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The Isoperimetric and Willmore Problems

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Abstract. We introduce some methods to study the isoperimetric problem in 3-dimensional Riemannian manifolds and we show that in the positive curvature case we can control the topology of the isoperimetric regions. We consider the case of the projective space, which was first solved by Ritoré and Ros, and we apply it to the Willmore problem.

1. Introduction

The isoperimetric problem is a classical topic in geometry but at the same time many basic questions about it remain unsolved. In this paper we first introduce some of the methods used in the study of that problem, although we will not try to be exhaustive at all. We will explain some relatively flexible ideas, like symmetrization or stability, which can be adapted to a certain number of situations. As an example we will study the problem for radial metrics on the 3-sphere.

Then we will show that in 3-manifolds with positive Ricci curvature the topology of the isoperimetric regions can be controled. In particular we will prove that, when the volume of the ambient space is large, any isoperimetric surface must be either an sphere or a torus. As consequence, we will solve the isoperimetric problem in the real projective space or, equivalently, the isoperimetric problem for antipodal invariant regions in the 3-sphere. That result was first obtained by Ritoré and Ros [**35**], but here we will give a somewhat different proof.

Finally, as application of the above results, we will solve the Willmore conjecture for tori in euclidean space which are symmetric with respect to a point.

2. The isoperimetric problem

In this paper we will only consider the three dimensional case. Let M be a Riemaniann 3-dimensional manifold with or without boundary and volume $\mathrm{V}(M) \in]0,\infty]$. Given a positive number $v < \mathrm{V}(M)$, we want to study the compact surfaces $\Sigma \subset M$ such that

(1) Σ encloses a region of volume v, and
(2) Σ minimizes area under the constraint (1),

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see Figure 1. If $\partial M \neq \emptyset$ we do not count the area coming from the boundary of M. Existence and regularity for the isoperimetric problem has been considered by several authors. From the works of Almgren [**5**], Grüter [**19**] and Gonzalez, Massari, Tamanini [**16**], see also [**32**], we have the following satisfactory result.

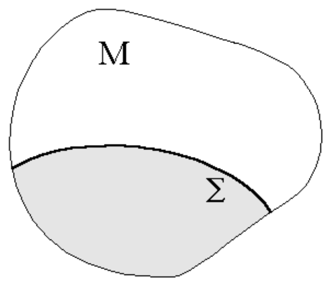

Figure 1

Theorem 1. *If M is either compact or homogeneous, then for any given v there exists a surface which solves the isoperimetric problem above. Any solution Σ is a smooth embedded surface with constant mean curvature and, if $\partial M \cup \Sigma \neq \emptyset$, then ∂M and Σ meet orthogonally.*

The surfaces which solve the problem and the regions enclosed by them will be called *isoperimetric surfaces* and *isoperimetric regions* respectively.

Isoperimetric inequalities, that is, estimates of the area of the isoperimetric surfaces have been also studied by many authors. Later on we will give a couple of examples of this type. This kind of results have a wide number of applications, see for instance [**14**].

However, the study of the geometry and the topology of solutions and the explicit description of these surfaces when the ambient space is simple enough, is an almost unexplored field. In this paper we will put the emphasis just in this kind of properties.

If M is compact, the *isoperimetric profile* of M is defined by

$$I = I_M : [0, \mathrm{V}(M)] \to \mathbb{R}, \qquad I(v) = \min\{\mathrm{A}(\partial\Omega) \,|\, \Omega \subset M \,\text{region},\, \mathrm{V}(\Omega) = v\}.$$

This function is continuous, vanishes at 0 and $\mathrm{V}(M)$, and is positive elsewhere. Moreover, as the complement of an isoperimetric region is an isoperimetric region too, I is always symmetric $I(\mathrm{V}(M) - 2v) = I(v)$.

3. Symmetrization

The method which is more clearly connected with our problem is the symmetrization one. Classically, symmetrization was used in several ways, see Schwarz [**42**] and Steiner [**44**], to show that the round sphere solves the isoperimetric problem in $\mathbb{R}^3$. Since we have existence and regularity results, we can use another version of the idea which was first introduced by Hsiang [**24, 25**]. Roughly speaking, we can say that if M has enough isometries then any isoperimetric surface Σ is connected and if M has many symmetries then Σ is symmetric. Instead of stating general results we will show how the method works in the Euclidean case.

Theorem 2. *Isoperimetric surfaces in $\mathbb{R}^3$ are round spheres.*

Proof. Let Ω be an isoperimetric region and $\Sigma = \partial\Omega$. If Ω is not connected, then we move one of its connected components until we touch for the first time the remaining part of Ω. At that moment, we have a new isoperimetric region with a singular point at its boundary, which is impossible by the regularity results above.

