

THE GAUSS MAP OF MINIMAL SURFACES

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We give a new approach to the study of relations between the Gauss map and compactness properties for families of minimal surfaces in the Euclidean three space. In particular, we give a simple and unified proof of the curvature estimates for stable minimal surfaces and for minimal surfaces whose Gauss map image omits five points.

The Gauss map of a minimal surface in the Euclidean space \mathbb{R}^3 is a conformal map. This fact has deep consequences in the behavior of these surfaces and has allowed a massive presence of complex variable techniques in the classical theory of minimal surfaces. In 1959 Osserman¹⁹ started a systematic study of this map, showing that the Gauss map of a non-flat complete minimal surface must be dense in the unit sphere \mathbf{S}^2 . He also proved²⁰ the following curvature estimate, which clearly implies the result above: There exists a positive constant $C > 0$ such that, for all minimal surfaces $\psi : \Sigma \rightarrow \mathbb{R}^3$ whose Gauss map omits a fixed geodesic disc in \mathbf{S}^2 and for all point $p \in \Sigma$, we have

$$|K(p)|d(p)^2 \leq C, \quad (1)$$

where K is the Gauss curvature of ψ and $d(p)$ is the geodesic distance from p to the boundary of Σ , that is, $d(p)$ is the infimum of the lengths of the divergent curves in Σ leaving from p .

Improving some intermediate results^{21,16} and specially the paper by Xavier³², Fujimoto⁶ proved that given five distinct points of the sphere, there is a positive constant C such that any minimal surface whose Gauss map omits these points satisfies the estimate (1). In particular, the only complete minimal surface satisfying this property is the plane. Since examples are known²² of complete minimal surfaces whose Gauss map omits precisely any arbitrarily prescribed set of \mathbf{S}^2 with less than five points, the result of Fujimoto is sharp.

Another interesting property of minimal surfaces which has been studied in detail is stability. Minimal surfaces are characterized as critical points of the area functional. A minimal surface is said to be *stable* if its second variation formula for the area is non negative, for any compactly supported infinitesimal deformation, see⁴ for more information. It is a remarkable fact

that stability depends only of the Gauss map of the surface. Some results, involving stability, which look like those obtained by Osserman and Fujimoto have been showed. Do Carmo and Peng¹, Fisher-Colbrie and Schoen⁴ and Pogorelov²⁷ have proved independently that the only complete stable minimal surface in \mathbb{R}^3 is the plane. Later Schoen³⁰ proved, the stronger result, that the stability assumption implies the estimate (1).

In this paper we develop an unified approach to the study of these two problems: the Gauss map image and the stability of minimal surfaces in \mathbb{R}^3 . We prove that, modulo general arguments, the curvature estimates for minimal surfaces which are either stable or whose Gauss map omits five points, follows from the uniqueness of the plane among complete surfaces in the above families. In particular it we give a new proof of the Schoen's curvature estimate for stable minimal surfaces in R^3 , see ³⁰. An unexpected consequence of our work is the fact that in order to prove the above uniqueness of the plane in the complete case, it is enough to show it for surfaces with bounded Gauss curvature. Using this property we simplify the proof of Fujimoto results in ⁶.

Our approach is inspired in a paper by Zalcman³⁴. He takes a similar point of view for the study of properties of meromorphic maps. In §1 we give Zalcman's result, adapted to our setting, and using it we obtain a new proof of Picard theorems. Another important remark is that all the results stated in §1 are true for harmonic maps from Riemann surfaces to compact n -dimensional Riemannian manifolds.

In this paper we will only work with orientable surfaces.

1 Compact properties for meromorphic maps

Let Σ be a Riemann surface, \mathbb{C} the finite plane, \mathbf{S}^2 the unit sphere in \mathbb{R}^3 and $D(r) = \{z \in \mathbb{C} : |z| < r\}$ the Euclidean disc of radius $r > 0$. The unit disc will be noted simply by D . Let $\mathcal{M}(\Sigma)$ be the set of meromorphic maps defined over Σ , that is the set of holomorphic maps from Σ to $\mathbf{S}^2 \subset \mathbb{R}^3$. Projecting stereographically from the north pole of the sphere we obtain an identification of \mathbf{S}^2 with $\overline{\mathbb{C}}$. Unless otherwise is stated, we will consider meromorphic maps as \mathbf{S}^2 -valued. If $F \in \mathcal{M}(\Sigma)$ and f is its associated $\overline{\mathbb{C}}$ -valued map, then we have the relation

$$F = \left(\frac{2f}{|f|^2 + 1}, \frac{|f|^2 - 1}{|f|^2 + 1} \right) \in \mathbf{S}^2 \subset \mathbb{C} \times \mathbb{R} = \mathbb{R}^3. \quad (2)$$

