

RECENT ADVANCES IN ISOPERIMETRIC PROBLEMS

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The origin of isoperimetric problems is lost in the beginning of the history of Mathematics. We know that Greek mathematicians treated the isoperimetric properties of the circle and the sphere, the last of which can be formulated in two equivalent ways: (i) among all bodies of the same volume, the round ball has the least boundary area, (ii) among all surfaces of the same area, the round sphere encloses the largest volume.

The first proof of the isoperimetric property of the circle is due to Zenodorus, who wrote a lost treatise on isoperimetric figures, known through the fifth book of the *Mathematical Collection* by Pappus of Alexandria [13]. Zenodorus proved that among polygons enclosing a given area, the regular ones have the least possible length. This property implies the isoperimetric property of the circle by an standard approximation argument. Since then many proofs have been given, some of them incomplete, although employing interesting and fruitful ideas. Without even trying to be exhaustive, the list of mathematicians that have considered these problems includes Euler, the Bernoullis, Gauss, Steiner, Weierstrass, Schwarz, Levy, Schmidt, and many others.

Nowadays by an isoperimetric problem we mean one in a wide class in which we try to find a perimeter-minimizing surface (or hypersurface) under one or more volume constraints and with possibly additional boundary and symmetry conditions. Thanks to the development of Geometric Measure Theory in the past century (see, for instance, the text [15] and the references therein) we have existence and regularity results for most of the “natural” isoperimetric problems we can think about. Of course by regularity we mean that the solution of the problem is either a smooth surface, or that has well-understood singularities, as in the double bubble problem. Despite these extraordinary developments, only a few results characterizing the isoperimetric solutions in a given ambient manifold are known.

In these notes we will describe how to seek the solutions of some isoperimetric problems in the Euclidean space \mathbb{R}^3 , including the double bubble problem. For other ambient manifolds such as n -dimensional spheres or hyperbolic spaces we refer to the reader to Burago and Zalgaller’s treatise [4] on geometric inequalities, where an extensive bibliography can be found. We will not treat either of some recent interesting advances in the study of isoperimetric domains in surfaces.

1. THE CLASSICAL ISOPERIMETRIC PROBLEM IN \mathbb{R}^3

We wish to find, among the surfaces in \mathbb{R}^3 enclosing a fixed volume $V > 0$, the ones with the least area. From general results of Geometric Measure Theory [15], this problem has at least a smooth compact solution. Moreover, from variation formulae for area and volume, the mean curvature of such a surface must be constant. The mean curvature at some point of the surface is the arithmetic mean of the principal curvatures, which indicates how the surface is bent in space. It is not difficult to show, from the second variation formula for the area while

keeping constant the volume enclosed, that the solution surface (and so the enclosed domain) has to be connected.

There are several ways to prove that the sphere is the only solution to this problem. Perhaps the most geometrical ones are the various symmetrization methods due to Steiner and Schwarz [4] and Hsiang [14]. Let us explain briefly their arguments. Consider an isoperimetric body Ω .

Steiner's method applies to the family of lines orthogonal to a given plane P and, for every line L in this family, replaces $L \cap \Omega$ by the segment in L centered at $P \cap L$ of the same length. This procedure yields another body Ω' with the same volume as that of Ω , and strictly less boundary area unless the original body Ω were symmetric with respect to a plane parallel to P . This implies that Ω is symmetric with respect to a plane parallel to P .

Schwarz considers a family of planes parallel to a given line L . For every plane P orthogonal to L , the intersection $P \cap \Omega$ is replaced by the disc in P centered at $P \cap L$ of the same area. Again a new body Ω' is obtained with the same volume as that of Ω , and least boundary area unless Ω were rotationally symmetric with respect to a line parallel to L . In a similar way, one can use a family of concentric spheres instead of parallel planes to obtain a new symmetrization known as *spherical symmetrization*.

A third symmetrization was used by Hsiang. He considered a plane P dividing Ω into two equal volume parts Ω^+ and Ω^- . Assuming that $\text{area}(\Omega^+) \leq \text{area}(\Omega^-)$, he took the domains $\Omega = \Omega^+ \cup \Omega^-$ and $\Omega' = \Omega^+ \cup r(\Omega^+)$, where r is the reflection with respect to P . Then Ω' is also an isoperimetric domain, from which we conclude that $\text{area}(\Omega^+) = \text{area}(\Omega^-)$. We also have by regularity that $\partial\Omega$ and $\partial\Omega'$ are constant mean curvature surfaces and, by construction, they coincide in $\partial\Omega^+$. By general properties of constant mean curvature surfaces we conclude $\partial\Omega = \partial\Omega'$, and so $\Omega = \Omega'$, from which we deduce that Ω was symmetric with respect to P .