The same argument applies if Ω has nonconnected boundary. By moving one of the boundary components we get a new isoperimetric region with a singular point. So we conclude that Σ is connected. Consider now a plane $P \subset \mathbb{R}^3$ cutting Ω in

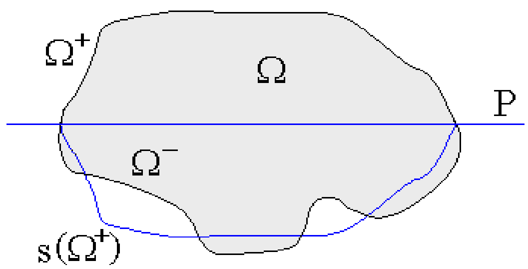

Figure 2

two pieces with the same volume, Ω^+ and Ω^-, see Figure 2. Assume that the area of the pieces induced on Σ verifies $\mathrm{A}(\Sigma \cap \Omega^+) \leq \mathrm{A}(\Sigma \cap \Omega^-)$. The new domain $\Omega^* = \Omega^+ \cup s(\Omega^+)$, where s denotes the symmetry with respect to P, is symmetric and satisfies $\mathrm{V}(\Omega^*) = \mathrm{V}(\Omega)$ and $\mathrm{A}(\partial\Omega^*) \leq \mathrm{A}(\Sigma)$. So $\mathrm{A}(\partial\Omega^*) = \mathrm{A}(\Sigma)$ and Ω^* is also an isoperimetric region. Therefore, using again the regularity of minimizers, we have that $\partial\Omega^*$ and Σ are two closed surfaces in $\mathbb{R}^3$ with constant mean curvature which coincide in a nonempty open set. As surfaces of this kind are analytic, we get finally that $\partial\Omega^* = \Sigma$ or, in other words, Σ is symmetric with respect to P (instead of analitycity we can invoke the unique continuation property [**7**] which works also when the ambient space is not analytic but only C^∞).

As each family of parallel planes in $\mathbb{R}^3$ contains one plane which divides Ω in two equal volumes, we conclude easily that Σ is a round sphere. □

Of course, Theorem 2 follows also from the well-known Alexandrov uniqueness theorem [**3**] for closed constant mean curvature surfaces in $\mathbb{R}^3$. However, when we try to apply these methods to other related situations, symmetrization gives stronger results than Alexandrov reflection technique.

The above symmetrization argument works without any change in the hyperbolic 3-space and in the unit 3-sphere $\mathbf{S}^3$, and gives that isoperimetric surfaces in these spaces are round spheres too. The method works also in higher dimension.

If M is either a halfspace or a slab bounded by parallel planes in $\mathbb{R}^3$ then, using the same idea again, we get that any isoperimetric surface Σ is a connected surface of revolution which meets the boundary of M orthogonally. Note that Σ must meet ∂M: otherwise by moving Σ until we touch the first time the boundary of M, we get an isoperimetric surface which meets ∂M tangentially, a contradiction. If M is a halfspace with $P = \partial M$ and s denotes the symmetry with respect to P, we have that $\Sigma \cup s(\Sigma)$ is an embedded closed surface with constant mean curvature, which must be a round sphere by [**3**]. So, Σ is a halfsphere. When M is a slab, using symmetrization with respect to planes orthogonal to P, we get that Σ is a surface of revolution and, then, it must be a piece of a sphere, an unduloid or a cylinder, see Figure 3. It is known, by a different argument, that unduloids are not isoperimetric surfaces, thus only halfspheres and cylinders appear as minimizers, see [**2**] and [**35**]. However if M is an slab in $\mathbb{R}^n$, $n \geq 10$, Pedrosa and Ritoré [**33**] have shown that unduloids are solutions for some prescribed volumes.

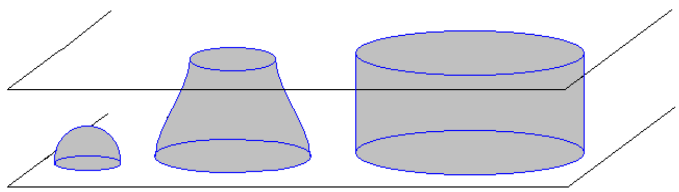

FIGURE 3

4. Stability

Isoperimetric surfaces are *stable*, that is, they minimize area up to the second order (under volume constraint). This means that for any function u in the Sobolev space $H^1(\Sigma)$ with $\int_\Sigma u = 0$ we have $Q(u,u) \geq 0$, where Q is defined by the second variation formula of the area functional. The quadratic form Q is given by

$$Q(u,u) = \int_\Sigma |\nabla u|^2 - (Ric(N) + |\sigma|^2)u^2 - \int_{\partial\Sigma} \kappa\, u^2, \tag{1}$$

where $Ric(N)$ denotes Ricci curvature of M along the unit normal direction N of Σ, σ is the second fundamental form of Σ and κ is the normal curvature of ∂M along N, see for instance [**8**] and [**39**]. Stability plays an important role in the theory of minimal surfaces. In the context of the isoperimetric problem, the interest of this notion was first pointed out by Barbosa and do Carmo, who gave a proof of the minimizing property of the sphere in $\mathbb{R}^3$ by looking at the stability of the solutions.

