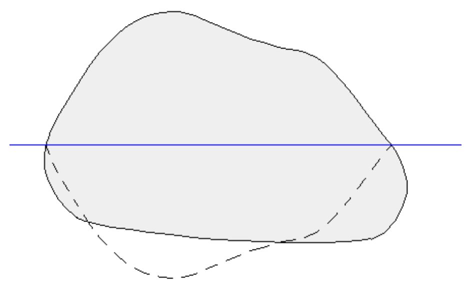

FIGURE 4. If an hyperplane bisects the volume of an isoperimetric region, then the region is symmetric with respect to the hyperplane

Theorem 2. *Isoperimetric hypersurfaces in $\mathbb{R}^n$ are spheres.*

Proof. Let Ω be an isoperimetric region in $\mathbb{R}^n$ with $\Sigma = \partial\Omega$. First we observe that a stability argument gives that the set $R(\Sigma)$ of regular points of Σ is connected; see Lemma 1 in section §2.1 below.

Consider a hyperplane $P \subset \mathbb{R}^n$ so that the regions $\Omega^+ = \Omega \cap P^+$ and $\Omega^- = \Omega \cap P^-$ have the same volume, $P^\pm$ being the halfspaces determinated by P, as in Figure 4. Assuming $A(\Sigma \cap P^+) \leq A(\Sigma \cap P^-)$ one can see that the symmetric region $\Omega' = \Omega^+ \cup s(\Omega^+)$, where s denotes the reflection with respect to P, is an isoperimetric region too. In particular $A(\Sigma \cap P^+) = A(\Sigma \cap P^-)$ and $\Sigma' = \partial\Omega'$ verifies the regularity properties stated in Theorem 1.

If $R(\Sigma) \cap P = \emptyset$, then the regular set of Σ' would be either empty or nonconnected. As both options are impossible we conclude that P meets the regular set and, so, it follows from the unique continuation property [4] applied to the constant mean curvature equation that $R(\Sigma) = R(\Sigma')$. Then $\Sigma = \Sigma'$ and we get that Ω is symmetric with respect to P.

As each family of parallel hyperplanes in $\mathbb{R}^n$ contains an hyperplane P which bisects the volume of Ω, we conclude easily that Ω is a ball. $\square$

The arguments above also show that the isoperimetric regions in the n-dimensional sphere are metric balls. In the low dimensional case (when the solutions are smooth) Alexandrov reflection technique [1], can also be used to solve the isoperimetric problem in the Euclidean space. However, this technique does not work in the three sphere.

The easy modifications needed to reproduce the argument in the other situations considered in this paper are left to the reader. The only extra fact we need to use in the cases below is that any isoperimetric hypersurface of revolution is necessarily nonsingular. Outside of the axis of revolution this follows from Theorem 1 (otherwise the singular set would be too large). Using that the unique minimal cone of revolution in $\mathbb{R}^n$ is a hyperplane it follows that points in the axis are regular too.

Theorem 3. *Isoperimetric hypersurfaces in a halfspace are halfspheres.*

Proof. Hsiang symmetrization gives that any isoperimetric hypersurface has an axis of revolution. To conclude the theorem from that point, we can repeat the argument in the proof of the case 1 of Theorem 5 below. □

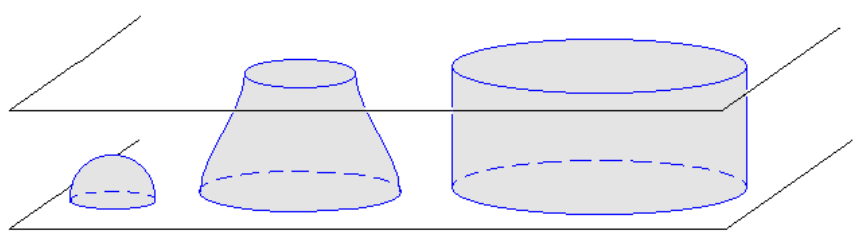

FIGURE 5. Tentative solutions of the isoperimetric problem in the slab

The slab presents interesting and surprising behavior with respect to the isoperimetric problem. The symmetrization argument in Theorem 2 shows that any solution is an hypersurface of revolution. But the complete answer is the following (the case $n = 9$ is still open); see Figure 5.

Theorem 4. (Pedrosa & Ritore, [58]). *If $n \leq 8$, isoperimetric surfaces in the slab $\mathbb{R}^{n-1} \times [0, 1]$ are halfspheres and cylinders. However, if $n \geq 10$ unduloids solve the isoperimetric problem for certain intermediate values of the volume.*

By using the existence and regularity Theorem 1, we are going to give a new proof of the isoperimetric property of spherical caps in the ball $B = \{x \in \mathbb{R}^n \ : \ |x| \leq 1\}$.

Theorem 5. (Almgren [2]; Bokowski & Sperner [16]). *Isoperimetric hypersurfaces in a ball are either hyperplanes through the origin or spherical caps.*

Proof. Although the arguments below work in any dimension, in order to explain the ideas more intuitively, we will assume that B is a ball in $\mathbb{R}^3$. Let Ω be an isoperimetric region of the ball B and $\Sigma = \partial\Omega$. First observe that $\Sigma \cap \partial B \neq \emptyset$: otherwise

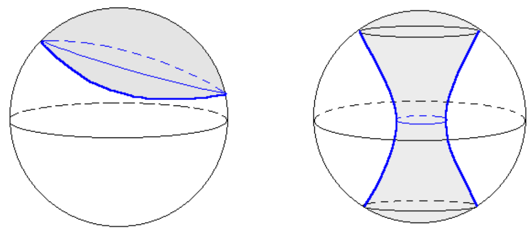

FIGURE 6. Candidates to solve the isoperimetric problem in the ball

by moving Σ until we touch the first time ∂B we obtain a new isoperimetric region which meets the boundary of the ball tangentially and we contradict Theorem 1. The symmetrization argument in Theorem 2 shows that Σ is connected and has an axis of revolution. We remark that in the ball there are constant mean curvature surfaces satisfying these conditions other than the spherical caps, like in Figure 6. In fact one of the difficulties of the problem, which appears also in most situations, is how to exclude these alternative candidates. We discuss two different possibilities.

1. Σ *is a disk*. In this case we will show that Σ is spherical or planar by using an argument from [18], see Figure 7.

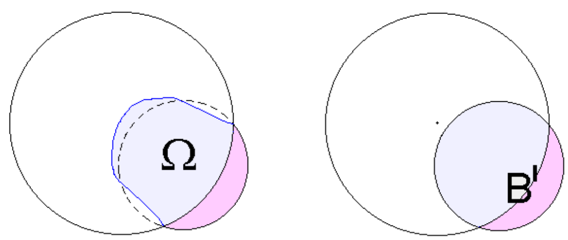

FIGURE 7. Discoidal solutions are spherical

After changing Ω by its complementary region if necessary, we can consider an auxiliary second ball B' such that $V(B\cap B') = V(\Omega)$ and $\partial B'\cap\partial B = \partial\Sigma$. This is always possible unless Σ encloses the same volume as the plane section through $\partial\Sigma$, a case for which the result is clear.

Define the new regions $B\cap B'$ in the ball B and $W = \Omega\cup(B'-B)$ in $\mathbb{R}^3$. As $V(B\cap B') = V(\Omega)$, from the minimizing property of Ω we have $A(\Sigma)\leq A(B\cap\partial B')$. Thus $A(\partial W) = A(\Sigma) + A(\partial B' - B) \leq A(B')$ and, using that $V(W) = V(\Omega) + V(B'-B) = V(B')$, the isoperimetric property of the ball in $\mathbb{R}^3$ implies that W is a ball. In particular, as $\Sigma\subset W$, we conclude that Σ is an spherical cap.

2. Σ *is an annulus*. We will prove that this second case is impossible, see Figure 8. Let's rotate Ω a little bit around an axis of the ball orthogonal to the axis of revolution to get a new region Ω' with boundary $\Sigma' = \partial\Omega'$. Among the regions defined in $\Omega\cup\Omega'$ by $\Sigma\cup\Sigma'$, let's consider the following ones, see Figure 8:

$\Omega_1 = \Omega \cap \Omega'$, which contains the center of the ball,
Ω_3 and Ω_4, the two components of $\Omega - \Omega'$ which meet ∂B, and
Ω_2 and Ω_5, components of $\Omega' - \Omega$ which intersect the boundary of B.

We claim that the five regions above are pairwise disjoint. The only non trivial cases to be considered are the couples Ω_3, Ω_4 and Ω_2, Ω_5. As Ω is rotationally invariant, Ω' can be also obtained as the symmetric image of Ω with respect to a certain plane P through the center of B. Clearly P intersects Ω_1 and is disjoint from Ω_i, $i = 2, 3, 4, 5$. As $\Omega_3 \cap \partial B$ and $\Omega_4 \cap \partial B$ lie at different sides of P we conclude that $\Omega_3 \neq \Omega_4$ and, in the same way, $\Omega_2 \neq \Omega_5$.

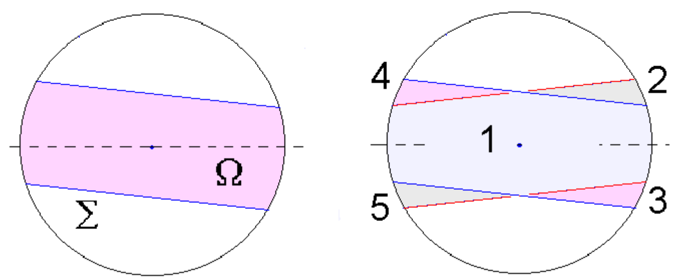

FIGURE 8. Topological annuli cannot be solutions

Finally define in B the region $\Omega'' = (\Omega - \Omega_3) \cup \Omega_2$. As $\partial\Omega''$ contains pieces coming from both Σ and Σ', it follows that Ω'' is not smooth. On the other hand, as Ω_2 and Ω_3 are symmetric with respect to P we see that $V(\Omega'') = V(\Omega)$ and $A(\partial\Omega'') = A(\Sigma)$. Thus Ω'' is also an isoperimetric region, which contradicts the regularity of Theorem 1. $\square$

Another proof of Theorem 5, which essentially uses an infinitesimal version of the argument above, is given in Ros [66].

1.4. **The isoperimetric problem for the Gaussian measure.** We want to extend the family of objects where we are studying the isoperimetric problem. Let M^n be a Riemannian manifold with metric ds^2 and distance d. For any closed set $R \subset M$ and $r > 0$ we can consider the *r-enlargement* $R_r = \{x \in M : d(x, R) \leq r\}$. Given a (positive, Borel) measure μ on M, we define the *boundary measure* (also called *μ-area* or *Minkowski content*) μ^+ as follow

$$\mu^+(R) = \liminf_{r \to 0} \frac{\mu(R_r) - \mu(R)}{r}.$$

If R is smooth and μ is the Riemannian measure, then $\mu^+(R)$ is just the area of ∂R. Now we can formulate the isoperimetric problem for the measure μ: we want to minimize $\mu^+(R)$ among regions with $\mu(R) = t$. Note that in order to study the isoperimetric problem we only need to have a measure and a distance (not necessarily a Riemannian one). Although we will not follow this idea, this more abstract formulation is useful in some areas such as discrete geometry, probability and Banach space theory; see for instance [36] and [49].

The *Isoperimetric profile* $I_\mu = I_{(ds^2,\mu)}$ of (M,μ) is defined, as above, by

$$I_\mu(t) = \inf\,\{\mu^+(R) : \mu(R) = t\}, \qquad 0 \le t \le \mu(M).$$

If we change the metric or the measure by a positive factor a, it follows easily that

$$I_{a\mu}(t) = aI_\mu(\frac{t}{a}) \quad \text{and} \quad I_{(a^2ds^2,\mu)} = \frac{1}{a}I_{(ds^2,\mu)}.$$

Among the new examples which appear in our extended setting, we remark the following ones (in these notes we will be interested only in nice new situations and we will only consider absolutely continuous measures).

1. *Smooth measures* of the type $d\mu = fdV$, with $f \in C^\infty(M)$, $f > 0$. If R is good enough, then

$$\mu(R) = \int_R fdV \qquad \text{and} \quad \mu^+(R) = \int_{\partial R} f\; dA.$$

2. If $V(M) < \infty$ we can consider the *normalized Riemannian measure* (or *Riemannian probability*) $\mu_M = V/V(M)$.

3. The *Gaussian measure* $\gamma = \gamma_n$ in $\mathbb{R}^n$ defined by

$$d\gamma = \exp\,(-\pi|x|^2)dV,$$

Note that γ is a probability measure, that is $\gamma(\mathbb{R}^3) = 1$. It is well-known that the Gaussian measure appears as the limit of the binomial distribution. We will see in section §3 that it appears also as a suitable limit of the spherical measure when the dimension goes to infinity. For $n \ge 2$, γ_n is characterized (up to scaling) as the unique probability measure on $\mathbb{R}^n$ which is both rotationally invariant and a product measure.

The isoperimetric problem for the Gaussian measure has remarkably simple solutions; see Theorem 20 for a proof.

Theorem 6. ([17],[73]) *Among regions $R \subset \mathbb{R}^n$ with prefixed $\gamma(R)$, halfspaces minimize the Gaussian area.*

In particular, $I_\gamma(1/2) = 1$ and the corresponding isoperimetric hypersurfaces are hyperplanes passing through the origin.

Next we give a comparison result for the isoperimetric profiles of two spaces which are related by means of a Lipschitz map.