By applying Steiner or Hsiang symmetrization it follows that Ω is symmetric with respect to a plane parallel to any given one. By Schwarz symmetrization, that Ω is symmetric with respect to a line parallel to an arbitrary one. It is not difficult to see from these properties (and the compactness and connectedness of $\partial\Omega$) that $\partial\Omega$ must be a sphere.

Hence a symmetrization method suffices to characterize the isoperimetric domains in Euclidean space \mathbb{R}^3 due to the large group of isometries of this space. As we will see in the next sections, this is not enough to characterize the isoperimetric domains in some other situations.

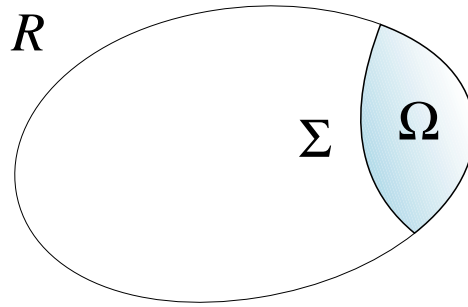
Perhaps it is worth noting that there is also a symmetrization method for embedded constant mean curvature surfaces, known as the Alexandrov reflection method [22], which shows that such a surface embedded in \mathbb{R}^3 is symmetric with respect to a plane parallel to a given one, and hence it has to be a sphere.

2. SOME OTHER ISOPERIMETRIC PROBLEMS IN EUCLIDEAN SPACE

We consider in this section a modified version of the classical isoperimetric problem in \mathbb{R}^3 . For a regular region $R \subset \mathbb{R}^3$ and for $V \leq \text{vol} R$ we want to find a least area surface $\Sigma \subset R$ *separating* a region $\Omega \subset R$ of volume V . The considered surfaces can have boundary, which is contained in the boundary of R . The region Ω is bounded by Σ and eventually by a piece of ∂R .

This problem is often referred to as a *free boundary* problem (with a volume constraint). We strongly remark that the area of $\partial\Omega \cap \partial R$ is not considered in this problem.

Geometric Measure Theory [15] ensures the existence of the solution Σ at least for compact R , and its regularity, at least in low dimensions. Moreover any solution Σ has constant mean curvature and meets the boundary of R at $\partial\Sigma$ orthogonally.

FIGURE 1. Isoperimetric domains in a region R

When R is strictly convex the surface Σ is connected, and bounds on their genus and on the number of components of $\partial\Sigma$ are known [21]. For a given R it is certainly difficult to characterize the isoperimetric solutions, but the following conjecture is probably true

Conjecture. *Any solution to the isoperimetric problem in an strictly convex region is homeomorphic to a disc.*

Let us now consider some other choices of the region R .

2.1. The isoperimetric problem in a halfspace. Let us assume that R is the halfspace $z \geq 0$. We will find the surfaces Σ that separate a region $\Omega \subset R$ of fixed volume with the least perimeter. Since R is noncompact one could have problems in proving existence of isoperimetric domains as a minimizing sequence could diverge, but this is solved by using translations. So we have existence and also regularity, which is a local matter. In this case we have

Theorem. *Isoperimetric domains in the halfspace $z \geq 0$ are halfballs centered at the plane $z = 0$.*

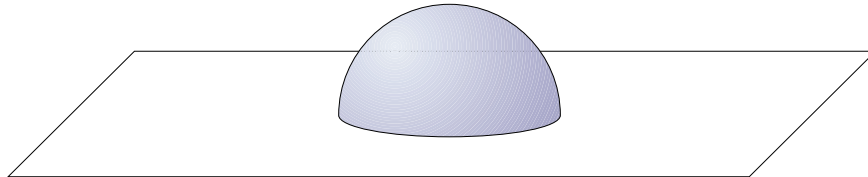


FIGURE 2. Isoperimetric domains in a half plane

For the proof of this theorem we first observe that the isoperimetric region Ω must touch the plane $z = 0$. Otherwise moving Ω until it becomes tangent to the plane $z = 0$ we get an isoperimetric region such that $\Sigma = \partial\Omega$ touches $z = 0$, but neither at $\partial\Sigma$ nor orthogonally. This proves in general that an isoperimetric solution must touch the boundary of the domain.