THEOREM 3. (Barbosa & do Carmo [**8**]) *Let Σ be a compact orientable surface immersed in $\mathbb{R}^3$ with constant mean curvature. If Σ is stable then it must be a round sphere.*

If M has positive Ricci curvature and ∂M is convex then it follows, using locally constant test functions in (1), that any isoperimetric surface of M must be connected.

The symmetries of M have been used in the symmetrization process and can be used also in the following natural and useful stability argument, see for instance Alías, López, Palmer [**4**], Hutchings, Morgan, Ritoré, Ros [**26**], Ros, Souam [**38**] and Ros, Vergasta [**39**], for concrete applications to the study of constant mean curvature surfaces.

Consider a 1-parameter family $\{f_\theta : |\theta| < \varepsilon\}$ of isometries of M. Then the associated infinitesimal rotation $u = \langle \frac{df}{d\theta}, N \rangle$ is a Jacobi function, that is, $Q(u, v) = 0$ for any $v \in H^1(\Sigma)$ or, equivalently,

$$\Delta u + (Ric(N) + |\sigma|^2)u = 0 \text{ on } \Sigma \qquad \text{and} \qquad \frac{\partial u}{\partial \nu} = \kappa u \text{ along } \partial\Sigma.$$

The well-known Courant nodal argument, [**14**], gives the following restriction on u.

PROPOSITION 1. *Let Σ be an isoperimetric surface in M and u an infinitesimal rotation of Σ. Then either $u \equiv 0$ or $\{u \neq 0\}$ has at most two connected components.*

PROOF. Assume $\{u \neq 0\}$ has at least 3 nodal domains Σ_i, $i = 1, 2, 3$ and define the functions $u_i \in H^1(\Sigma)$ by $u_i = u$ in Σ_i, and $u_i = 0$ in $\Sigma - \Sigma_i$.

So, we have $Q(u_i, u_j) = 0$ for $i \neq j$ and, using that u is a Jacobi function, $Q(u_i, u_i) = Q(u_i, u) = 0$. Therefore there exist nonzero constants a_1 and a_2 such that $v = a_1 u_1 + a_2 u_2$ verifies $\int_M v = 0$ and $Q(v, v) = 0$. Stability then implies that v is Jacobi function too. As v vanishes on Σ_3, the unique continuation property [**7**] says that $v \equiv 0$. This contradiction proves the result. □

By joining the two ways we dispose now to use the symmetries of the problem, we can give a new proof of the following result. A similar approach has been used in the solution of the double bubble conjecture in $\mathbb{R}^3$, see [**26**].

THEOREM 4. (Bokowski & Sperner [**13**], Almgren [**6**]) *Isoperimetric surfaces in a Euclidean ball are either spherical caps or planar discs.*

PROOF. First observe that, as in the case in which M is $\mathbb{R}^3$ or a halfspace, any isoperimetric surface Σ is connected and meets the boundary of the ball.

Given a line through the center of the ball we have, by continuity, a plane which contains the line and cut the region enclosed by Σ in two equal volumes. Symmetrizacion gives that this plane is a plane of symmetry of Σ. As this implies that Σ as infinitely many symmetries, we conclude that it must be a surface of revolution. In particular, Σ is either a disc or an annulus. In the first case it follows from properties of Delaunay surfaces that Σ is an umbilical surface.