Proposition 1. *Let (M,μ) and (M',μ') be Riemannian manifolds with measures and $\phi : M \to M'$ a Lipschitz map with Lipschitz constant $c > 0$. If $\phi(\mu) = \mu'$, then $I_\mu \le cI_{\mu'}$.*

Proof. Let $R' \subset M'$ a closed set and $R = \phi^{-1}(R')$. The hypothesis $\phi(\mu) = \mu'$ means that $\mu'(R') = \mu(R)$. The Lipschitz assumption implies that, for any $r > 0$, $\phi(R_r) \subset R'_{cr}$ and therefore $\mu'(R'_{cr}) \geq \mu(\phi^{-1}(\phi(R_r)) \geq \mu(R_r)$. Thus we conclude that

$$\frac{\mu'(R'_{cr}) - \mu'(R')}{cr} \geq \frac{\mu(R_r) - \mu(R)}{cr}$$

and the proposition follows by taking $r \to 0$. $\square$

If we assume that μ and μ' are smooth, ϕ is a diffeomorphism and there exists a μ-isoperimetric hypersurface $\Sigma \subset M$ enclosing a region Ω with $\mu(\Omega) = t$, then one can see easily that the equality $I_\mu(t) = cI_{\mu'}(t)$ is equivalent to the condition $|d\phi(N)| = c$, where N is the unit normal vector along a Σ.

As a first application of the proposition we get, by taking a suitable linear map ϕ and using Theorem 5, the following fact.

Corollary 1. *Let B be the unit ball in $\mathbb{R}^n$ and $E = \{(x_1, \dots, x_n) \mid \frac{x_1^2}{a_1^2} + \cdots + \frac{x_n^2}{a_n^2} \leq 1\}$ an ellipsoid with $0 < a_1 \leq \cdots \leq a_n$ and $a_1 \cdots a_n = 1$, both spaces endowed with the Lebesgue measure. Then $a_n I_E \geq I_B$ and the section with the hyperplane $\{x_n = 0\}$ minimizes area among hypersurfaces which separate E in two equal volumes.*

1.5. **Cubes and boxes.** Consider the box $W =]0, a_1[\times \dots]0, a_n[\subset \mathbb{R}^n$, with $0 < a_1 \leq \cdots \leq a_n$. The unit cube corresponds to the case $a_1 = \cdots = a_n = 1$. First we observe that the symmetrization argument of section §1.3 implies that the isoperimetric problem in W is equivalent (after reflection through the faces of the box) to the isoperimetric problem in the rectangular torus $T^n = \mathbb{R}^n/\Gamma$, where Γ is the lattice generated by the vectors $(2a_1, 0, \dots, 0), \dots, (0, \dots, 0, 2a_n)$.

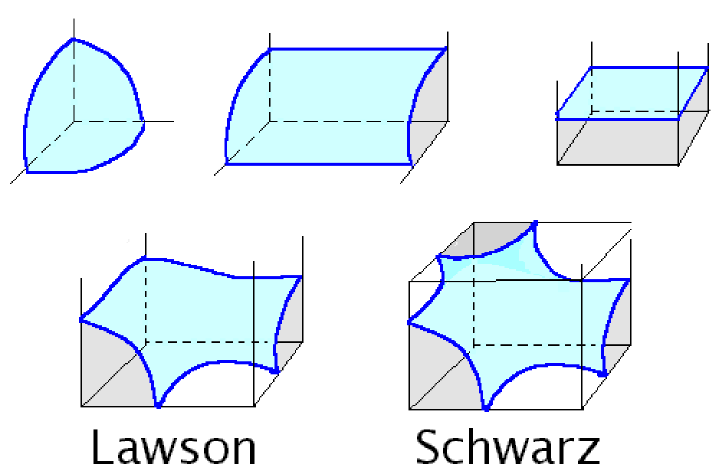

FIGURE 9. Candidates to isoperimetric surfaces in the cube

For $n = 3$ (from Theorem 1 the case $n = 2$ is trivial) Ritoré [60] proves that any isoperimetric surface belongs to some of the types described in Figure 9; see §2.1

below (see also Ritoré and Ros [64] for other related results). It is natural to hope that isoperimetric surfaces in a 3-dimensional box are spheres, cylinders and planes. However the problem remains open. Lawson type surfaces sometimes come close to beating those simpler candidates. Indeed, by using Brakke's Surface Evolver we can see that in the unit cube there exists a Lawson surface of area 1.017 which encloses a volume $1/\pi$ (as compared to the conjectured spheres-cylinders-planes area which is equal to 1, see Figure 10). In higher dimensions the corresponding conjecture is probably wrong.

The next result is a nice application of the Gaussian isoperimetric inequality (see for instance [47]): it gives a sharp isoperimetric inequality in the cube and solves explicitly the isoperimetric problem when the prescribed volume is one half of the total volume.

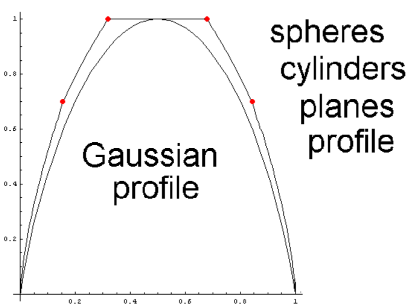

FIGURE 10. Comparison between Gaussian and tentative cubical profiles

Theorem 7. *Let $W =]0,1[^n$ be the open unit cube in $\mathbb{R}^n$. Then $I_W \geq I_\gamma$.*

Proof. Consider the diffeomorphism $f : \mathbb{R}^n \to W$ defined by

$$df_{(x_1,\dots,x_n)} = \begin{pmatrix} e^{-\pi x_1^2} & & \\ & \ddots & \\ & & e^{-\pi x_n^2} \end{pmatrix}.$$

As f is a 1-Lipschitz map which transforms the Gaussian measure into the Lebesgue one, the result follows directly from Proposition 1. $\square$

In Figure 10 we have drawn both the Gaussian profile and the candidate profile of the unit cube in $\mathbb{R}^3$ (which is given by pieces of spheres, cylinders and planes). Note how sharp the estimate in Theorem 7 is. As both curves coincide for $t = 1/2$, we can conclude the following fact.

Corollary 2. (Hadwiger, [34]) *Among surfaces which divide the cube in two equal volumes, the least area is given (precisely) by hyperplanes parallel to the faces of the cube.*

The argument used in the proof of Theorem 7 gives the following more general result.

Theorem 8. *Given $0 < a_1 \leq \cdots \leq a_n$ with $a_1 \cdots a_n = 1$, consider the box $W =]0, a_1[\times \cdots \times]0, a_n[\subset \mathbb{R}^n$. Then $a_n I_W \geq I_\gamma$ and $\{x_n = a_n/2\}$ has least area among hypersurfaces which divide the box in two equal volumes.*

As isoperimetric problems in boxes and rectangular tori are equivalent, we can write the following consequence of the last theorem.

Corollary 3. *Consider the lattice $\Gamma \subset \mathbb{R}^n$ spanned by the vectors $(a_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, a_n)$, with $0 < a_1 \leq \cdots \leq a_n$. Among hypersurfaces which divide the torus $\mathbb{R}^n/\Gamma$ in two equal volumes, the minimum of the area is given by $2a_1 \cdots a_{n-1}$. This area is attained by the pair of parallel $(n-1)$-tori $\{x_n = 0\}$ and $\{x_n = a_n/2\}$.*

Below we have two applications of the isoperimetric inequality in the cube. The first one concerns the connections between lattices and convex bodies and it is related with the classical Minkowski theorem about lattice points in a symmetric convex body of given volume. We present the result for the case of the cubic lattice, but the proof extends without changes to other orthogonal lattices.

Theorem 9. (Nosarzewska [56]; Schmidt [71]; Bokowski, Hadwiger, Wills [15])
Let $\Gamma = \mathbb{Z}^n$ be the integer cubic lattice and $K \subset \mathbb{R}^n$ a convex body. Then

$$\#(\Gamma \cap K) \geq V(K) - \frac{1}{2} A(\partial K),$$

where $\#(X)$ denotes the number of points of the set X.

Proof. Denote by C a generic unit cube in $\mathbb{R}^n$ of the type

$$C = \{(x_1, \ldots, x_n) \mid k_i - \frac{1}{2} \leq x_i \leq k_i + \frac{1}{2},\ i = 1, \ldots, n\},$$

where $k_1, \ldots, k_n$ are arbitrary integer numbers. Note that the centers $p = (k_1, \ldots, k_n)$ of the cubes are just the points of the lattice Γ.

If for some of these cubes C we have $V(K \cap C) > \frac{1}{2}$, then the center p of C belongs to K: to see that observe that otherwise, by separation properties of convex sets, it should exist a plane through p such that $K \cap C$ lies at one side of this plane, which would imply that the volume of $K \cap C$ is at most $\frac{1}{2}$ as in Figure 11.

On the other hand, if $v = V(K \cap C) \leq \frac{1}{2}$, the isoperimetric inequality for the cube (see Theorem 7) gives

$$A(\partial K \cap C) \geq I_\gamma(v) \geq 4v(1-v) \geq 2v = 2V(K \cap C),$$

where we have used that the Gaussian profile satisfies the estimate $I_\gamma(t) \geq 4t(1-t)$, see §3.1 for more details.

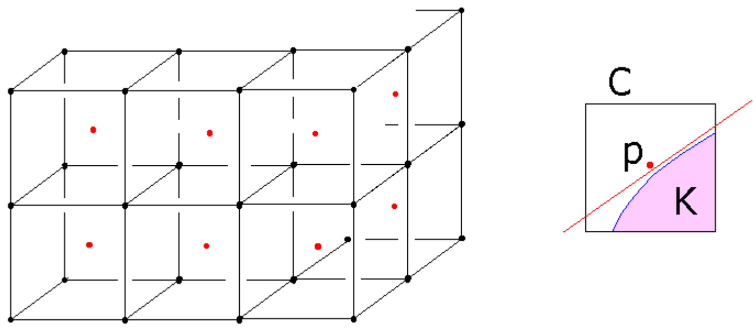

FIGURE 11. Lattices and convex bodies

Using the properties we have for the above two families of cubes, we conclude the proof as follows:

$$V(K) = \sum_C V(C\cap K) = \sum_{V(C\cap K)>\frac{1}{2}} V(C\cap K) \ + \sum_{V(C\cap K)\le\frac{1}{2}} V(C\cap K)$$

$$\le \#(\Gamma\cap K) + \sum_C A(\partial K\cap C)/2 \ \le\ \#(\Gamma\cap K) + A(\partial K)/2.$$

□

For arbitrary lattices, an extended version of the above result is given in [72].

To finish this section we give an estimate for the area of triple periodic minimal surfaces. The result (and the proof) extend to higher dimensional rectangular tori.

Theorem 10. *Let Γ be the orthogonal lattice generated by $\{(a,0,0),(0,b,0),(0,0,c)\}$, $0<a\le b\le c$, and $\Sigma\subset\mathbf{R}^3/\Gamma$ an embedded nonflat closed orientable minimal surface. Then $A(\Sigma)>2ab$.*

Proof. First observe that Σ separates the torus $\mathbb{R}^3/\Gamma$: otherwise the pullback image of Σ in $\mathbb{R}^3$ would be nonconnected, which contradicts the maximum principle (see [53]).

Let Ω be one of the regions in the torus with $\partial\Omega=\Sigma$. Assume that $V(\Omega)\le abc/2$ and take the enlargement Ω_r, $r\ge 0$, of Ω such that $V(\Omega_r)=abc/2$. Parameterizing $\mathbf{R}^3/\Gamma$ by the geodesics leaving Σ orthogonally we get $dV=(1-tk_1)(1-tk_2)dtdA$ and, if we define $C(r)$ as the set of points in Σ whose distance to $\partial\Omega_r$ is equal to r, we have

$$A(\partial\Omega_r)\le\int_{C(r)}(1-rk_1)(1-rk_2)dA.$$

We remark that $(1 - rk_1)(1 - rk_2) \geq 0$ on $C(r)$. From Corollary 3 and the integral inequality above, combined with Schwarz's inequality, we obtain

$$2ab \leq A(\partial\Omega_r) \leq \int_\Sigma (1 - rH)^2 dA = A(\Sigma),$$

where we have used that the mean curvature H of Σ vanishes.

Suppose the equality holds. If $r > 0$ Schwarz's inequality gives that Σ is flat. If $r = 0$, then Σ would be an isoperimetric region, which contradicts Corollary 3. As both options are impossible, we conclude that $A(\Sigma) > 2ab$. $\square$

Note that $\mathbb{R}^3/\Gamma$ contains flat 2-tori of area ab. The area of P-Schwarz minimal surface in the unit cubic torus is ~ 2.34 (the theorem gives that this area is greater than 2). Note that, as the P minimal surface is stable [69] and (by symmetry arguments) separates the torus in two equal volumes, it follows that in the cubic torus there are local minima of the isoperimetric problem which are not global ones. R. Kusner pointed out to me that by desingularizing the union of the two flat 2-tori induced by the planes $x = 0$ and $y = 0$ in the cubic unit torus (see Traizet [77]), we obtain embedded *nonorientable* nonflat compact minimal surfaces with area smaller than 2.

1.6. **The periodic isoperimetric problem.** The explicit description of the solutions of the isoperimetric problem in flat 3-tori (the so called *periodic isoperimetric problem*) is one of the nicest open problems in classical geometry. The natural conjecture in this case says that any solution is a sphere, a (quotient) of a cylinder or two parallel flat tori. For rectangular 3-tori $T^3 = \mathbf{R}^3/\Gamma$, where Γ is a lattice generated by three pairwise orthogonal vectors, we have a good control on the shape of the best candidates (see §1.5 and §2.1) and the conjecture seems to be well tested. For general flat 3-tori the information we have is weaker and the natural conjecture is not so evident (though we believe it). The problem extends naturally to flat 3-manifolds and to the quotient of $\mathbb{R}^3$ by a given crystallographic group.

A different interesting extension of the problem is the study of stable surfaces (i.e. local minima of the isoperimetric problem, see section §2.1) in flat three tori. It is known that this family contains some beautiful examples (other than the sphere, the cylinder and the plane) such as the Schwarz P and D triply periodic minimal surfaces and Schoen minimal Gyroid, see Ross [69], but exhaustive work in this direction has never been done. The most important restriction we know at this moment about a stable surface is that its genus is at most 4, see [63] and [62]. The situation where the lattice is not fixed, but is allowed to be deformed, could be geometrically significant too.