We also have that Ω is connected: otherwise we could move two components of Ω until they touch, producing a singularity in the boundary. We now apply Hsiang symmetrization, but only for the planes which are orthogonal to $z = 0$, to conclude that Ω is rotationally symmetric with respect to a line L orthogonal to $z = 0$.

Hence Σ is obtained by rotating a plane curve to get a constant mean curvature surface. It turns out that there are only a few types of curves that produce, when rotated, a constant

mean curvature surface. They were studied by Ch. Delaunay in 1841 [6] and they are the ones depicted in Figure 3. Since our curve touches the line of revolution (it has a maximum of the z -coordinate), looking at the list, we conclude that it is part of a circle and so Σ is a halfsphere.

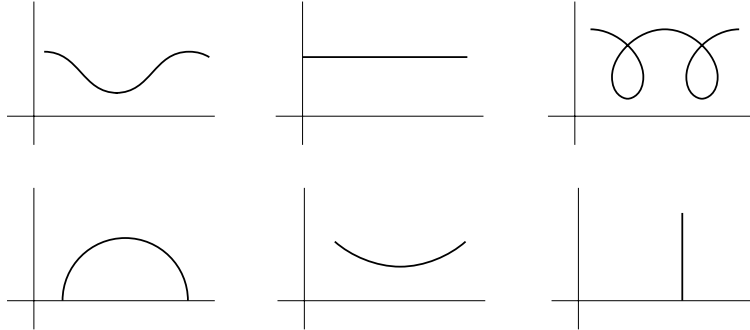


FIGURE 3. Generating curves of surfaces of revolution with constant mean curvature. The horizontal line is the axis of revolution. From left to right and above to below, the generated constant mean curvature surfaces are unduloids, cylinders, nodoids, spheres, catenoids and planes orthogonal to the axis of revolution.

2.2. The isoperimetric problem in a ball. Let us now assume that R is a ball. Spherical symmetrization proves that an isoperimetric surface Σ is of revolution around some line L containing the center of the ball. As shown in Figure 4, there are surfaces of this kind which are not spheres. What we know is that Σ is a piece of a sphere or a flat disc if Σ touches L .

Let us prove

Theorem ([21]). *Isoperimetric domains in a ball are the ones bounded by a flat disc passing through the center of the ball or by spherical caps meeting orthogonally the boundary of the ball.*

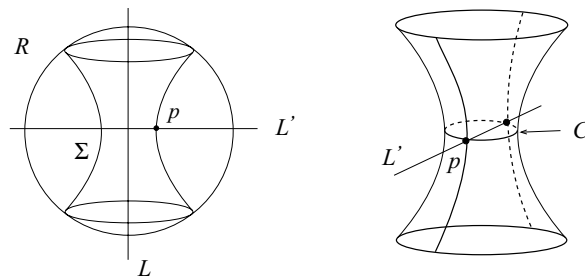


FIGURE 4. There are candidates to isoperimetric domains in a ball which are not spheres nor flat discs. The last two pictures illustrate the argument showing that isoperimetric domains must touch the axis of revolution

To prove the theorem, assume that Σ is neither a piece of a sphere nor a flat disc, so that Σ does not touch L . Choose $p \in \Sigma$ at minimum distance from L . Consider the Killing field X of rotations around the axis L' orthogonal to L passing through p . The set C of points

of Σ where X is tangent to Σ can be shown to consist on a finite set of closed curves. This set includes $\partial\Sigma$ and the intersection of the plane $\langle L, L' \rangle$, generated by L and L' , with Σ . By the special properties of the field X there is another curve in C passing through p apart from $\langle L, L' \rangle \cap \Sigma$. We conclude that $\Sigma - C$ has at least four connected components.

But this is enough to show that Σ cannot be an isoperimetric surface by using Courant's Nodal Domain Theorem [5]. Perhaps the most intuitive argument to convince the reader is that we can rotate (at least infinitesimally) slightly two of these components to get a nonsmooth surface which encloses the same volume and have the same area of Σ . The new surface should be also isoperimetric, which is a contradiction since it is not regular.