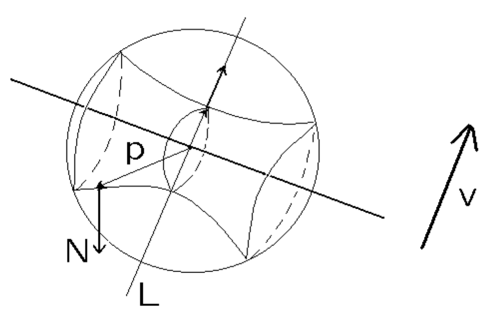

FIGURE 4

Now we prove that the second case is impossible. Take L a line through the origin meeting Σ orthogonally (take for instance the line passing through the point

$q \in \Sigma$ closest to the origin) and consider the family $\{f_\theta\}$ of rotations around L, see Figure 4. Note that, as $\Sigma \cap L \neq \emptyset$, L cannot be the axis of revolution of Σ. The associated infinitesimal rotation on Σ is given by $u = det(p, N, v)$, where p, N and v are the position vector, the unit normal vector of Σ and the unit direction of L respectively. By looking at Figure 4, we see that u vanishes along the curves $\Sigma \cap P$, P being the plane which contains the axis of revolution and the line L, and along the circle in Σ through q. Therefore we see that either $u \equiv 0$ or u has at least 4 nodal domains. From Proposition 1 we conclude that both options are impossible and this contradiction proves the theorem. □

The proof of the above theorem gives without any change the following more general facts.

THEOREM 5. (i) *If the ambient space is the sphere* $M = \mathbf{S}^3$ *with a* $O(3)$*-invariant metric, then each connected component of an isoperimetric surface is a (topological) sphere of revolution.*

(ii) *If* M *is a Euclidean ball with a radial metric, i. e. a* $O(3)$*-invariant metric, then the components of any isoperimetric surface are either spheres or discs of revolution.*

In the first case $O(3)$ is viewed as the linear subgroup $1 \times O(3) \subset O(4)$. By a surface of revolution we mean a surface which is $O(2)$-invariant.

5. Nonconstant curvature

In this section M will be a complete 3-dimensional Riemannian manifold without boundary. We will denote its Ricci curvature by Ric. When the ambient space M has nonconstant curvature, we have the following basic results.

THEOREM 6. (Kleiner [**27**]) *Assume* M *is complete, simply-connected and has nonpositive sectional curvature. Let* $\Omega \subset M$ *be a domain and* $B \subset \mathbb{R}^3$ *an Euclidean ball. If* $V(\Omega) = \mathrm{V}(B)$ *then* $\mathrm{A}(\partial\Omega) \geq \mathrm{A}(\partial B)$*. The equality holds if and only if* $\Omega = B$.

THEOREM 7. (Lévy [**28**], Gromov [**18**]) *Assume* M *is compact and has Ricci curvature* $Ric \geq 2$*. Let* $\Omega \subset M$ *be a domain and* $B \subset \mathbf{S}^3$ *a metric ball. Then*

$$\frac{\mathrm{V}(\Omega)}{\mathrm{V}(B)} = \frac{\mathrm{V}(M)}{\mathrm{V}(\mathbf{S}^3)} \Rightarrow \frac{\mathrm{A}(\partial\Omega)}{\mathrm{A}(\partial B)} \geq \frac{\mathrm{V}(M)}{\mathrm{V}(\mathbf{S}^3)},$$

and the equality implies that $M = \mathbf{S}^3$ *and* $\Omega = B$.

Kleiner's theorem says that the Euclidean isoperimetric inequality extends to Cartan-Hadamard manifolds. The result does not hold for nonsimply-connected ambient spaces. This is a general fact in our setting: most results are sensitive to changes on the fundamental group. In Theorem 7, if the fundamental group of M were larger, then $\mathrm{V}(M)$ would be smaller and so the isoperimetric inequality would be weaker. An improved version of Theorem 7 can be found in [**12**]. Theorem 7 extends to higher dimension but the corresponding version of Theorem 6 remains open.

In the positive curvature case we can control the topology of the isoperimetric surfaces. The result below is based on an idea of Hersch [**22**] which has been extended and used in several related contexts first by Yang and Yau [**48**] and then by many people, see [**15, 29, 30, 31, 38, 39, 41, 47**] and in particular [**35**].

THEOREM 8. *Let Σ be an isoperimetric surface in a compact orientable 3-manifold M with $Ric \geq 2$. Then Σ is connected and genus$(\Sigma) \leq 3$. Moreover, in the cases genus$(\Sigma) = 2$ or 3 we have that $(1 + H^2)A(\Sigma) \leq 2\pi$.*

If M is closed and has positive Ricci curvature, then its isoperimetric profile I_M must be concave, see [**11**]. Therefore, from the symmetry of I_M, we get that the maximum value of the isoperimetric profile is given by $I_M(\mathrm{V}(M)/2)$. Now we show that this maximum gives a lower bound for certain total curvature on closed surfaces Σ bounding a domain in M.

PROPOSITION 2. *If $Ric \geq 2$ on M, then any closed surface $\Sigma \subset M$ which separates the ambient space verifies*

$$\int_\Sigma (1 + H^2) dA \geq \max I_M.$$

If the equality holds, then either Σ is a minimal surface which separates M in two pieces with the same volume or Σ is a round sphere embedded umbilically in M.