Nanogeometry. The periodic stability condition is the simplest geometric model for a group of important phenomena appearing in material sciences. Take two liquids which repel each other (like oil and water) and put them together in a vessel. The equilibrium position of the mixture corresponds to a local minimum of the energy

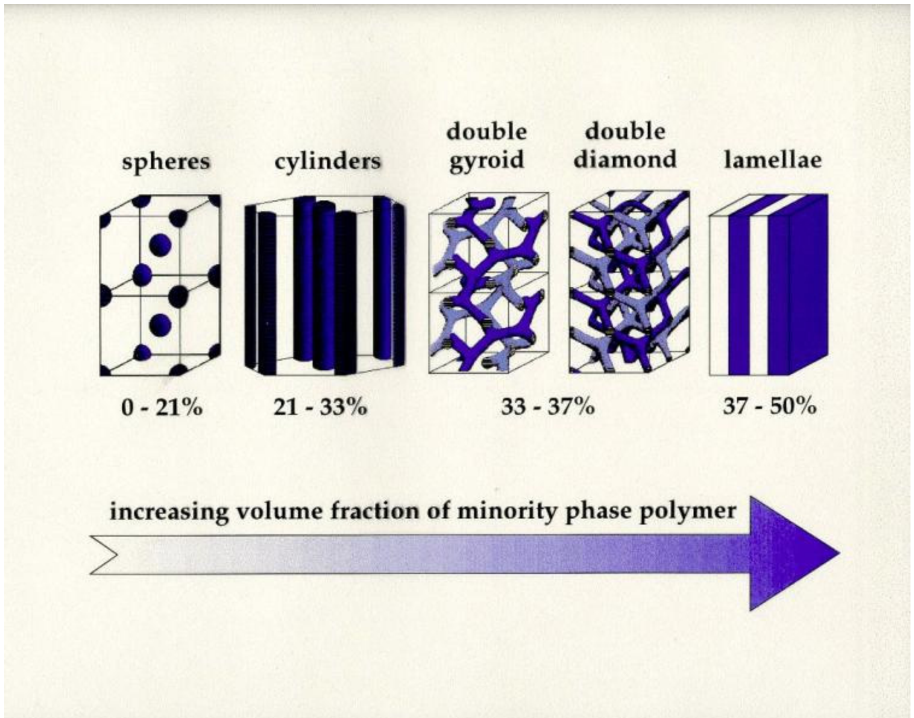

FIGURE 12. Edwin Thomas talk on material science, MSRI (Berkeley), 1999

of the system, which, assuming suitable ideal conditions, is given by the area of the interface Σ separating both materials. As the volume fraction of the liquids is given, we see that Σ is a local solution of the isoperimetric problem (i. e. a stable surface). It happens that there are several pairs of substances of this type that (for *chemical reasons*) behave periodically. These periods hold at a very small scale (on the scale of nanometers) and the main shapes which appear in the laboratory, under our conditions, are described in Figure 12. The parallelism between these shapes and the periodic stable surfaces listed above is clear. For more details see [75] and [35]. Instead of the area, the bending energy, see §2.3, is sometimes considered to explain these phenomena. However, from the theoretical point of view, this last functional is harder to study, and it does not seem possible at this moment to classify its extremal surfaces.

2. THE ISOPERIMETRIC PROBLEM FOR RIEMANNIAN 3-MANIFOLDS

In this section M will be a 3-dimensional orientable compact manifold without boundary. The volume of M will be denoted by $v = V(M)$. Some of the results below are also true in higher dimension. However we will only consider the 3-dimensional case for the sake of clarity.

First we explain several facts about *stable* constant mean curvature surfaces. Some of these stability arguments allows us to solve the isoperimetric problem in the real projective space. Then we will study the *bending energy*, an interesting functional which can be related, in several ways, to the isoperimetric problem. The general properties of the isoperimetric profile and the isoperimetric inequality of Lévy and Gromov will be considered too.

2.1. **The stability condition.** Stability is a key notion in the study of the isoperimetric problem. In the following we will see several interesting consequences of this idea.

A closed orientable surface $\Sigma \subset M$ is an *stable surface* if it minimizes the area up to second order under a volume constraint. The stability of Σ is equivalent to the following facts (see [5]):

1) Σ has constant mean curvature and

2) $Q(u,u) \geq 0$ for any function u in the Sobolev space $H^1(\Sigma)$ with $\int_\Sigma u dA = 0$, Q being quadratic form defined by the second variation formula for the area.

This quadratic form is given by

$$Q(u,u) = \int_\Sigma \left(|\nabla u|^2 - (Ric(N) + |\sigma|^2)u^2 \right) dA, \tag{1}$$

where $Ric(N)$ is the Ricci curvature of M in the direction of the unit normal vector to Σ and σ is the second fundamental form of the inmersion. The function u corresponds to an infinitesimal normal deformation of the surface, and the zero mean value condition means that the deformation preserves the volume infinitesimally. The *Jacobi operator* $L = \Delta + Ric(N) + |\sigma|^2$ is related to Q by the formula $Q(u,v) = -\int_\Sigma uLvdA$.

There are several notions of stability in the constant mean curvature context, but in this paper we will only use the one above. The most important class of stable surfaces is given by the isoperimetric surfaces.

If Σ had boundary $\partial\Sigma \subset \partial M$, then the quadratic form Q should be modified to

$$Q(u,u) = \int_\Sigma \left(|\nabla u|^2 - (Ric(N) + |\sigma|^2)u^2 \right) dA \; - \int_{\partial\Sigma} \kappa \; u^2 ds, \tag{2}$$

where κ is the normal curvature of ∂M along N, see for instance [68].

If W is a region in $\mathbb{R}^3$ and $\Sigma \subset W$ has constant mean curvature and meets ∂W orthogonally, stability corresponds with soap bubbles which can really be blown in the experiment of Figure 3.

A simple consequence of (1) and (2) is that if $Ric > 0$ and ∂M is convex, then any stable surface is connected. In the case $Ric \geq 0$, Σ is either connected or totally geodesic. In fact, if Σ is not connected we can use as a test function u a locally constant function, and the above assertion follows directly.

Formulae (1) and (2) are valid in higher dimension and isoperimetric hypersurfaces $\Sigma \subset M$ satisfy the stability condition. Moreover locally constant functions on the regular set $R(\Sigma)$ of Σ belong to the Sobolev space $H^1(\Sigma) = H^1(R(\Sigma))$. The equality of the above spaces follows because the Hausdorff codimension of the singular set is large, see Theorem 1. Therefore we have the following result, which was used in the proof of Theorem 2.

Lemma 1. *Let M be an n-dimensional Riemannian manifold and Σ an isoperimetric hypersurface. If $Ric \geq 0$ and ∂M is convex, then either the regular set $R(\Sigma)$ of Σ is connected or Σ is totally geodesic.*

Although the idea of stability is naturally connected with the isoperimetric problem, it has not been widely used until recently. The pioneer result in this direction was the following one.

Theorem 11. (Barbosa & do Carmo [5]). *Let Σ be a compact orientable surface immersed in $\mathbb{R}^3$ with constant mean curvature. If Σ is stable, then it is a round sphere.*

Consider the 3-dimensional box $W = [0, a_1] \times [0, a_2] \times [0, a_3]$, $0 < a_1 \leq a_2 \leq a_3$. Remind that any isoperimetric surface Σ in W gives an isoperimetric surface Σ' in the torus $T = \mathbb{R}^3/\text{span}\{(2a_1, 0, 0), (0, 2a_2, 0), (0, 0, 2a_3)\}$ by taking reflections with respect to the faces of W. We remark that the argument below only uses the *stability* of Σ'.

Proposition 2. (Ritoré [60]). *Let Σ be a nonflat isoperimetric surface in a box W. Then Σ has 1-1 projections over the three faces of W (as in Figure 9).*

Proof. We work not with Σ but with Σ', which is also connected and orientable. Denote by s_i the symmetry in T induced by the symmetry with respect to the plane $x_i = 0$ in $\mathbb{R}^3$. Note that the set of points fixed by s_i consists of the tori $\{x_i = 0\}$ and $\{x_i = a_i\}$ and so it separates T. In particular, as Σ' is connected, $C_i = \{p \in \Sigma' \,:\, s_i(p) = p\}$ is non empty and separates Σ' too.

Let $L = \Delta + |\sigma|^2$ be the Jacobi operator and $N = (N_1, N_2, N_3)$ the unit normal vector of Σ'. Then $u = N_i$ satisfies $u \not\equiv 0$, $Lu = 0$ and $u \circ s_i = -u$. Hence u vanishes along the curve of fixed points C_i. Assume, reasoning by contradiction, that u vanishes at some point outside C_i. Then u will have at least 3 nodal domains, that is $\{u \neq 0\}$ has at least three connected components Σ_α, $\alpha = 1, 2, 3$: to conclude this, use that nonzero solutions of the Jacobi equation $Lu = 0$ change sign near each point p with $u(p) = 0$.

Consider the functions u_α on Σ' defined by $u_\alpha = u$ on Σ_α and $u_\alpha = 0$ on $\Sigma' - \Sigma_\alpha$. As $Q(u_1, u_2) = 0$, we can construct a nonzero function of the type $v = a_1 u_1 + a_2 u_2$ satisfying $Q(v, v) = 0$ and $\int_{\Sigma'} v\, dA = 0$. The stability of Σ' then implies that $Lv = 0$. As v vanishes on Σ_3 we contradict the unique continuation property [4]. Therefore we have that u has no zeros in the interior of Σ and the proposition follows easily. $\square$

See Proposition 1 in [66] for a more general argument of this type.

It can be shown, see [60, 61], that any nonflat constant mean curvature surface in a box W with 1-1 projections into the faces of the box and which meets ∂W orthogonally, is either a piece of a sphere or lies in one of the two other families of nonflat surfaces that appear in Figure 9. The classical Schwarz P minimal surface in the cube is the simplest hexagonal example. The first example of pentagonal type was constructed by Lawson [46]. Moreover Ritoré [60, 61] has proved that the moduli space of Schwarz (resp. Lawson) type surfaces is parameterized by the moduli space of hyperbolic hexagons (resp. pentagons) with right angles at each vertex. If we do not consider the restriction on the projections of the surface, then the family of constant mean curvature surfaces which meet ∂W orthogonally is much larger, see Grosse-Brauckmann [31].

Ross [69] has shown that Schwarz P minimal surface is stable in the cube. Lawson type surfaces can be constructed by soap bubbles experiments as in Figure 3. So, apparently some of them are stable too.

Another important stability argument depends on the existence of conformal spherical maps on compact Riemann surfaces. It was introduced by Hersch [38] and extended by Yang and Yau [79]. The version below can be found in Ritoré and Ros [63].

Theorem 12. *Let M^3 be a compact orientable 3-manifold with $\partial M = \emptyset$ and Ricci curvature $Ric \geq 2$. If Σ is an isoperimetric surface, then Σ is compact, connected and orientable of genus less than or equal to 3. Moreover, if genus$(\Sigma) = 2$ or 3, then $(1 + H^2)A(\Sigma) \leq 2\pi$.*

Proof. Assume that there is a conformal (i.e. meromorphic) map $\phi : \Sigma \longrightarrow \mathbf{S}^2 \subset \mathbb{R}^3$ whose degree verifies

$$\deg(\phi) \leq 1 + \left[\frac{g+1}{2}\right] \text{ and } \int_\Sigma \phi dA = 0, \tag{3}$$

where $[x]$ denotes the integer part of x and $g = \text{genus}(\Sigma)$. We will take ϕ as a test function for the stability condition. The Gauss equation transforms $Ric(N) + |\sigma|^2$ into $Ric(e_1) + Ric(e_2) + 4H^2 - 2K$, where e_1, e_2 is an orthonormal basis of the tangent plane to Σ and H and K are the mean curvature and the Gauss curvature, respectively. Using The Gauss-Bonnet theorem and the facts that $|\phi| = 1$ and $|\nabla\phi|^2 = 2\text{Jacobian}(\phi)$, we get

$$0 \leq Q(\phi, \phi) = \int_\Sigma |\nabla\phi|^2 - (Ric(N) + |\sigma|^2)$$

$$= 8\pi\deg(\phi) - \int_\Sigma \{Ric(e_1) + Ric(e_2) + 4H^2 - 2K\}$$

$$\leq 8\pi(1+[\frac{g+1}{2}]) - 4(1+H^2)A(\Sigma) + 8\pi(1-g),$$

which implies the desired result.

It remains to show that a map ϕ verifying (3) exists. The Brill-Noether theorem, see for instance [27], guarantees the existence of a nonconstant meromorphic map $\phi : \Sigma \to \mathbf{S}^2$ whose degree satisfies (3). Now we prove that ϕ can be modified, by composition with a suitable conformal transformation of the sphere, in order to get a map with mean value zero. Let G be the group of conformal transformations of $\mathbf{S}^2$ (which coincides with the conformal group of $O = \{x \in \mathbb{R}^3 : |x| < 1\}$). It is easy to see that there exists a unique smooth map $f : x \mapsto f_x$ from O into G verifying:

(i) $f_0 = \mathrm{Id}$ is the identity map

(ii) For any $x \neq 0$, $f_x(0) = x$ and $d(f_x)_0 = \lambda_x \mathrm{Id}$ for some $\lambda_x > 0$.

(iii) Given $y \in \mathbf{S}^2$, we have that, when $x \to y$, f_x converges almost everywhere to the constant map y.