Observe that the isoperimetric domains in a ball are never symmetric with respect to the center of the ball. We may complicate the problem by imposing this symmetry. The following problem is still open.

Problem . *Find the least area surfaces in a ball separating a fixed volume, which are symmetric with respect to the center of the ball.*

2.3. The isoperimetric problem in a box. The convex region R given by $[a, a'] \times [b, b'] \times [c, c']$ will be called a *box*. For this region no symmetrization can be applied to the isoperimetric domains. The most reasonable conjecture for such a region is

Conjecture . *The surfaces bounding an isoperimetric domain in a box R are*

- (i) *an octant of a sphere centered at one vertex of R , or*
- (ii) *a quarter of a cylinder whose axis is one of the edges of R , or*
- (iii) *a piece of a plane parallel to some of the faces of R .*

The type of solution depends on the shape of the box R and on the values of the enclosed volume.

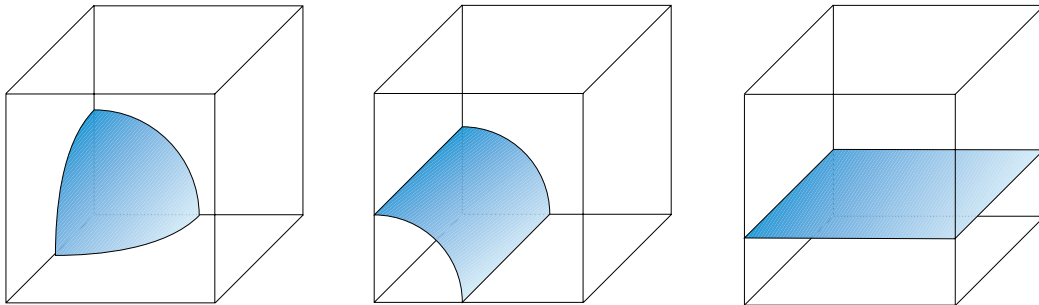


FIGURE 5. Probable solutions of the isoperimetric problem in a box

What is known at this moment? Some partial results, but not the complete answer. We know that the conjecture is true for a compact subset of the space of the boxes (modulo dilations) [20], [18]. Also that the candidates are constant mean curvature surfaces which are graphs over the three faces of the box. Apart from the ones stated in the above conjecture we have two families of constant mean curvature surfaces which could be isoperimetric solutions [19]. They are depicted in Figure 6. The right hand side family is a three-parameter one and includes a part of the classical Schwarz \mathcal{P} -minimal surface. This surface has been shown to be stable (nonnegative second variation of area enclosing a fixed volume) by M. Ross,

although it cannot be a solution of the isoperimetric problem by results of Hadwiger [8], see also Barthélemy [3]. The left hand side family is a two-parameter one. It is also known that the isoperimetric solution for half of the volume is a plane in the case of the cube.

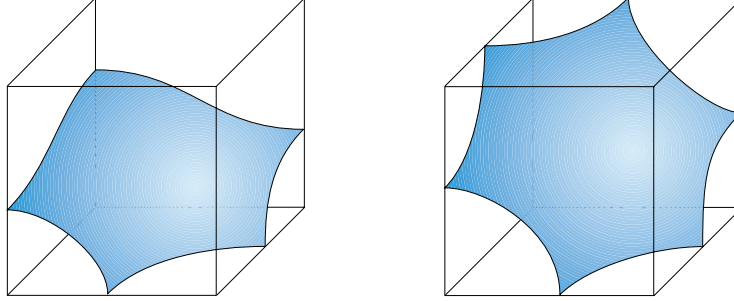


FIGURE 6. Candidates to be solutions of the isoperimetric problem in a box

2.4. The isoperimetric problem in a slab. Let us assume now that R is a slab bounded by two parallel planes P_1 and P_2 in \mathbb{R}^3 . Existence in this noncompact region is ensured by applying translations parallel to the planes P_i to any minimizing sequence. One can also apply symmetrization (with respect to planes orthogonal to P_i) to conclude that an isoperimetric solution is symmetric with respect to some line L orthogonal to P_i . Possible solutions in this case are half-spheres centered at some of the planes P_i , tubes and unduloids (see Figure 3). A careful analysis of the stability of the generating curves is required to discard unduloids to get

Theorem ([2], [24], [16]). *The surfaces bounding an isoperimetric domain in a slab in \mathbb{R}^3 are*