PROOF. Let Ω be one of the regions enclosed by Ω and assume $\mathrm{V}(\Omega) \leq \mathrm{V}(M)/2$. Consider $t \geq 0$ such that $\mathrm{V}(\Omega_t) = \mathrm{V}(M)/2$, where $\Omega_t = \{p \in M \mid \mathrm{dist}(p, \Omega) \leq t\}$. If we parametrize M by normal geodesics leaving Σ, then the Riemannian measure on M can be writen as $dV = g(p,t) dA dt$, where dA is the canonical measure on Σ and $g : \Sigma \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is the Jacobian of the normal exponential map. Define $C(t) = \{p \in \Sigma \mid \text{cut value of } p \ \geq t\}$, that is $p \in C(t)$ iff the geodesic γ leaving p in the direction $N(p)$ minimizes length between Σ and $\gamma(t)$. Given $p \in C(t)$, the Jacobian can be estimated as $g(p,t) \leq (\cos t - H \sin t)^2 \leq 1 + H^2$, where the first inequality follows from Heintze and Karcher [**21**] and the second one follows from Schwarz inequality. Therefore

$$\max I_M = I_M(\mathrm{V}(M)/2) \leq \mathrm{A}(\partial \Omega_t) \leq \int_{C(t)} g(p,t) \leq \int_\Sigma (1 + H^2).$$

Suppose the equality holds. If $t = 0$, it follows directly that $\mathrm{V}(\Omega) = \mathrm{V}(M)/2$ and $H \equiv 0$ (note that $g(p,0) = 1$). If $t > 0$, then the conclusion follows from [**21**]. □

THEOREM 9. *Let M be a compact 3-manifold with Ricci curvature $Ric \geq 2$ and volume $\mathrm{V}(M) \geq \mathrm{V}(\mathbf{S}^3)/2$. Then, any isoperimetric domain of M is bounded by either a sphere or a torus.*

PROOF. The hypothesis on the volume of M implies that M is a topological 3-sphere or a 3-dimensional projective space. Thus M is orientable. Assume that M is not a round sphere (if $M = \mathbf{S}^3$ the statement is clear). Let Σ be an isoperimetric surface of M and H its mean curvature. From Theorem 8 we know that Σ is a connected orientable surface of genus at most 3. Proposition 2 gives

$$(1 + H^2)\mathrm{A}(\Sigma) \geq I_M(\mathrm{V}(M)/2).$$

Using the volume assumption plus the fact that M is not a metric sphere, we get from Theorem 7 that

$$I_M\left(\frac{\mathrm{V}(M)}{2}\right) > \ 4\pi \frac{\mathrm{V}(M)}{\mathrm{V}(\mathbf{S}^3)} \geq 2\pi.$$

Therefore $(1 + H^2)\mathrm{A}(\Sigma) > 2\pi$ and Theorem 8 implies that genus$(\Sigma) \leq 1$. □

If the ambient space M has nonnegative curvature and the simplest topology, then it is natural to expect that the solutions of our problem will be also as simple as possible. The following conjecture gives two precise versions of this idea.

Conjecture. *Let Σ be an isoperimetric surface in M.*

(i) *If $M \subset \mathbb{R}^3$ is a strictly convex domain, then Σ is a topological disk.*

(ii) *If M is a 3-sphere with a metric of positive Ricci curvature, then Σ is a topological sphere.*

6. Space-forms

Assume now that the ambient space is an orientable 3-dimensional space form $\overline{M}(c)$, with constant sectional curvature c. Only a few results are known about the isoperimetric problem in a nonsimply-connected 3-space-form. The correct idea at this point is that when the fundamental group becomes larger, then the isoperimetric problem gets harder. The first restriction we have for general space-forms is the following.

THEOREM 10. *Let Σ be an isoperimetric surface of $\overline{M}(c)$.*

(i) *If Σ has genus zero, then Σ is an umbilical sphere.*

(ii) *If* $\text{genus}(\Sigma) = 1$, *then Σ is flat.*

In the first case, Σ lifts to the simply-connected covering of $\overline{M}(c)$ and the result follows from Hopf uniqueness theorem for constant mean curvature spheres, [**23**]. Assertion (ii) is proved in [**35**] by using a stability argument.

Now we give the complete solution of the isoperimetric problem on the real projective space $\mathbf{P}^3 = \mathbf{S}^3/\{\pm\}$ obtained as a metric quotient of the unit 3-sphere under the antipodal map. The result was first proved by Ritoré and Ros [**35**]. The proof here uses Theorem 9 and differs somewhat from [**35**]. At the present, $\mathbf{S}^3$ and $\mathbf{P}^3$ are the only elliptic 3-space forms where the problem has been solved. The problem is also open for higher dimensional projective spaces. We remark that in the projective space symmetrization does not work. We must solve the problem by different arguments.