Define the continuous map $F : O \to O$ by

$$F(x) = \frac{1}{A(\Sigma)} \int_\Sigma f_x \circ \phi \, dA.$$

Property (iii) implies that F extends continuously to the closed balls, $F : B^3 \to B^3$ such that $F(y) = y$ for all y with $|y| = 1$. Then, by a clear topological argument, F is onto and, in particular, there exists $x \in O$ such that

$$\int_\Sigma f_x \circ \phi \, dA = 0.$$

□

2.2. **The 3-dimensional projective space.** The isoperimetric problem for 3-dimensional space-forms, i. e. complete 3-spaces with constant sectional curvature, is a very interesting particular situation. The simply connected case is solved by symmetrization, but if the fundamental group is nontrivial this method does not work or gives little information. It is natural to hope that the complexity of the solutions will depend closely on the complexity of the fundamental group. However the following theorem gives restrictions which are valid in any 3-space-form.

Theorem 13. *Let Σ be an isoperimetric surface of an orientable 3-dimensional space-form $M(c)$ with constant curvature c.*

a) *If Σ has genus zero, then Σ is an umbilical sphere.*

b) *If* $\mathrm{genus}(\Sigma) = 1$*, then Σ is flat.*

Proof. In the first case, the result follows from Hopf uniqueness theorem [39] for constant mean curvature spheres. Assertion b) is proved in [63] as follows. Assume Σ is nonflat. It is well known that since Σ is of genus one, its geometry is controlled by the sinh-Gordon equation: there is a flat metric ds_0^2 on Σ and a positive constant a such that the metric on Σ can be written as $ds^2 = a\, e^{2w} ds_0^2$ where w verifies

$\Delta_o w + \sinh w \cosh w = 0$ (Δ_0 is the Laplacian of the metric ds_0^2). Moreover the quadratic form Q, in term of the flat metric is given by

$$Q(u,u) = \int_\Sigma \{|\nabla_0 u|^2 - (\cosh^2 w + \sinh^2 w)u^2\} dA_0.$$

Take $\Omega \subset \Sigma$ a nodal domain of w (i.e. a connected component of $\{w \neq 0\}$) and consider the function $u = \sinh w$ in Ω and $u = 0$ in $\Sigma - \Omega$. Direct computation, using the sinh-Gordon equation, gives $-u\Delta_0 u = (\cosh^2 w - |\nabla_0 w|^2)\sinh^2 w$ and therefore

$$Q(u,u) = \int_\Omega \{-u\Delta_0 u - (\cosh^2 w + \sinh^2 w)u^2\} dA_0$$

$$= -\int_\Omega \{|\nabla_0 w|^2 + \sinh^2 w)\sinh^2 w\} dA_0 < 0.$$

On the other hand, at every point, the sign of w coincides with the sign of the Gauss curvature of Σ and, as Σ is nonflat, the Gauss-Bonnet theorem gives that w has at least two nodal domains. Thus we have on Σ two functions u_1 and u_2 satisfying $Q(u_1,u_1) < 0$ and $Q(u_2,u_2) < 0$. As the support of these functions are disjoint we get $Q(u_1,u_2) = 0$ and so a linear combination of the u_i will contradict the stability of Σ. □

As an application of the last theorem we solve the isoperimetric problem in a slab.

Corollary 4. ([63]). *Isoperimetric surfaces in a slab of* $\mathbb{R}^3$ *and in the flat three manifold* $\mathbf{S}^1 \times \mathbb{R}^2$ *are round spheres and flat cylinders.*

Proof. The symmetrization argument in §1.3 gives that these two isoperimetric problems are equivalent and that any solution must be a surface of revolution. In particular, any isoperimetric surface in $\mathbf{S}^1 \times \mathbb{R}$ must have genus 0 or 1, and the result follows from Theorem 13 above. □

By combining various of the results we have already proved, we can give a complete solution of the isoperimetric problem in the projective space $\mathbf{P}^3 = \mathbf{S}^3/\{\pm\}$, or in other words, we can solve the isoperimetric problem for antipodally symmetric regions on the 3-sphere. As far as I know the only nontrivial 3-space forms where we can solve the problem at this moment are $\mathbf{P}^3$, the Lens space obtained as a quotient of $\mathbf{S}^3$ by the cube roots of the unity (see §2.3) and the quotient of $\mathbb{R}^3$ by a screw motion (see [63]). In the first two cases we cannot use symmetrization. The solution in the projective space depends on stability arguments.

Theorem 14. (Ritore & Ros [63]). *Isoperimetric surfaces of* $\mathbf{P}^3$ *are geodesic spheres and tubes around geodesics.*

Proof. Let $\Sigma \subset \mathbf{P}^3$ be and isoperimetric surface. From Theorem 12, Σ must be connected, orientable and genus$(\Sigma) \leq 3$.

If genus$(\Sigma) = 0, 1$, then, thanks to Theorem 13, Σ must be umbilical or flat. Such surfaces are easily classified and coincide with the ones in the statement of the theorem.

Let us see that the genus$(\Sigma) = 2, 3$ case cannot hold. The last part of Theorem 12 gives $(1 + H^2)A(\Sigma) \leq 2\pi$. On the other hand, for any closed surface in the 3-sphere $\widetilde{\Sigma} \subset \mathbf{S}^3$ with mean curvature $\widetilde{H}$ we have the Willmore inequality $\int_{\widetilde{\Sigma}}(1 + \widetilde{H}^2) \geq 4\pi$ with the equality holding only if $\widetilde{\Sigma}$ is an umbilical sphere (see [78] and §2.3 below). If we take as $\widetilde{\Sigma}$ the pullback image of Σ to $\mathbf{S}^3$, we obtain

$$4\pi \geq 2(1 + H^2)A(\Sigma) = \int_{\widetilde{\Sigma}}(1 + H^2) \geq 4\pi.$$

Then $\widetilde{\Sigma}$ is a round sphere, which is impossible because $\widetilde{\Sigma}$ must enclose an antipodally symmetric region. □

Direct computations give that for small (resp. large) volumes the solution is a geodesic sphere (resp. the complement of a geodesic sphere). For intermediate values of the volume the isoperimetric solution is a tube around a geodesic. In particular we have the following equivalent results.

Corollary 5. a) *If Σ divides $\mathbf{P}^3$ in two pieces of equal volume, then $A(\Sigma) \geq \pi^2$ and equality the equality holds if and only if Σ is the minimal Clifford torus.*

b) *If Σ divides $\mathbf{S}^3$ in two antipodally symmetric pieces of equal volume, then $A(\Sigma) \geq 2\pi^2$ and equality holds if and only if Σ is the minimal Clifford torus.*

2.3. **Isoperimetry and bending energy.** Consider the functional which associates to each compact surface Σ in the Euclidean space $\mathbb{R}^3$ the integral of its squared mean curvature, $\int_\Sigma H^2 dA$. This expression is called the *bending energy* of Σ and has a rich structure. It is natural to ask what are its minima under various topological restrictions. The global minimum is 4π, which is attained only for the round sphere. An interesting open question is to decide if the minimum for tori is given by a certain anchor ring, which has energy equal to $2\pi^2$ (Willmore [78]). As the functional is compatible with conformal geometry, if Σ is viewed as a surface in the sphere $\mathbf{S}^3$ (via the stereographic projection), then the bending energy is given by

$$\int_\Sigma (1 + H^2) dA, \tag{4}$$

where now H denotes the mean curvature in the sphere. The Willmore conjecture says that the minimum of (4) for tori equals $2\pi^2$ and is attained by the minimal Clifford torus in $\mathbf{S}^3$; see [66] for more information about the current state of this question. Here we are interested in the connections between bending energy and the isoperimetric problem. We have seen in the proof of Theorem 14 how the first one is useful in understanding the second one, and we will see relations in the other direction too.

Given a smooth region Ω on M with $\partial\Omega = \Sigma$, we can parameterize $M - \Omega$ (outside a closed subset of measure zero) by geodesics leaving Σ orthogonally, i.e. by the normal exponential map $\exp_\Sigma$, see [37]. Then $dV = g(p,t)dtdA$, where $g(p,t)$ is the Jacobian of $\exp_\Sigma$, $p \in \Sigma$ and $t \in \mathbb{R}$. The *cut function* $c : \Sigma \to \mathbf{R}$ is a continuous function which associates to each point p the largest $t > 0$ such that $d(\alpha(t), \Sigma) = t$, where $\alpha(t)$ is the geodesic with $\alpha(0) = p$ which leaves Ω orthogonally and d denotes the Riemannian distance in M.

For any $r \geq 0$ we can estimate the area of the boundary of the enlargement $\Omega_r = \{p \in M \mid d(p,\Omega) \leq r\}$ as follows: if we introduce the set $C(r) = \{p \in \Sigma \ : \ d(p, \partial\Omega_r) = r\} \subset \{p \in \Sigma \ : \ c(p) \geq r\}$, then we have

$$A(\partial\Omega_r) \leq \int_{C(r)} g(p,r)dA. \tag{5}$$

Proposition 3. [65]. *Let $\Sigma \subset M$ be a closed surface which separates an ambient three manifold M with $Ric \geq 2$. Then*

$$\int_\Sigma (1+H^2)dA \geq \max I_M = I_M(v/2).$$

If the equality holds, then either M is minimal or totally umbilical.

Proof. The last equality will be proved in section §2.4. Let $\Omega \subset M$ be one of the regions bounded by Σ and assume that $V(\Omega) \leq v/2$. Consider the enlargement Ω_r, $r \geq 0$, such that $\mathrm{V}(\Omega_r) = v/2$. The comparation theorem of Heintze and Karcher [37], combined with Schwarz's inequality, gives that, if $p \in C(r)$, then the Jacobian of the normal exponential map verifies $g(p,r) \leq (\cos r - H \sin r)^2 \leq 1 + H^2$. Therefore, formula (5) gives

$$A(\partial\Omega_r) \leq \int_M (1+H^2)dA.$$

If the equality holds, then Schwarz's inequality implies that M is umbilical (in case $r > 0$) or minimal (in case $r = 0$).

□

Proposition 4. [66] *Let M be a compact orientable 3-manifold with $Ric \geq 2$ whose profile satisfies $\max I_M > 2\pi$. Then any isoperimetric surface of M is either a sphere or a torus.*

Proof. Let Σ be an isoperimetric surface of M with mean curvature H. From Proposition 3 we obtain

$$(1+H^2)A(\Sigma) \geq \max I_M > 2\pi.$$

If $\mathrm{genus}(\Sigma) \geq 2$, then Theorem 12 implies $(1+H^2)A(\Sigma) \leq 2\pi$, which contradicts the last inequality. □

Let $\mathbb{Z}_n$ be the group of the n^{th} roots of unity. This group acts on the unit 3-sphere $\mathbf{S}^3 \subset \mathbb{C}^2$ by complex multiplication $\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2)$, for any $\lambda \in \mathbb{Z}_n$ and $(z_1, z_2) \in \mathbf{S}^3$. The quotient spaces $\mathbf{L}_n = \mathbf{S}^3/\mathbb{Z}_n$ are called *lens spaces*. Note that $\mathbf{L}_2$ is just the projective space. The lens space $\mathbf{L}_3$ is the unique elliptic 3-space-form with fundamental group equal to $\mathbb{Z}_3$.

In section §3.3 we will show certain isoperimetric inequalities for product measures. These results can be extended to fiber bundles, see [67], and imply in particular that $\max I_{\mathbf{L}_3} > 2\pi$. Therefore Proposition 4 combined with Theorem 13 allows us to solve completely the isoperimetric problem for the lens space $\mathbf{L}_3$.

Theorem 15. [67]. *The isoperimetric surfaces of the lens space $\mathbf{L}_3 = \mathbf{S}^3/\mathbb{Z}_3$ are geodesic spheres and tubes around geodesics. In particular, the (quotient) of the minimal Clifford torus has least area among surfaces which separate $\mathbf{L}_3$ in two equal volumes.*

Theorem 16. (Willmore [78]). *For any compact surface Σ immersed in the unit 3-sphere we have $\int_\Sigma (1 + H^2) dA \geq 4\pi$ and the equality holds if and only if Σ is a metric sphere.*

Proof. If Σ is not embedded it was proved by Li and Yau [51] that the bending energy is larger than 8π. If Σ is embedded, it separates $\mathbf{S}^3$. As the maximum of the isoperimetric profile of the 3-sphere is 4π, the claim follows from Proposition 3. □

In a similar way, using the solution of the isoperimetric problem in the projective space, we can prove the Willmore conjecture in the antipodally invariant case.

Theorem 17. (Ros [65]). *Let $\Sigma \subset \mathbf{S}^3$ be an antipodally invariant closed surface of odd genus. Then*

$$\int_\Sigma (1 + H^2) dA \geq 2\pi^2.$$

If the equality holds, then Σ is the minimal Clifford torus.

Proof. The odd genus assumption implies that the quotient surface $\Sigma' = \Sigma/\pm$ separates projective space $\mathbf{P}^3$, see [65]. Therefore using Proposition 3 and Theorem 14 we obtain

$$\int_\Sigma (1 + H^2) dA = 2 \int_{\Sigma'} (1 + H^2) dA \geq 2 \max I_{\mathbf{P}^3} = 2\pi^2.$$

The equality case follows from Proposition 3 and Corollary 5. □

Another proof of this theorem has been given by Topping [76]. The Willmore conjecture has been also proved for tori in $\mathbb{R}^3$ which are symmetric with respect to a point, see Ros [66].

2.4. **The isoperimetric profile.** In this section we will consider some general properties of the isoperimetric profile $I = I_M :]0, v[\to \mathbf{R}$ of a compact Riemannian 3-manifold M, $I(t) = \inf\{\mathrm{A}(\partial\Omega) : \Omega \subset M \text{ region}, \ V(\Omega) = t\}$, where $v = V(M)$. As the complement of an isoperimetric region is also an isoperimetric region, we get $I(v-2t) = I(t)$, for any t. Items a) and b) in the following theorem were first proved by Bavard and Pansu [9]. Berard and Meyer [10] proved (8) by using a different argument. The isoperimetric profile has been also studied by Gallot [25] and Hsiang [41].