- (i) *half-spheres centered at some of the boundary planes, or*
- (ii) *tubes around a line orthogonal to the boundary planes.*

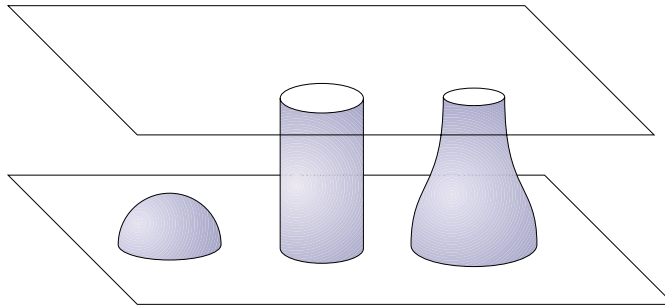


FIGURE 7. Isoperimetric problems in a slab. The one on the right is an unduloid, which appears in large dimensions

Perhaps it worths saying that this result is true in \mathbb{R}^{n+1} , for $n \leq 7$, but not for $n \geq 9$ (case $n = 8$ remains open). In high dimensions it can be showed the existence of unduloids which are solutions to the isoperimetric problem [16]. The argument is a simple comparison: for $n \geq 9$, a half-sphere with center in one of the boundary planes and tangent to the other cannot be an isoperimetric solution by regularity. But it has less perimeter than a tube of the

same volume. We conclude that there is an isoperimetric solution which is neither a sphere nor a tube. The only remaining possibility is an unduloid.

3. MULTIPLE BUBBLES

The standard double bubble is obtained by joining two spherical soap bubbles. It is composed of three spherical caps (one of which may degenerate to a flat disc) spanning the same circle. The caps meet along the circle in an equiangular way. The whole configuration is rotationally invariant around a line. Standard bubbles are candidates to be solutions of the following isoperimetric problem, known as the double bubble problem

Problem. *Amongst surfaces enclosing and separating two given volumes, find the ones with the least possible area.*

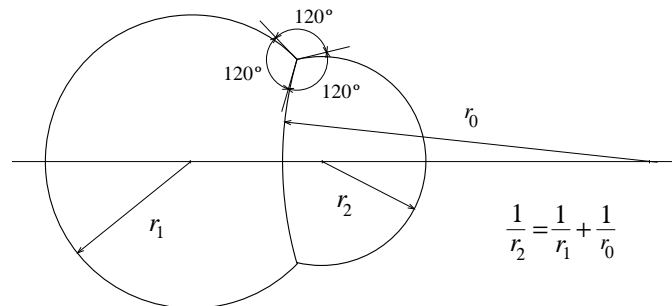


FIGURE 8. The standard double bubble

For existence we refer to Almgren's work [1], for regularity to Taylor [23], who showed that any solution consists on constant mean curvature sheets in such a way that (i) either three sheets meet along a curve at equal angles of 120 degrees, or (ii) four of these curves and six sheets meet at some point like the segments joining the barycenter of a regular tetrahedron with the vertexes (sheets correspond to the triangles determined by two of such segments and the edge of the tetrahedron needed to close the triangle). Natural candidates to be solutions of this isoperimetric problem are the standard double bubbles (there is precisely one for every pair of volumes), and in fact we have

Theorem. *The standard double bubble is the least perimeter way to enclose and separate two given volumes in \mathbb{R}^3 .*

This result was first proved by Hass and Schlafly [9] for the two equal volumes case, see also [9]. The general case was solved by Hutchings, Morgan, Ritoré and Ros [11] and announced in [12].

As in the previous examples, one tries to find some kind of symmetry in the problem. This was done by Foisy [7] and Hutchings [10] following an idea of Brian White: for up to three volumes in \mathbb{R}^3 Borsuk-Ulam's Theorem (more precisely one of its corollaries known as the ham sandwich theorem) shows that we can find a plane P_1 dividing each region of a solution Σ of the double bubble problem in two equal volume parts. Hutchings [10] proved that such a plane is a symmetry plane. A second application of Borsuk-Ulam's shows that there is another plane P_2 , orthogonal to P_1 , which divides each region again in two equal volume parts, and it is again a symmetry plane. But now it is easy to conclude that any plane which contains the line $L = P_1 \cap P_2$ divides each region of the bubble in two equal volume parts



FIGURE 9. Double bubbles. The one on the right was shown to be unstable, and hence it does not appear in nature. Pictures by John Sullivan, University of Illinois (<http://www.math.uiuc.edu/~jms>)

and so it is a plane of symmetry. We conclude that Σ is of revolution around the line L . So in fact we have some curves that rotated around a certain axis, give us the whole bubble. Since these curves generate constant mean curvature surfaces, they are the Delaunay curves in Figure 3.