THEOREM 11. ([**35**]). *Isoperimetric surfaces in $\mathbf{P}^3$ are either geodesic spheres or tubes around a line.*

PROOF. Let $\Sigma \subset \mathbf{P}^3$ be an isoperimetric surface. As $\mathrm{V}(\mathbf{P}^3) = \mathrm{V}(\mathbf{S}^3)/2$, Theorem 9 gives that $\text{genus}(\Sigma) = 0$ or 1. Then Theorem 10 implies that Σ is an umbilical sphere or a flat torus. The explicit classification of this kind of surfaces in $\mathbf{P}^3$ is an elementary and well-known fact and this concludes the proof. □

Direct computations give that for small (resp. large) prescribed volume, the isoperimetric domain in $\mathbf{P}^3$ is a geodesic ball (resp. the complementary of a geodesic ball). If the prescribed volume is near $\mathrm{V}(\mathbf{P}^3)/2$, then the isoperimetric solution is a tubular neighborhood of a geodesic line. In particular, if we consider just one half of the volume of $\mathbf{P}^3$, the solution is the domain enclosed by the quotient of the Clifford torus (i.e. a tube of radius $\pi/4$).

COROLLARY 1. *If $\Sigma \subset \mathbf{S}^3$ is a compact surface which divides the 3-sphere in two antipodally invariant pieces of equal volume, then $A(\Sigma) \geq 2\pi^2$. The equality holds if and only if Σ is the minimal Clifford torus.*

If M is a complete flat space with cyclic infinite fundamental group, then Ritoré and Ros [**35**] proved that any isoperimetric surface in M is either an umbilical sphere or a flat cylinder. The problem remains open for other quotients of $\mathbb{R}^3$. If $M = \mathbf{S}^1(r_1) \times \mathbf{S}^1(r_2) \times \mathbf{S}^1(r_3)$ is a rectangular flat 3-torus, then a symmetrization argument gives that the isoperimetric problem in M is equivalent to the isoperimetric problem in a box of $\mathbb{R}^3$. In that case we know the following.

THEOREM 12. *Let $M = [0,1] \times [0,a] \times [0,b] \subset \mathbb{R}^3$ be a box with $1 \le a \le b$. Then the rectangle $M \cap \{z = b/2\}$ has least area among surfaces which divide the box in two equal volumes.*

This result was proved by Hadwiger [**20**] for the case of the cube and extended to boxes by Schnell [**43**]. A related general property for the isoperimetric problem in a product space has been obtained recently by Barthe and Maurey [**9**], [**10**]. The natural conjecture is that isoperimetric surfaces in a box must be pieces of spheres, cylinders or planes. This conjecture is known to be true for boxes of the type $M = [0,1] \times [0,a] \times [0,\infty[$, with $a \ge 1$ large enough, see [**36**]. Ross [**41**] has shown that the classical Schwarz P minimal surface is stable in the cubic 3-torus of volume 1. So in this space we have a local minimum of the isoperimetric problem which is not a global one: Schwarz surface divides the 3-torus in two equal pieces and, thus, theorem 12 implies that its area is bigger than 2.

7. The Willmore conjecture

Now we want to study the functional $\int_\Sigma H^2 dA$ on closed surfaces Σ immersed in $\mathbb{R}^3$, where H denotes the mean curvature of the immersion. This functional is the simplest global invariant under conformal transformations of the space. In this sense, it can be viewed as the *area* in the conformal geometry setting. The functional takes its minimum at the round sphere and a natural problem is to try to understand its minima when we restrict to surfaces with fixed topology or which belong to a given isotopy class or under other kind of topological restrictions. From this point of view even the simple next situation is still open: what is the minimum of the functional on tori? This problem was first proposed by Willmore [**46**], who conjectured that the minimum is given by certain anchor ring Σ_0, and today we have a lot of partial results about it, see [**37**] and references therein. I will only mention that the conjecture is known to be true for non embedded tori and for embedded tori outside the standard isotopy class.

As the functional is compatible with conformal geometry, the problem can be stated as a problem in the 3-sphere: if Σ is viewed, via the stereographic projection, as a surface in the unit sphere $\mathbf{S}^3$, then the functional above transforms into

$$\int_\Sigma (1 + H^2) dA,$$

where now H denotes the mean curvature of Σ in $\mathbf{S}^3$. Thus Proposition 2 gives us a relation between the isoperimetric problem and the Willmore problem. For instance, in the 3-sphere we know that the maximum of the isoperimetric profile is 4π, which gives a well-known universal bound for the Willmore functional, see [**46**].