Theorem 18. *Given t, $0 < t < v$, let Ω be any isoperimetric region in M with $V(\Omega) = t$ and $\partial\Omega = \Sigma$. The isoperimetric profile I of M verifies the following properties.*

a) *I has left and right derivatives $I'_+(t)$ and $I'_-(t)$, for any $0 < t < v$. Moreover if H is the mean curvature of Σ, then*

$$I'_+(t) \leq 2H \leq I'_-(t). \tag{6}$$

b) *If σ denotes the second fundamental form of Σ, then the second derivative of I satisfies (in the weak sense)*

$$I''(t)I(t)^2 + \int_\Sigma \left(Ric(N) + |\sigma|^2\right) \leq 0. \tag{7}$$

c) *For small t, any isoperimetric region of volume t is a convex region contained in a small neighborhood of some point of M. In particular*

$$I(t) \sim (36\pi t^2)^{1/3}, \ \text{when } t \to 0. \tag{8}$$

Proof. We will only show the assertion c). Let Σ_n be a sequence of isoperimetric surfaces enclosing volumes $t_n \to 0$. We discuss the following possibilities.

1. *The curvature of the sequence Σ_n is unbounded.* By homothetically expanding the surfaces Σ_n, we obtain a sequence $\Sigma'_n \subset B(r_n)$ of properly embedded constant mean curvature surfaces in $B(r_n) = \{x \in \mathbb{R}^3 : |x| \leq r_n\}$, endowed with a metric ds_n^2 which converges smoothly to the usual Euclidean metric. Moreover $r_n \to \infty$, $0 \in \Sigma'_n$ for each n and the second fundamental form of the surface verifies $\max|\sigma_{\Sigma'_n}|^2 = |\sigma_{\Sigma'_n}|^2(0) = 2$.

As $\sigma_{\Sigma'_n}$ is bounded, then locally the surface Σ'_n consists of a certain number of sheets like in Figure 13. Each one of these sheets is a graph over a planar domain of a function with bounded derivatives. If two sheets become arbitrarily close nearby some point when n goes to infinity, then we can modify easily the surface Σ_n in order to get a new surface with least area and enclosing the same volume. In Figure 14 we have a scheme of these modifications in various cases. In the cases (a) and (b), two sheets come together and the enclosed region lies at different sides of the surface. If several sheets of the surface are involved, then we can use (c).

Hence, as Σ'_n has constant mean curvature, standard compactness results, see for instance [59], give that (after taking a subsequence) Σ'_n converges smoothly and

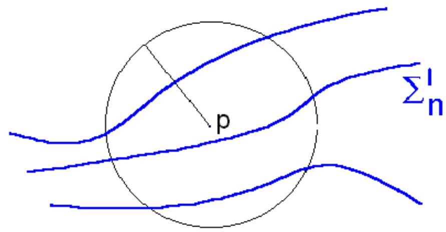

FIGURE 13. The surface Σ'_n is given locally, and independently of n, by graphs of functions with bounded derivatives

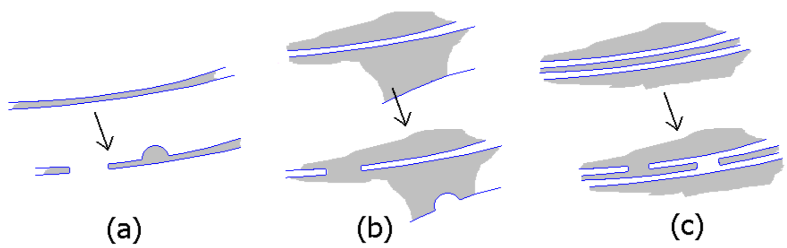

FIGURE 14. If two sheets of Σ'_n become arbitrarily close we can reduce area without modifying the enclosed volume, which contradicts the minimizing property of the surface Σ_n

with multiplicity one to a properly embedded surface $\Sigma' \subset \mathbf{R}^3$ of constant mean curvature $H_{\Sigma'}$, where now $\mathbb{R}^3$ is endowed with the Euclidean metric, with $0 \in \Sigma'$ and $|\sigma_{\Sigma'}|^2(0) = 2$.

The fact that the surfaces Σ_n minimize area under volume constraint allow us to conclude that the surface Σ' (which is orientable) satisfies the following form of the stability condition (which applies also to noncompact surfaces):

$$Q_{\Sigma'}(u,u) \geq 0, \quad \forall\, u \in C^\infty(\Sigma') \text{ with compact support and } \int_{\Sigma'} u\,dA = 0, \tag{9}$$

$Q_{\Sigma'}$ being the quadratic form defined in (1). This condition is known to imply that Σ' is either a union of parallel planes or a round sphere, see da Silveira [23] or López Ros [52]. As the component of Σ' passing through the origin is nonflat, we have that Σ' is a sphere (of radius 1). Therefore, for n large enough, a certain component of Σ_n can be written, after suitable rescaling, as a graph over a round sphere and the function which defines the graph converges to zero. If the surfaces Σ_n were connected assertion c) would be proved.

If Σ_n were disconnected we could repeat the argument above and produce another limit sphere by using a different component of Σ_n. Note that the union of these two limit spheres is unstable and this contradiction proves that Σ_n is connected for n large enough.

2. *The curvature of $\{\Sigma_n\}$ is bounded.* In this case we rescale the surfaces so that they enclose a volume equal to 1. By taking limits in this situation we will produce a family of pairwise disjoint planes in $\mathbb{R}^3$ enclosing a finite volume and this contradiction finishes the proof. $\square$

Below we list some direct consequences of the Theorem.

a) I is locally Lipschitz in $]0, v[$ and extends continuously to 0 and v, $I(0) = I(v) = 0$.

b) If $Ric_M \geq 0$, then I is concave (use (7)). As the profile is symmetric, it follows in particular that I attains its maximum at $t = v/2$.

c) In 3-space forms, solutions of the isoperimetric problem for small prescribed volumes are round spheres: lift these solutions to the universal covering and either compare area-volume with the ones of the metric spheres or use Alexandrov theorem [1].

d) Theorem 18 extends to 3-manifolds M such that M/G is compact, where G is the group of isometries of M. Using this extension we can prove the isoperimetric property of the spheres in $\mathbb{R}^3$ in a new way: It is clear that in $\mathbb{R}^3$ solutions for different volumes differ by an homothety. Therefore the isoperimetric profile of the Euclidean space given by $I(t) = ct^{\frac{2}{3}}$ for a certain constant $c > 0$. If Σ is an isoperimetric region enclosing a volume t, from (6) and Schwarz inequality we have $I'(t)^2 = 4H^2 \leq 2|\sigma|^2$, which combined with (7) gives

$$I''(t)I(t)^2 + \frac{1}{2}I'(t)^2 I(t) \leq 0. \tag{10}$$

But using the explicit expression of I we obtain in fact the equality in (10), and this implies that Σ is totally umbilical.

The arguments used in the proof of item c) in Theorem 18 also show the following result.

Proposition 5. *If Σ_n is a sequence of isoperimetric surfaces of M enclosing volumes $t_n \to t$, $0 < t < v$, then (up to a subsequence) Σ_n converges smoothly and with multiplicity one to an isoperimetric surface Σ enclosing a volume t.*

As a consequence of this proposition, for each t, $0 < t < v$, there exist isoperimetric surfaces Σ_+ and Σ_- enclosing regions of volume t and whose mean curvatures are given by $I'_+(t)/2$ and $I'_-(t)/2$, respectively. In particular, if for a certain prescribed volume t, there is a unique solution of the isoperimetric problem, then the profile I is differentiable at t.

2.5. **Lévy-Gromov isoperimetric inequality.** We present here one of the basic isoperimetric inequalities in Riemannian geometry. It was first proved by Lévy for convex hypersurfaces in Euclidean space. The proof below contains, in particular, the solution of the isoperimetric problem in the sphere and therefore, by a easy limit argument, gives the sharp Euclidean isoperimetric inequality too. An improved version, which involves bounds on the diameter of the manifolds and also works partially for the negative curvature case, is given in Berard, Besson and Gallot [11]. The argument applies in any dimension, in spite of the possible existence of singularities for solutions of the isoperimetric problem.

Theorem 19. (Lévy [50]; Gromov [28]). *Let M^3 be a compact Riemannian manifold with Ricci curvature greater than or equal to the one of the sphere $\mathbf{S}^3(r)$ of radius r, $Ric \geq 2/r^2$. Denote by μ_M and ϑ the normalized Riemannian measures of M and $\mathbf{S}^3(r)$ respectively. Then $I_{\mu_M} \geq I_\vartheta$. Moreover, $I_{\mu_M}(s) = I_\vartheta(s)$ for some $0 < s < 1$ implies $M = \mathbf{S}^3(r)$.*

Proof. Normalize so that $r = 1$. Let $\Sigma \subset M$ be an isoperimetric surface enclosing a region Ω with $V(\Omega) = sV(M)$, for some $s \in]0,1[$, and mean curvature H (with respect to the inner normal). Consider a metric sphere $\Sigma_0 \subset \mathbf{S}^3$ bounding a spherical ball Ω_0 with $V(\Omega_0) = sV(\mathbf{S}^3)$ and mean curvature H_0.

Assume $H \geq H_0$ and parameterize both regions Ω, Ω_0 by geodesics leaving the boundary orthogonally. Then $dV = g(p,t)dtdA$ and $dV_0 = g_0(t)dtdA_0$, where $()_0$ means the spherical corresponding of $()$. In the sphere, the Jacobian of the exponential map is given by $g_0(t) = (\cos t - H_0 \sin t)^2$.

Consider the cut functions c and c_0 of Σ and Σ_0 respectively (note that now we are not parameterizing $M - \Omega$, as in §2.3, but Ω itself). Clearly c_0 is constant. Under our hypothesis, the comparison theorem of Heintze and Karcher [37] says that $c(p) \leq c_0$ and $g(p,t) \leq g_0(t)$. Therefore

$$V(\Omega) = \int_\Sigma \int_0^{c(p)} g(p,t)dtdA \leq \int_\Sigma \int_0^{c_0} g_0(t)dtdA = A(\Sigma) \int_0^{c_0} g_0(t)dt = \frac{A(\Sigma)V(\Omega_0)}{A(\Sigma_0)}$$

which just says that $I_\vartheta(s) \leq I_{\mu_M}(s)$. In the case $H \leq H_0$ the argument above applies to the complementary regions $M - \Omega$ and $\mathbf{S}^3 - \Omega_0$ (whose mean curvatures are $-H$ and $-H_0$, respectively) and gives $I_\vartheta(1-s) \leq I_{\mu_M}(1-s)$. So we obtain the desired inequality by using the symmetry of the profile.

If the equality holds we conclude that $H = H_0$ and, so, the argument applies to both Ω and $M - \Omega$. Moreover we get $c = c_0$ and $g = g_0$. The remaining part of the proof follows from [37]. □

If $Ric_M \geq 2$, the comparison result in [37] implies that $\mathrm{V}(M) \leq \mathrm{V}(\mathbf{S}^3)$ and the equality characterizes the round sphere. We can prove that, when the volume of M is large, we have strong control on the topology of isoperimetric surfaces.

Corollary 6. (Ros [66]). *Let M be a compact 3-manifold with $Ric \geq 2$. If $V(M) \geq V(\mathbf{S}^3)/2$, then any isoperimetric surface of M is homeomorphic either to a sphere or to a torus.*

Proof. Recall that I_M denotes the profile with respect to the (unnormalized) Riemannian measure. Assuming M is not a round sphere, Theorem 19 gives for $t = 1/2$,

$$\frac{\max I_M}{v} = \frac{I_M(v/2)}{v} > \frac{4\pi}{V(\mathbf{S}^3)} = 2\pi,$$

where $v = V(M)$, and the corollary follows from Proposition 4. $\square$

Concerning the topology of isoperimetric surfaces Σ in a positively curved ambient space M, we conjecture that if M is a 3-sphere with a metric of positive Ricci curvature (resp. a strictly convex domain $\mathbb{R}^3$), then Σ is homeomorphic to a sphere (resp. a disc).

3. The isoperimetric problem for measures

In this section we first study the Gaussian measure and then we prove, by extending the symmetrization of Steiner [74] and Schwarz [70], a comparison isoperimetric inequality for product spaces. This comparison is sharp and allows us to obtain important geometric consequences. In particular, we will see that the isoperimetric problem for a product of two compact Riemannian manifolds always admits some solution of slab type [8]. It is well-known that, in a Riemannian manifold, the isoperimetric inequality is equivalent to the Sobolev inequality and gives bounds on the eigenvalues of the Laplacian, see [19, 25]. Here we will see that isoperimetric inequalities of Gaussian type imply three fundamental analytic inequalities.

In this section μ will be an absolutely continuous probability measure on a Riemannian manifold M^n: $d\mu = \varphi dV$ and $\mu(M) = 1$. Moreover, we will assume that $\mu(D) > 0$ for any nonempty open subset $D \subset M$. We will say that (M, μ) is a *probability space* and we will denote its isoperimetric profile by I_μ. Note that $I_\mu(0) = I_\mu(1) = 0$. The hypotheses about μ could be relaxed in some of the results below, but we have chosen to work in a comfortable context for expository reasons.

If M has finite volume, we can consider its *Riemannian probability* μ_M defined as the normalized Riemannian measure $d\mu_M = dV/V(M)$.

3.1. **The Gaussian measure.**

The Gaussian measure $\gamma = \gamma_n$ on $\mathbb{R}^n$ is the probability measure defined by

$$d\gamma = \exp(-\pi|x|^2)dV.$$

Classically, γ_1 appears in the *central limit theorem*: consider in the discrete n-dimensional cube $\{1, -1\}^n \subset \mathbb{R}^n$ the normalized counting measure μ_n. Then the orthogonal projection of μ_n onto the main diagonal of the cube $\mathbb{R} = \mathrm{span}\{(1, \ldots, 1)\}$ converges to the Gaussian measure on $\mathbb{R}$ (up to suitable renormalization).

Let us see that the same holds if we project the normalized spherical measure in dimension m onto a given n-dimensional subspace and take a limit as m goes to ∞.