Like in the previously discussed isoperimetric problems, symmetrization is not enough to classify the isoperimetric solutions. Using again Hutching's results and stability techniques we are able to reduce the number of candidates different of the standard double bubble to the possibilities depicted in Figure 10

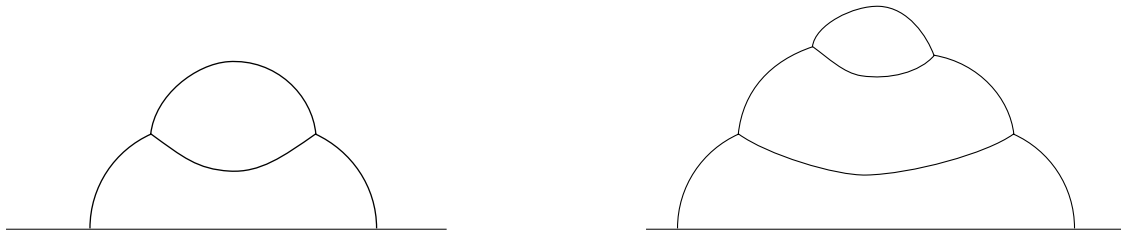


FIGURE 10. The horizontal line is the axis of revolution. When rotated the curves give the whole bubble. Each curve is a piece of a Delaunay one. The ones touching the axis are circles. When three curves meet, they do it at 120 degree angles. For equal volumes just the first configuration has to be considered. In the second configuration one of the regions is disconnected.

The final argument is again an stability one. By using rotations orthogonal to the axis of revolution of the double bubble we prove

Proposition. *Consider a configuration of curves that generate a solution of the double bubble problem by rotation. Assume there are points $\{p_1, \dots, p_n\}$ in the regular part of the curves so that the normal lines meet at some point p , possibly ∞ , in the axis of revolution.*

Then $\{p_1, \dots, p_n\}$ cannot separate the configuration.

We illustrate the power of this Proposition by easily discarding the first type of candidates. Pick the line L equidistant from a and b . Assume that this line meets the axis of revolution at some point p . In each one of the curves joining a and b there is at least a point at maximum

or minimum distance from p . Call them p_1 and p_2 . Then p_1 and p_2 separate the configuration so that the generated bubble cannot be a solution of the double bubble problem.

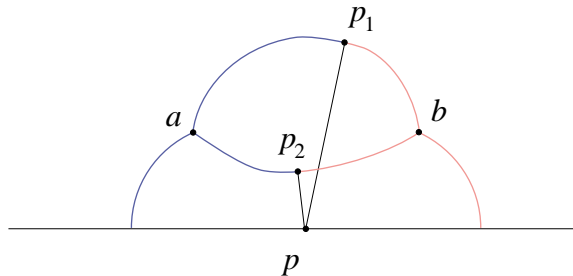


FIGURE 11. The partition method

In order to discard the second type of candidates some more work is needed, but it has already been done in [11].

Of course we can ask about the surfaces of least area which enclose and separate n regions in \mathbb{R}^3 . Existence and regularity follow from Almgren and Taylor results. For $n = 3, 4$ there are two natural candidates, see Figure 12, which we shall call again standard bubbles. For these volumes we also have the following

Conjecture. *The standard n -bubble, $n \leq 4$, is the least perimeter way to enclose and separate n given volumes in \mathbb{R}^3 .*

However the situation is extremely complicated when we consider $n > 4$ regions since in this case we even don't have an applicant to solve the problem.



FIGURE 12. A standard triple bubble. Picture by John Sullivan, University of Illinois (<http://www.math.uiuc.edu/~jms>)

Symmetrization works for double bubbles in Euclidean spaces of any dimension. It seems natural to hope that the standard double bubble be the least perimeter way to enclose and separate two given regions in \mathbb{R}^n , for any $n \geq 3$. In case $n = 4$ this has been proved, by using the arguments in [11], in [17].

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