In the same way, Corollary 1 implies the following result.

THEOREM 13. (Ros [**37**]) *For any compact surface $\Sigma \subset \mathbf{S}^3$ of odd genus which is antipodal invariant, we have $\int_\Sigma (1 + H^2) dA \ge 2\pi^2$ and the equality holds if and only if Σ is the minimal Clifford torus.*

Proof. As the genus of Σ is odd, it follows from [**37**] that the antipodal map preserves the components of $\mathbf{S}^3 - \Sigma$ and, therefore, the quotient surface $\Sigma' = \Sigma/\pm$ separates the projective space $\mathbf{P}^3$. Using Proposition 2 and Corollary 1 we obtain

$$\int_\Sigma (1+H^2)dA = 2\int_{\Sigma'}(1+H^2)dA \geq 2\pi^2$$

and the equality holds just for the Clifford torus. □

Recently Topping [**45**] has given another proof of Theorem 13 by using an integral geometric approach. For another recent result see Ammann [**1**]. Finally I will give a proof of the Willmore conjecture in another case which is more natural from the Euclidean point of view. We show that the result is true for any torus in $\mathbb{R}^3$ with a center of symmetry.

Theorem 14. *Let $\Sigma \subset \mathbb{R}^3$ be a compact surface of odd genus which is symmetric with respect to a point. Then*

$$\int_\Sigma H^2 dA \geq 2\pi^2$$

and the equality holds if and only if Σ is homothetic to the anchor ring Σ_0.

Proof. Let s be the symmetry $s(x_1,x_2,x_3,x_4) = (-x_1,-x_2,-x_3,x_4)$ on $\mathbf{S}^3$. The fix points of s are $\pm p$, where $p = (0,0,0,1)$. Using the conformal nature of the Willmore functional, the result we want is equivalent to the following one.

Claim. *If $\Sigma \subset \mathbf{S}^3$ is a compact surface with odd genus and $s(\Sigma) = \Sigma$, then*

$$\int_\Sigma (1+H^2)dA \geq 2\pi^2$$

and the equality holds if and only if Σ is ambiently conformal to the minimal Clifford torus.

Now we prove the claim. Lemma 1 below gives that the points $\pm p$ lie in the same components of $\mathbf{S}^3 - \Sigma$ (in particular s preserves these components). Denote by Ω and Ω' the two domains enclosed by Σ and suppose that $\pm p \in \Omega'$.

Given a smooth domain $W \subset \mathbf{S}^3$ and $0 \leq t$, we consider the enlarged domain $W_t = \{x \in \mathbf{S}^3 : \mathrm{dist}(x,W) \leq t\}$. For $t < 0$, we define W_t as the points in W at distance at least t from the boundary, $W_t = \{x \in W : \mathrm{dist}(x,\partial W) \geq |t|\}$. In particular, there exist constants a and b, $a < 0 < b$, such that $\mathrm{V}(W_a) = 0$, $\mathrm{V}(W_b) = \mathrm{V}(\mathbf{S}^3)$ and $\mathrm{V}(W_t)$ is a strictly increasing continuous function, for $a \leq t \leq b$.

Let $\{f_\lambda : 0 < \lambda < \infty\}$ be the 1-parameter group of conformal transformations of $\mathbf{S}^3$ which corresponds, via stereographic projection centered at p, to the homotheties centered at the origin of $\mathbb{R}^3$. Each f_λ fixes the points $\pm p$ and verifies $f_\lambda \circ s = s \circ f_\lambda$. Moreover f_1 is the identity map. For any λ, $0 < \lambda < \infty$, we can construct a region $\Omega(\lambda) = (f_\lambda(\Omega))_t$, where t is uniquely determined by the condition $\mathrm{V}(\Omega(\lambda)) = \mathrm{V}(\mathbf{S}^3)/2$. Note that $s(\Omega(\lambda)) = \Omega(\lambda)$.

When λ is very small, the domain $f_\lambda(\Omega)$ is close to the point $-p$ and so the enlarged domain $\Omega(\lambda)$ almost coincides with the halfsphere centered at that point, see Figure 5. The same holds when λ is very big: in this case $\Omega(\lambda)$ is close to the halfsphere centered at the point p. Clearly our construction is continuous enough to conclude that, for certain intermediate value λ_0, the hyperplane $x_4 = 0$ cuts the region $\Omega(\lambda_0)$ in 2 pieces with the same volume.