Let $c_m = V(\mathbf{S}^m)$ be the volume of the m-dimensional unit sphere and $\rho_m = c_{m-1}/c_m$. Among the properties of the the sequence $\{c_m\}$ we mention that

$$(m-1)c_m = 2\pi c_{m-2} \quad \text{and} \quad \rho_m \sim \sqrt{\frac{m}{2\pi}} \quad \text{when } m \to \infty. \tag{11}$$

Denote by ϑ_m the Riemannian probability on the sphere $\mathbf{S}^m(\rho_m)$. The radius ρ_m is chosen so that the profile of ϑ_m satisfies $I_{\vartheta_m}(1/2) = 1$. Given $n \leq m$, we consider the projection $p : \mathbf{S}^m(\rho_m) \to B^n(\rho_m)$, $p(x,y) = x$, and the image measure $\sigma_{m,n} = p(\vartheta_m)$ induced on the ball $B^n(\rho_m) = \{x \in \mathbb{R}^n : |x| \leq \rho_m\}$. Then, for any region R in the ball $B^n(\rho_m)$, we have

$$\sigma_{m,n}(R) = \vartheta_m(p^{-1}(R)) = \frac{c_{m-n}}{c_m \rho_m^n} \int_R \left(1 - \frac{|x|^2}{\rho_m^2}\right)^{-\frac{n+1}{2}} \left(1 - \frac{|x|^2}{\rho_m^2}\right)^{\frac{m}{2}} dV. \tag{12}$$

Taking limits in the expression (12), and using the relation on the right in (11), we obtain the following result (sometimes attributed to Poincaré):

Proposition 6. *When $m \to \infty$, the projection $\sigma_{m,n} = p(\vartheta_m)$ of the spherical probability ϑ_m converges to the Gaussian measure γ_n on $\mathbb{R}^n$.*

Note that, in the case $n = m-1$, the integrand in (12) is constant. Thus we get the following known result.

Lemma 2. *The projection $p : \mathbf{S}^m(\rho_m) \to B^{m-1}(\rho_m)$ transforms the spherical probability ϑ_m in the Riemannian probability β_{m-1} of the ball $B^{m-1}(\rho_m)$.*

In particular, from Proposition 1 we get that the profiles of the sphere and the ball are related by $I_{\beta_{m-1}} \geq I_{\vartheta_m}$. This inequality is sharp because $I_{\beta_{m-1}}(1/2) = I_{\vartheta_m}(1/2)$.

Proposition 6 allows us to solve the isoperimetric problem in the Gaussian space as a limit of the spherical isoperimetric problem (note that the same holds for the Euclidean space).

Theorem 20. (Borell [17]; Sudakov & Tsirel'son [73]). *Among regions $R \subset \mathbb{R}^n$ with prefixed $\gamma(R)$, half spaces minimize the Gaussian area.*

Proof. Let $R \subset \mathbb{R}^n$ be a closed subset and $E = \{x_n \leq a\} \subset \mathbb{R}^n$ a halfspace with $\gamma(E) = \gamma(R)$. If we fix $b < a$ and we consider the halfspace $F = \{x_n \leq b\}$, then for m large enough we have

$$\vartheta_m(p^{-1}(R)) > \vartheta_m(p^{-1}(F)), \tag{13}$$

where $p : \mathbf{S}^m(\rho_m) \to \mathbb{R}^n$ is the orthogonal projection of the sphere to the linear subspace $\mathbb{R}^n$.

Note that $p^{-1}(F)$ is an isoperimetric region of the spherical probability ϑ_m. Moreover, for this measure, the isoperimetric inequality can be writen in the following integral form (see §3.2 below): if $Q \subset \mathbf{S}^n(\rho_m)$ is a closed subset and G is a metric ball in the same sphere with $\vartheta_m(Q) \geq \vartheta_m(G)$, then, for any $r > 0$, the r-enlargements of Q and G verify

$$\vartheta_m(Q_r) \geq \vartheta_m(G_r). \tag{14}$$

As the projection p reduces the distance we see that the r-enlargement of R and the one of its pullback image are related by

$$p^{-1}(R_r) \supset \{p^{-1}(R)\}_r. \tag{15}$$

Therefore, (13), (14) and (15) give

$$\vartheta_m(p^{-1}(R_r)) \geq \vartheta_m(\{(p^{-1}(F)\}_r). \tag{16}$$

Taking limits in (16) when m goes to infinity we obtain, from Proposition 6,

$$\gamma_n(R_r) \geq \gamma_n(F_r).$$

The left term of this inequality is clear. To get the right one we have used that, when m is large, the sphere $\mathbf{S}^m(\rho_m)$ becomes arbitrarily flat and the normal vector of the boundary of the ball $p^{-1}(F)$ in $\mathbb{S}^m(\rho_m)$ becomes almost parallel to $\mathbb{R}^n$. Finally, if $b \to a$ we conclude $\gamma_n(R_r) \geq \gamma_n(E_r)$ and that gives directly the isoperimetric inequality we desired. $\square$

If (M, μ) and (M_1, μ_1) are two probability spaces, then it is clear that $I_\mu \geq I_{\mu \otimes \mu_1}$. In particular we have $I_\mu \geq I_{\mu^{\otimes 2}} \geq \cdots I_{\mu^{\otimes n}}$. The theorem above implies that $I_{\gamma_1} = I_{\gamma_2} = \cdots = I_{\gamma_n}$ (note that $\gamma_n = \gamma_1^{\otimes n}$). This is another remarkable property of the Gaussian measure that in fact characterizes γ among probability measures on $\mathbb{R}$, see Bobkov and Houdré [13].

Given a probability space (M, μ), its *Gaussian constant* is defined as

$$c = c(\mu) = \inf_t \frac{I_\mu(t)}{I_\gamma(t)}. \tag{17}$$

This constant gives the best possible Gaussian isoperimetric inequality in (M, μ). If M is compact and μ_M is the normalized Riemannian measure, then $c(\mu_M) > 0$ and there exists $t_0 \in \,]0, 1[$ such that $I_{\mu_M}(t_0) = cI_\gamma(t_0)$. This follows because the asymptotic behavior of the above functions at $t = 0$ is known to be given by

$$I_{\mu_M}(t) \sim a\ t^{\frac{n-1}{n}}, \text{ and } I_\gamma(t) \sim t(4\pi \log 1/t)^{1/2},$$

for some $a > 0$, see [10] and [6]. Note the parallelism between the Gaussian constant and the usual Cheeger constant (see for instance [19]), which is defined as in (17), but replacing I_γ with the piecewise linear function $J(t) = \min\{t, 1-t\}$. As we will see in the next section, in several applications of the isoperimetric inequalities, the Gaussian constant gives sharper results than the Cheeger one.

The isoperimetric profile of the spheres satisfies the following extraordinary sequence of inequalities.

Theorem 21. (Barthe [6]).

$$I_{\vartheta_m} \geq I_{\vartheta_{m+1}} \geq I_\gamma \geq (I_{\vartheta_{m+1}})^{\frac{m+1}{m}} \geq (I_{\vartheta_m})^{\frac{m}{m-1}}. \tag{18}$$

Here we list some remarks related with the result above.

1. Specializing the theorem to $m = 2$ we have $2\sqrt{t(1-t)} \geq I_\gamma(t) \geq 4t(1-t)$.

2. As $I_{\vartheta_m}(1/2) = I_\gamma(1/2) = 1$, we see that the inequalities (18) are sharp.

3. $I_{\vartheta_m} \to I_\gamma$ when m goes to ∞.

4. The normalized Riemannian measure β_n on $B^n(\rho_{n+1})$ satisfies $I_{\beta_n} \geq I_\gamma$ and the equality holds at $t = \frac{1}{2}$ (use Lemma 2).

5. If M^n is compact with $\partial M = \emptyset$, $Ric_M \geq \frac{n-1}{\rho_n^2}$ and μ_M is its Riemannian probability, then $I_{\mu_M} \geq I_\gamma$. This follows from the high dimensional version of Theorem 19 which, under our hypothesis gives that $I_{\mu_M} \geq I_{\vartheta_n}$. Note that $\frac{n-1}{\rho_n^2}$ converges to 2π when $n \to \infty$.

3.2. **Symmetrization with respect to a model measure.** We say that a probability measure μ_0 on a n_0-dimensional Riemannian manifold M_0 is a *model measure* if there exists a continuous family (in the sense of the Hausdorff distance on compact subsets) $\mathcal{D} = \{D^t \,|\, 0 \leq t \leq 1\}$ of closed subsets of M_0 satisfying the following conditions:

i) $\mu_0(D^t) = t$ and $D^s \subset D^t$, for $0 \leq s \leq t \leq 1$,

ii) D^t is a smooth isoperimetric domain of μ_0 and $I_{\mu_0}(t) = \mu_0^+(D^t)$ is positive and smooth for $0 < t < 1$.

iii) The r-enlargement of D^t verifies $(D^t)_r = D^s$ for some $s = s(t,r)$, $0 \leq t \leq 1$, and

iv) $D^1 = M_0$ and D^0 is either a point or the empty set.

The essential property is that enlargements of isoperimetric regions in $\mathcal{D}$ are also isoperimetric regions. Concerning the enlargements of the empty set, we are using the convention $\emptyset_r = \emptyset$.

Among the examples of model measures we have the Riemannian probability in $\mathbf{S}^n$. A second important group is given by the Gaussian measure $(\mathbb{R}^n, \gamma)$ and symmetric log-concave probability measures on $\mathbb{R}$, that is measures of the type $d\mu = e^{-\tau} dt$, where τ is a convex function with $\tau(-t) = \tau(t)$ for any $t \in \mathbb{R}$, in this last case the isoperimetric regions are the intervals $\{t \,:\, t \leq a\}$, $a \in \mathbb{R}$, see Bobkov and Houdré [14].

Proposition 7. *Let (M,μ) be a probability space and (M_0,μ_0) a model measure. Then $I_\mu \geq I_{\mu_0}$ if and only if for any nonempty closed set $\Omega \subset M$ and for all $r \geq 0$ its r-enlargement verifies $\mu(\Omega_r) \geq \mu_0(D_r)$, where $D \in \mathcal{D}$ with $\mu_0(D) = \mu(\Omega)$.*

Proof. Suppose that $I_\mu \geq I_{\mu_0}$ and take $f(r) = \mu(\Omega_r)$ and $h(r) = \mu_0(D_r)$. Observe that f and h are continuous: the right continuity is clear. To prove the continuity to the left use that, for $r > 0$, $\partial\Omega_r$ is a set of measure zero. Moreover $h(r)$ is smooth for $0 < h(r) < 1$ and, in this range, $h'(r) = I_{\mu_0}(h(r))$. Consider the continuous function $F : [0, 1] \to [0, \infty]$ defined by the conditions $F(0) = 0$ and $F'(t) = 1/I_{\mu_0}(t)$ for $0 < t < 1$. Then $(F \circ h)'(r) = 1$ whenever $0 < h(r) < 1$.

We want to show that $f(r) \geq h(r)$ for $r > 0$. If either $f(r) = 1$ or D is empty, that is trivial. So, we only need to prove the inequality in the case $D \neq \emptyset$ and $0 < r < r_0 = \min\{\rho : f(\rho) = 1\}$. For these r we have $f(r) > 0$ and the mean value theorem gives

$$\liminf_{\rho\to r^+} \frac{F(f(\rho)) - F(f(r))}{\rho - r} = \liminf F'(t_\rho)\frac{f(\rho) - f(r)}{\rho - r}$$

$$= F'(f(r))\mu^+(\Omega(r)) \geq F'(f(r))I_\mu(f(r)) = \frac{I_\mu(f(r))}{I_{\mu_0}(f(r))} \geq 1,$$

where t_ρ is an intermediate value between $f(r)$ and $f(\rho)$. As $f(0) = h(0)$ we deduce that $F(f(r)) \geq F(h(r))$ for $0 < r < r_0$. In fact, if that were false, we could find a number $0 < a < r_0$ so that $F(f(a)) - F(h(a)) < 0$ and $0 < h(r) < 1$ for $0 < r < a$. Hence, for some $\varepsilon > 0$, the continuous function

$$g(r) = F(f(r)) - F(h(r)) - \frac{r}{a}\{F(f(a)) - F(h(a))\}, \quad 0 \leq r \leq a$$

would verify $g(0) = g(a) = 0$ and $\liminf_{\rho\to r^+} g(\rho) > \varepsilon$, $0 < r < a$, which is impossible. Finally, F being an increasing function, we get $f(r) \geq h(r)$ as we claimed. □

A modified version of the above result (under weaker hypotheses) can be found in [14].

Symmetrization is a well-known and useful construction in Euclidean geometry which, among other things, gives the Euclidean isoperimetric inequality; see Steiner [74], Schwarz [70] and [18]. Steiner-Schwarz symmetrization differs from Hsiang symmetrization introduced in §1.3. In the simple planar case, the idea is as follows: given a region Ω in the plane, like in Figure 15, its symmetrization is the figure Ω^S meeting vertical lines L in closed intervals symmetric with respect to the x-axis and with the same length that $\Omega \cap L$.

Now we will see how this construction, joint with its main properties, can be extended to general product spaces (in fact, the idea works also in the fiber bundles setting, see [67]). A proof of the isoperimetric property of halfspaces in the Gaussian space using this symmetrization was given by Ehrhard [21].

Consider three probability spaces $(M_0^{n_0}, \mu_0)$, $(M_1^{n_1}, \mu_1)$ and $(M_2^{n_2}, \mu_2)$ and assume that μ_0 is a model measure. Given a closed subset $\Omega \subset M_1 \times M_2$, the section of Ω through the point $x \in M_1$ will be denoted by $\Omega(x) = \Omega \cap (\{x\} \times M_2)$. The same notation will be used to denote the sections of regions in $M_1 \times M_0$.

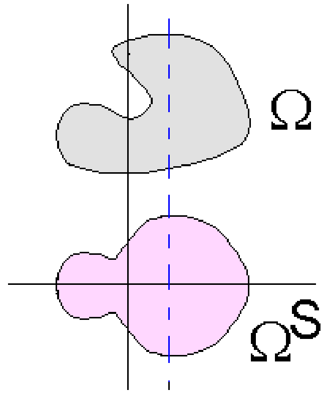

FIGURE 15. The classical symmetrization was used by Steiner and Schwarz to prove the Euclidean isoperimetric inequality

The *symmetrization with respect to* μ_0 of Ω is the subset $\Omega^S \subset M_1 \times M_0$ defined as follows. Take an arbitrary point $x \in M_1$.

If $\Omega(x) = \emptyset$, then $\Omega^S(x) = \emptyset$.

If $\Omega(x) \neq \emptyset$, then $\Omega^S(x) = \{x\} \times \ D$, where $D \in \mathcal{D}$ with $\mu_0(D) = \mu_2(\Omega(x))$.

Proposition 8. *Assume $I_{\mu_2} \geq I_{\mu_0}$. If $\Omega \subset M_1 \times M_2$ is a closed subset, then*

a) Ω^S *is a closed set in* $M_1 \times M_0$.

b) $\mu_1 \otimes \mu_0(\Omega^S) = \mu_1 \otimes \mu_2(\Omega)$

c) $\Omega \subset \Omega'$ *implies* $\Omega^S \subset (\Omega')^S$

d) $(\Omega_r)^S \supset (\Omega^S)_r$

e) $(\mu_1 \otimes \mu_2)^+(\Omega) \geq (\mu_1 \otimes \mu_0)^+(\Omega^S)$

Proof. To prove a) take a sequence $\{(x_k, z_k)\}_k$ in Ω^S which converges to a point $(x, z) \in M_1 \times M_0$. We can also assume that $t_k = \mu_2(\Omega(x_k)) = \mu_0(D^{t_k})$ converges to a number t. As z_k lies in D^{t_k} we have z belongs to D^t. So we only need to show that $t \leq \mu_2(\Omega(x)) = \mu_0(\Omega^S(x))$.

Consider the characteristic functions $\chi_k, \chi : M_2 \to \mathbb{R}$ of $\Omega(x_k)$ and $\Omega(x)$, respectively. Observe that $\limsup_k \chi_k \leq \chi$: in fact, if $\xi \in M_2$ verifies $\limsup \chi_k(\xi) = 1$, then $\chi(\xi) = 1$ by the closeness of Ω, and if $\limsup \chi_k(\xi) = 0$, then there is nothing to prove. So, applying Fatou's lemma we conclude a).

The definition of Ω^S implies clearly assertions b) and c).

Now we prove d). First we remark that, for $r > 0$, the formula $\cup_x \Omega^S(x) = \Omega^S$ implies that $(\Omega^S)_r$ is the closure of $\cup_x \{\Omega^S(x)\}_r$. On the other hand, as $\Omega(x)_r \subset \Omega_r$, item c) gives $\{\Omega(x)_r\}^S \subset (\Omega_r)^S$ and thus, $\cup_x \{\Omega(x)_r\}^S \subset (\Omega_r)^S$. Therefore, to conclude d) it is enough to check that, if $\Omega(x) \neq \emptyset$, then

$$\{\Omega(x)_r\}^S \supset \{\Omega^S(x)\}_r.$$

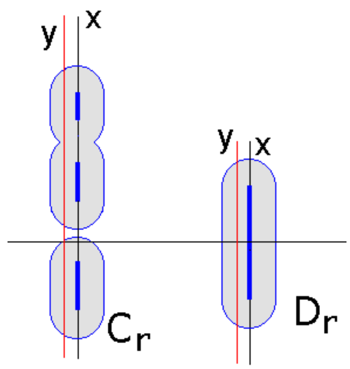

FIGURE 16. The sections through the point $y \in M_1$ of C_r and D_r

To prove this we use the notation $C = \Omega(x)$ and $D = \Omega^S(x) = C^S$. Thus $C \subset \{x\} \times M_2 \equiv M_2$, $D \subset \{x\} \times M_0 \equiv M_0$ and $\mu_2(C) = \mu_0(D)$. Denote by d_i the Riemannian distance in M_i, $i = 0, 1, 2$ and recall that the Riemannian distance d in the product manifold $M_1 \times M_2$ satisfies the identity $d = \sqrt{d_1^2 + d_2^2}$. Hence, if we take $y \in M_1$ with $d_1(x, y) \leq r$ and we define $r' = \sqrt{r^2 - d_1(x, y)^2}$, we can write $C_r(y) \equiv \{z \in M_2 \ : \ d_2(z, C) \leq r'\}$ and $D_r(y) \equiv \{z \in M_0 \ : \ d_0(z, D) \leq r'\}$, see Figure 16. So, Proposition 7 allows us to obtain $\mu_2(C_r(y)) \geq \mu_0(D_r(y))$, which implies, by definition of symmetrization, $(C_r)^S(y) \supset D_r(y)$. Then $(C_r)^S \supset D_r$, the desired inclusion.

The assertion in e) follows directly from b), c) and d). $\square$

3.3. **Isoperimetric problem for product spaces.** Symmetrization with respect to a model measure and its properties in Proposition 8 give immediately comparison results for isoperimetric inequalities in product probability spaces.

Theorem 22. *Consider three probability spaces (M_0, μ_0), (M_1, μ_1) and (M_2, μ_2). If μ_0 is a model measure and $I_{\mu_2} \geq I_{\mu_0}$, then $I_{\mu_1 \otimes \mu_2} \geq I_{\mu_1 \otimes \mu_0}$.*

As the n-dimensional Gaussian measure is a product measure $\gamma_n = (\gamma_1)^{\otimes n}$ and its isoperimetric profile does not depends on n, we obtain the interesting consequences of the theorem by taking $\mu_0 = \gamma$.

Theorem 23. (Barthe & Maurey [8]; Barthe [6]). a) *Let (M_1, μ_1) and (M_2, μ_2) be two probability spaces with $I_{\mu_i} \geq cI_\gamma$, $i = 1, 2$ and $c > 0$. Then, $I_{\mu_1 \otimes \mu_2} \geq cI_\gamma$. Moreover, if $\Omega \subset M_1$ is an isoperimetric region with $\mu_1(\Omega) = a$ and $I_{\mu_1}(a) = cI_\gamma(a)$, then $\Omega \times M_2$ isoperimetric region of $\mu_1 \otimes \mu_2$.*

b) *If the manifolds M_i, $i = 1, 2$, are compact (possibly with boundary), then the isoperimetric problem in the Riemannian product $M_1 \times M_2$ (endowed with its Riemannian measure) admits for some volume a solution of the type $\Omega \times M_2$ or $M_1 \times \Omega$, where Ω is an isoperimetric region in the corresponding factor.*

c) *Among hypersurfaces of* $\mathbf{S}^{n_1}(r_1) \times \cdots \times \mathbf{S}^{n_k}(r_k)$ *dividing the space in two equal volumes, the area is minimized by* $\mathbf{S}^{n_1-1}(r_1) \times \mathbf{S}^{n_2}(r_2) \times \cdots \times \mathbf{S}^{n_k}(r_k)$ *(up to order).*

d) *Among hypersurfaces in a product of Euclidean balls* $B^{n_1}(r_1) \times \cdots \times B^{n_k}(r_k)$ *dividing the space in two equal volumes, the area is minimized by* $B^{n_1-1}(r_1) \times B^{n_2}(r_2) \times \cdots \times B^{n_k}(r_k)$ *(up to order), see Figure 17.*

e) *If* (M, μ) *is a probability space with* $I_\mu \geq I_\gamma$*, then* $I_{\mu \otimes \gamma} = I_\gamma$*.*

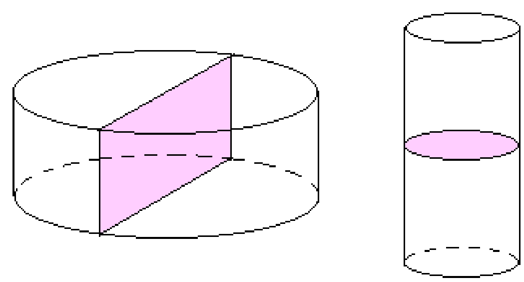

FIGURE 17. Surfaces of least area among the ones which divide a can $B^2(r) \times [0, 1]$ in two equal volumes

Proof. Given $c > 0$, the profile of $(M, ds^2/c^2, \mu)$ is c times the profile of (M, ds^2, μ). So, by rescaling, it is enough to prove the assertions above for $c = 1$. Items a) and e) follow from Theorem 22 and the isoperimetric properties of γ. To prove b), use that under these hypotheses the Gaussian constants of the normalized Riemannian measures are positive and take c the minimum of both constants. Items c) and d) follow from the sharp estimates for the profile of the sphere and the ball given in the comments after Theorem 18. □

The above symmetrization arguments work, without changes, when the spaces have infinite measure. Among the consequences which can be obtained in this case (using as model measure the Lebesgue one) we mention the following result, see [29] p. 335.

Theorem 24. *Let* $(M_1^{n_1}, \mu_1)$ *and* $(M_2^{n_2}, \mu_2)$ *be measure spaces with infinite total measure and denote by* ω_m *the Lebesgue measure in* $\mathbb{R}^m$*.*

a) *If* $I_{\mu_2} \geq I_{\omega_m}$*, then* $I_{\mu_1 \otimes \mu_2} \geq I_{\mu_1 \otimes \omega_m}$*.*

b) *If* $I_{\mu_1} \geq I_{\omega_{m_1}}$ *and* $I_{\mu_2} \geq I_{\omega_{m_2}}$*, then* $I_{\mu_1 \otimes \mu_2} \geq I_{\omega_{m_1+m_2}}$*.*

After the first version of these notes was writen, we received the paper Barthe [7] which contains a proof of Theorem 22 in the case that μ_0 is a symmetric log-concave measure on $\mathbb{R}$.

3.4. **Sobolev-type inequalities.** In this section we give three analytic inequalities for a probability space (M, μ) which admits a Gaussian isoperimetric inequality, that is, a space with positive Gaussian constant $c(\mu)$. We know that this family includes any compact manifold (with or without boundary) endowed with its Riemannian probability; see item c) in Theorem 18 and [10].

First we will see a functional inequality which, for a given probability space, is equivalent to the Gaussian isoperimetric inequality. In fact, the geometric properties in Theorem 23 were first proved in [8] by following this analytic formulation, instead of using symmetrization. Earlier versions of the next result were proved by Ehrhard [22] and Bobkov [12]. The general version below is due to Barthe and Maurey [8].

Theorem 25. *Let (M, μ) be a probability space and $c > 0$. Then, $I_\mu \geq cI_\gamma$ if and only if for any locally Lipschitz function $f : M \to [0, 1]$,*

$$I_\gamma\Big(\int_M f d\mu\Big) \leq \int_M \sqrt{I_\gamma(f)^2 + \frac{1}{c^2}|\nabla f|^2}\, d\mu. \tag{19}$$

Proof. Assume $c = 1$. If (19) holds, then for any closed subset $R \subset M$ we define $f_\varepsilon(x) = (1 - d(x, R)/\varepsilon)_+$, where d denotes the Riemannian distance in M and $h_+ = \max\{h, 0\}$. Putting $f = f_\varepsilon$ in (19) and taking $\varepsilon \to 0$ we obtain $\mu^+(\partial R) \geq I_\gamma(\mu(R))$.

Conversely, if $I_\mu \geq I_\gamma$ consider the probability measure $\mu \otimes \gamma$ in $M \times \mathbf{R}$ and the closed subset $R = R(f) = \{(x, t) \in M \times \mathbf{R} \mid \gamma(t) \leq f(x)\}$.

Then $(\mu \otimes \gamma)(R) = \int_M f d\mu$ and

$$(\mu \otimes \gamma)^+(\partial R) = \int_M \sqrt{I_\gamma(f)^2 + |\nabla f|^2}\, d\mu.$$

We can conclude the proof by using that $I_{\mu\otimes\gamma} \geq I_\gamma$ (see item e) in Theorem 23). □

The next inequality was first proved by Gross [30] for the Gaussian measure itself. It is an important tool in analysis with applications, among others, in evolution problems, see for instance [24, 44]. The proofs below have been taken from Ledoux [48].

Theorem 26. (Logarithmic Sobolev inequality). *Let (M, μ) be a probability space with $I_\mu \geq cI_\gamma$ for some $c > 0$. Then, for any smooth bounded positive function $g : M \to \mathbf{R}$,*

$$\int_M g \log g \; d\mu \leq \int_M g \; d\mu \log\left(\int_M g \; d\mu\right) + \int_M \frac{|\nabla g|^2}{4\pi c^2 g}\, d\mu. \tag{20}$$

Proof. We normalize so that $c = 1/\sqrt{2\pi}$ and we use that the asymptotic behavior of the Gaussian profile at $t = 0$ is given by $cI_\gamma(t) \sim t\sqrt{2\log\frac{1}{t}}$.