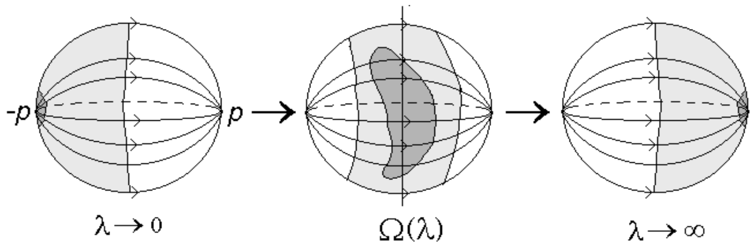

FIGURE 5

Consider the domain Ω^* obtained from $\Omega(\lambda_0)$ by symmetrization with respect to $x_4 = 0$: if the area of $\partial\Omega(\lambda_0)$ in $x_4 > 0$ is not greather than the one in $x_4 < 0$, take $\Omega^* = W \cup s'(W)$, where $W = \Omega(\lambda_0) \cap \{x_4 \geq 0\}$ and $s' = -s$ is the symmetry with respect to $x_4 = 0$. Note that Ω^* is invariant under s and s'. In particular $-\Omega^* = \Omega^*$, $\mathrm{V}(\Omega^*) = \mathrm{V}(\mathbf{S}^3)/2$ and $\mathrm{A}(\partial\Omega(\lambda_0)) \geq \mathrm{A}(\partial\Omega^*)$. As the Willmore functional is preserved by the transformations f_λ , using the argument in the proof of Proposition 2 and the isoperimetric inequality in Corollary 1, we get

$$\int_\Sigma (1+H^2) = \int_{f_{\lambda_0}(\Sigma)} (1+H_{\lambda_0}^2) \geq \mathrm{A}(\partial\Omega(\lambda_0)) \geq \mathrm{A}(\partial\Omega^*) \geq 2\pi^2,$$

where H_{λ_0} denotes the mean curvature of $f_{\lambda_0}(\Sigma)$. If the equality holds, we conclude easily, using again Proposition 2 and Corollary 1, that $f_{\lambda_0}(\Sigma)$ is the minimal Clifford torus. □

LEMMA 1. *Let $\Sigma \subset \mathbf{S}^3$ be a compact surface of odd genus and such that $s(\Sigma) = \Sigma$. Then the points $\pm p$ lie in the same connected component of $\mathbf{S}^3 - \Sigma$.*

PROOF. Denote by Ω and Ω' the components of $\mathbf{S}^3 - \Sigma$. If Σ contains some of the points $\pm p$, we conclude by looking at a neighborhood of this point that $s(\Omega) = \Omega'$ and therefore, both points must lie on Σ. As s reverses both the orientation on $\mathbf{S}^3$ and the normal vector on Σ, we have that the quotient surface $\Sigma' = \Sigma/\{1, s\}$ is orientable. Denote by g and g' the genus of the surfaces Σ and Σ' respectively. The quotient map $\Sigma \to \Sigma'$ has degree 2 and just 2 single branch points. So, Riemann-Hurwitz formula gives that $g = 2g'$ and this contradiction proves that $\Sigma \cap \{\pm p\} = \emptyset$.

Consider now a 2-dimensional equator S of $\mathbf{S}^3$ passing through the points $\pm p$ and meeting Σ transversally (after a small perturbation of Σ if necessary), and denote by S^+ and S^- the components of $\mathbf{S}^3 - S$. Note that $s(S) = S$ and $s(S^+) = S^-$. As Σ is obtained by gluing along their boundaries two copies of $\Sigma^+ = \Sigma \cap S^+$, we get that the Euler characteristic of these surfaces are related by the formula $\chi(\Sigma) = 2\chi(\Sigma^+)$ and thus $2 - 2g = 2(2m - r)$, where m is a certain integer number and r is the number of components of $\Gamma = \partial\Sigma^+ = \Sigma \cap S$. By assumption g is odd and so, the relation above implies that r is even.

Finally we study $\Gamma \subset S$, which consists of a disjoint union of r Jordan curves and verifies $s(\Gamma) = \Gamma$. Consider $\Gamma' \subset \Gamma$ (resp. $\Gamma'' \subset \Gamma$) the union of the Jordan curves of Γ which separate (resp. do not separate) the points $\pm p$. Clearly s preserves Γ' and Γ'' and transforms any Jordan curve $C \subset \Gamma''$ into a curve $s(C) \subset \Gamma''$ with $s(C) \neq C$. Hence Γ'' consists of an even number of closed curves and this implies that the number of components of Γ' is even too. Then we conclude that any arc in S transversal to Γ and joining the points $\pm p$ cuts Γ at an even number of times. Therefore, the points $\pm p$ lie in the same component of $\mathbf{S}^3 - \Sigma$, as we claimed.

□

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