Put $f = e^{-\lambda}g$ in (19), with $\lambda > 0$ large enough. Then

$$e^{-\lambda}\int g\sqrt{2\log\frac{e^\lambda}{\int_M g d\mu}}d\mu \leq e^{-\lambda}\int\sqrt{g^2(2\log\frac{e^\lambda}{g}) + |\nabla g|^2}\; d\mu \; + \text{ error term},$$

which can be written as

$$0 \leq \int \sqrt{2g^2(\lambda + \log \frac{1}{g} + \frac{|\nabla g|^2}{2g^2})} - \sqrt{2g^2(\lambda + \log \frac{1}{\int g})}\, d\mu \;+\; \text{error term.}$$

Multiplying by $\sqrt{\lambda}$, taking $\lambda \to \infty$ and using the formula

$$\lim_{\lambda\to\infty} \sqrt{\lambda}\left(\sqrt{a(\lambda+b_1)} - \sqrt{a(\lambda+b_2)}\right) = \sqrt{a}(b_1 - b_2),$$

we obtain,

$$0 \leq \sqrt{2} \int \left(g \log \frac{1}{g} + \frac{|\nabla g|^2}{2g} - g \log \frac{1}{\int g} \right)\, d\mu.$$

□

Theorem 27. (Poincaré inequality). *Assume the probability space (M, μ) verifies $I_\mu \geq cI_\gamma$, for some $c > 0$. Then, for any $h : M \to \mathbf{R}$ with $\int_M h d\mu = 0$,*

$$\int_M h^2\, d\mu \leq \frac{1}{2\pi c^2} \int_M |\nabla h|^2\, d\mu. \tag{21}$$

Proof. Put $g = (1 + \varepsilon h)^2$ in the log-Sobolev inequality (20) and take $\varepsilon \to 0$. □

To convince the reader of the sharpness of the above results, we consider a couple of examples. For the sphere $\mathbf{S}^n(\rho_n)$, Theorem 27 gives that the first eigenvalue of the Laplacian λ_1 satisfies $\lambda_1 \geq 2\pi$. The correct value is $\lambda_1 = \frac{n}{\rho_n^2}$, which converges to 2π when n goes to infinity.

For the flat torus $T^n = \mathbf{S}^1(\frac{1}{\pi}) \times \ldots \mathbf{S}^1(\frac{1}{\pi})$ Theorem 27 gives $\lambda_1 \geq 2\pi$ (note that $\rho_1 = 1/\pi$ and so the Gaussian constant of T^n is $c = 1$). The exact value is $\lambda_1 = \pi^2$. The Cheeger constant of T^n is $h = 1$ (by Theorem 7) and so the Cheeger theorem, see [19], gives the estimate $\lambda_1 \geq h^2/4 = 0.25$.

References

[1] A.D. Alexandrov, *Uniqueness theorems for surfaces in the large I*, Vestnik Leningrad Univ. Math. **11** (1956), 5-17.

[2] F. J. Almgren, *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Mem. AMS **165** (1976).

[3] F. J. Almgren, Spherical symmetrization, Proc. International workshop on integral functions in the calculus of variations, Trieste 1985, Red. Circ. Mat. Palermo (2) Supple. (1987), 11-25.

[4] N. Aronszajn, *A unique continuation theorem for solution of elliptic partial differential equations or inequalities of second order*, J. Math. Pures Appl. **36** (1957), 235-249.

[5] J. L. Barbosa & M. do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z. **185** (1984), 339-353.

[6] F. Barthe, *Extremal properties of central half-spaces for product measures*, J. Funct. Anal. **182** (2001), 81–107.

[7] —, *Log-concave and spherical models in isoperimetry*, preprint.

[8] F. Barthe & B. Maurey, *Some remarks on isoperimetry of Gaussian type*, Ann. Inst. H. Poincaré Probab. Statist. **36** (2000), 419-434.

[9] C. Bavard & P. Pansu, *Sur le volume minimal de* $\mathbf{R}^2$, Ann. Sci. École Norm. Sup. (4) **19** (1986), 479-490.

[10] P. Bérard & D. Meyer, *Inégalités isopérimétriques et applications*, Ann. Sci. cole Norm. Sup. (4) **15** (1982), 513–541.

[11] P. Berard, G. Besson & S. Gallot, *Sur une inegalité isopérimetrique qui généralise celle de Paul Lévy-Gromov*, Invent. Math. **80** (1985), 295-308.

[12] S. G. Bobkov, *An isoperimetric inequality on the discrete cube and an elementary proof of the isoperimetric inequality in Gauss space*, Ann. Probab. **24** (1) (1997), 206-214.

[13] Bobkov, S. G. & Udre, K (Houdre, C.), *Characterization of Gaussian measures in terms of the isoperimetric property of half-spaces* (Russian), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 228 (1996), Veroyatn. i Stat. 1, 31–38, 356; translation in J. Math. Sci. (New York) **93** (1999), 270–275.

[14] S. G. Bobkov & C. Houdré, *Some connections between isoperimetric and Sobolev-type inequalities*, Mem. Amer. Math. Soc. **129** (1997).

[15] J. Bokowski, H. Hadwiger & J. M. Wills, *Eine Ungleichung zwischen Volumen, Oberfläche und Gitterpunkzhal konvexer Körper in n-dimensionalen euklidischen Raum*, Math. Z. **127** (1972), 363-364.

[16] J. Bokowski & E. Sperner, *Zerlegung konvexer Körper durch minimale Trennflčhen*, J. Reine Angew. Math. **311/312** (1979), 80-100.

[17] C. Borell, *The Brunn-Minkowski inequality in Gauss space*, Invent. Math. **30** (1975), 207-216.

[18] Y. D. Burago & V. A. Zalgaller, *Geometric inequalities*, Springer-Verlag, Berlin (1988).

[19] I. Chavel, *Eigenvalues in Riemannian geometry*, Academic Press, 1984.

[20] —, *Isoperimetric Inequalities: Differential Geometric and Analytic Perspectives*, Cambridge Tracts in Mathematics 145, Cambridge University Press, Cambridge, 2001.

[21] A. Ehrhard, *Symetrisation dans l'espace de Gauss*, Math. Scand. **53** (1983), 281–301.

[22] —, *Inégalités isopérimétriques et intégrales de Dirichlet gaussienes*, Ann. Sci. Ecole Norm. Sup. **17** (1984), 317-332.

[23] A. da Silveira, *Stability of complete noncompact surfaces with constant mean curvature*, Math. Ann. **277** (1987), 629-638.

[24] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Univ. Press, (1989).

[25] S. Gallot, *Inégalités isopérimetriques et analytiques sur les variétés Riemannienes*, Asterisque **163-164** (1988), 31-91.

[26] E. Gonzalez, U. Massari & I. Tamanini, *On the regularity of boundaries of sets minimizing perimeter with a volume constraint*, Indiana Univ. Math. J. **32** (1983), 25-37.

[27] P. Griffiths & J. Harris, *Principles of algebraic geometry*, Wiley, New York, (1978).

[28] M. Gromov, *Paul Levy's isoperimetric inequality*, prepublications I.H.E.S. (1980).

[29] —, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Birkhäuser, Boston-Basel-Berlin (1999).

[30] L. Gross, *Logarithmic Sobolev inequalities* Amer. J. Math. **97** (1976), 1061-1083.

[31] K. Grosse-Brauckmann, *New surfaces of constant mean curvature* Math. Z. **214** (1993), 527–565.

[32] M. Grüter, *Boundary regularity for solutions of a partioning problem*, Arch. Rat. Mech. Anal. **97** (1987), 261-270.

[33] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer, Berlin (1957).

[34] —, *Gitterperiodische Punktmengen und Isoperimetrie*, Monatsh. Math. **76** (1972), 410-418.

[35] D.A. Hajduk, P.E. Harper, S.M. Gruner, C.C. Honecker, G. Kim, E. L. Thomas & L. J. Fetters, *The giroid: A new equilibrium morphology in weakly segregated diblock copolymers*, Macromolecules **27** (1994), 4063-4075.

[36] L. H. Harper, *Optimal numberings and isoperimetric problems on graphs*, J. Combinatorial Theory **1** (1966), 385–393.

[37] E. Heintze & H. Karcher, *A general comparison theorem with applications to volume estimates for submanifolds*, Ann. Sci. Ec. Norm. Super. **11** (1978), 451-470.

[38] J. Hersch, *Quatre propertiés isopérimétriques de membranes sphériques homogenes*, C. R. Acad. Sci. Paris Ser. A-B **270** (1970), A1645-A1648.

[39] H. Hopf, *Differential geometry in the large*, Lecture Notes in Mathematics, **1000** (1989), Springer-Verlag.

[40] W. Y. Hsiang, *A symmetry theorem on isoperimetric regions*, PAM-409 (1988), UC Berkeley.

[41] —, *On soap bubbles and isoperimetric regions in noncompact symmetric spaces. I*, Tohoku Math. J. (2) **44** (1992), 151–175.

[42] —, *Isoperimetric regions and soap bubbles*, Proceedings conference in honor to Manfredo do Carmo, Pitman survey in pure and. appl. math . **52** (1991), 229-240.

[43] H. Howards, M. Hutchings & F. Morgan, *The isoperimetric problem on surfaces*, Am. Math. Monthly **106** (1999), 430-439.

[44] G. Huisken, *Local and global behavior of hypersurfaces moving by mean curvature*, Differential geometry: partial differential equations on manifolds, Proc. Symp. Pure Math. **54** Part 1 (1993), 175–191.

[45] M. Hutchings, F. Morgan, M. Ritoré & A. Ros, *Proof of the double bubble conjecture* Ann. of Math. (to appear).

[46] H. B. Lawson, *Complete minimal surfaces in* $\mathbf{S}^3$, Ann. of Math. (2) **92** (1970), 335–374.

[47] M. Ledoux, *Concentration of measure and logarithmic Sobolev inequalities. Séminaire de Probabilités.* XXXIII, 120–216, Lecture Notes in Math., **1709**, Springer, Berlin, 1999.

[48] —, *The geometry of the Markov diffusion generators*, Probability theory, Ann. Fac. Sci. Toulouse Math. (6) **9** (2000), 305–366.

[49] M. Ledoux & M. Talagrand, *Probability in Banach spaces. Isoperimetry and processes*, Results in Mathematics and Related Areas (3), **23**, Springer-Verlag, Berlin, 1991.

[50] P. Lévy, *Leons dánalyse fonctionelle*, Gauthier–Villars, Paris 1922.

[51] P. Li & S. T. Yau, *A new conformal invariant and its applications to the Willmore conjecture and first eigenvalue of compact surfaces*, Invent. Math. **69**(1982), 269-291.

[52] F. J. López & A. Ros, *Complete minimal surfaces with index one and stable constant mean curvature surfaces*, Comment. Math. Helvet. **64** (1989), 34-43.

[53] W. H. Meeks, *The theory of triply periodic minimal surfaces*, Indiana Univ. Math. J. **39** (1990), 877–936.

[54] F. Morgan, *Geometric measure theory: a beginner's guide*, 2nd ed., Academic Press, 1998.

[55] —, *Regularity of isoperimetric hypersurfaces in Riemannian manifolds*, preprint.

[56] M. Nosarzewska, *Évaluation de la différence entre l'aire d'une région plane convexe et le nombre des points avec coordonnées entiéres couvert par elle*, Colloq. Math. **1** (1948), 305-311.

[57] R. Osserman, *The isoperimetric inequality*, Bull. Amer. Math. Soc. **84** (1978), 1182-1238.

[58] R. Pedrosa & M. Ritoré, *Isoperimetric domains in the Riemannian product of a circle with a simply connected space form and applications to free boundary problems*, Indiana Univ. Math. J. **48** (1999), 1357-1394.

[59] J. Perez & A. Ros, *Properly embedded minimal surfaces with finite total curvature*, Proceedings of the CIME course "Minimal surfaces in flat three-manifolds" at Martina-Franca, Italy, summer 1999 (to appear).

[60] M. Ritoré, *Superficies con curvatura media constante*, PhD Thesis, Granada, 1994, http://www.ugr.es/∼ritore.

[61] —, *Examples of constant mean curvature surfaces obtained from harmonic maps to the two sphere*, Math. Z. **226** (1997), 127-146.

[62] —, *Index one minimal surfaces in flat three-space forms*, Indiana Univ. Math. J. **46**, (1997), 1137-1153.

[63] M. Ritoré & A. Ros, *Stable Constant Mean Curvature Tori and the Isoperimetric Problem in Three Space Forms*, Comment. Math. Helvet. **67** (1992), 293-305.

[64] —, *The spaces of index one minimal surfaces and stable constant mean curvature surfaces embedded in flat three manifolds*, Trans. Amer. Math. Soc. **348** (1996), 391-410.

[65] A. Ros, *The Willmore conjecture in the real projective space*, Math. Research Letters **6** (1999), 487-494.

[66] —, *The isoperimetric and Willmore problems*, Proc. congress in Memory of A. Gray, Contemporary Math. (to appear).

[67] —, in preparation.

[68] A. Ros & E. Vergasta, *Stability for hypersurfaces of constant mean curvature with free boundary*, Geom. Dedicata **56** (1995), 19-33.

[69] M. Ross, *Schwarz' P and D surfaces are stable*, Differential Geom. Appl. **2** (1992), 179-195.

[70] H. A. Schwarz, *Gesammelte Mathematische Abhandlungen*, Springer Verlag, Berlin (1890).

[71] W. M. Schmidt, *Volume, surface area and the number of integer points covered by a convex set*, Arch. Math. (Basel) **23** (1972), 537-543.

[72] U. Schnell, *Lattice inequalities for convex bodies and arbitrary lattices*, Monatsh. Math. **116** (1993), 331-337.

[73] V. N. Sudakov & B. S. Tsirel'son, *Extremal properties of half-spaces for spherically invariant measures*, J. Soviet Math. (1978), 9-18.

[74] J. Steiner, *Sur le maximum et le minimum des figures dans le plan, sur la sphére et dans l'espace en géneral* J. Reine Angew. Math. **24** (1842), 93-152.

[75] E. L Thomas, D. M. Anderson, C.S. Henkee & D. Hoffman, *Periodic area-minimizing surfaces in block copolymers*, Nature **334** (1988), 598-602.

[76] P. Topping, *Towards the Willmore conjecture*, Calc. Var. and PDE, **11** (2000), 361–393.

[77] Traizet, *Construction of tripy periodic minimal surfaces*, preprint.

[78] T. J. Willmore, *Note on embedded surfaces*, An. Sti. Univ. "Al. I. Cuza" Iasi Sect. I a Mat. **11** (1965), 493-496.

[79] P. Yang & S. T. Yau, *Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7** (1980), 55-63.

aros@ugr.es
Departamento de Geometria y Topologia
Facultad de Ciencias, Universidad de Granada
18071 - Granada, Spain