We consider on $\mathcal{M}(\Sigma)$ the topology of the uniform convergence on compact subsets, which is equivalent to the C^k -convergence on compact subsets

for any natural number k . If \mathcal{P} is an arbitrary property for meromorphic maps, we put

$$\mathcal{P}(\Sigma) = \{F \in \mathcal{M}(\Sigma) : F \text{ satisfies the property } \mathcal{P}\}.$$

Given a property \mathcal{P} , we consider the following assertions:

I) For any two Riemann surfaces Σ and Σ_1 and for any holomorphic map without branch points $\phi : \Sigma \rightarrow \Sigma_1$,

$$\text{if } F \in \mathcal{P}(\Sigma_1), \text{ then } F \circ \phi \in \mathcal{P}(\Sigma).$$

II) Let Σ be any Riemann surface and $F \in \mathcal{M}(\Sigma)$. If for any relatively compact domain Ω of Σ one has

$$F|_{\Omega} \in \mathcal{P}(\Omega), \text{ then } F \in \mathcal{P}(\Sigma).$$

III) For any Riemann surface, $\mathcal{P}(\Sigma)$ is a closed subset of $\mathcal{M}(\Sigma)$.

IV) $\mathcal{P}(\Sigma)$ is compact for any Riemann surface.

If \mathcal{P} satisfies the axioms I), II) and III) we will say that \mathcal{P} is a *closed property*, and if it satisfies I), II) and IV) we will say that it is a *compact property*.

The following lemmata are known or reformulations of known results, see for example ^{13,14}. We include proofs for sake of completeness. If $F \in \mathcal{M}(\Omega)$, where Ω is a domain of \mathbb{C} , we denote the length of the Euclidean gradient of F by $|\nabla F|_e$. If f is the \mathbb{C} -valued map associated to F , using (2) we have the relation

$$|\nabla F|_e^2 = \frac{8|f'|^2}{(1 + |f|^2)^2}.$$

Lemma 1 *Let $\mathcal{F} \subset \mathcal{M}(D)$ be a family of meromorphic maps defined on the unit disc. Then \mathcal{F} is relatively compact if and only if the family $\{|\nabla F|_e : F \in \mathcal{F}\}$ is uniformly bounded on compact subsets of D .*

Proof. As our topology is equivalent to the C^1 -convergence on compact subsets, if \mathcal{F} is relatively compact, then the gradients of the maps of \mathcal{F} are uniformly bounded on compact subsets of D . Conversely, if $\{|\nabla F|_e : F \in \mathcal{F}\}$ is bounded uniformly on compact subsets, then it follows from Ascoli-Arzelà theorem that \mathcal{F} is relatively compact. \square

If Σ is a Riemann surface, then, by the uniformization theorem, there is on Σ a complete metric compatible with its Riemann structure and constant

Gauss curvature equal to 1, 0 or -1 . We will call this metric the *canonical metric* of Σ , and we will denote it by ds_c^2 . We write Δ_c and dA_c to denote the Laplacian and the canonical measure of this metric. If $F \in \mathcal{M}(\Sigma)$, we denote by $|\nabla F|_c$ the length of the gradient of F with respect to ds_c^2 . In particular for the complex plane $ds_c^2 = |dz|^2$, and for the unit disc we have

$$ds_c^2 = \left(\frac{2}{1-|z|^2} \right)^2 |dz|^2 \quad |\nabla F|_c = \frac{1-|z|^2}{2} |\nabla F|_e, \quad (3)$$

for any $F \in \mathcal{M}(D)$.

Lemma 2 *Let \mathcal{P} be a closed property. Then the following assertions are equivalents:*

- i) \mathcal{P} is a compact property.
- ii) $\mathcal{P}(D)$ is compact.
- iii) The family $\{|\nabla F|_c : F \in \mathcal{P}(\Sigma)\}$ is uniformly bounded by a constant which is independent of the Riemann surface Σ .

Proof. The equivalence between i) and ii) follows from axiom I) and the fact that a family of meromorphic maps over Σ is relatively compact if and only if its restriction to a neighborhood of each point of Σ is relatively compact.

If \mathcal{P} satisfies iii), then we have from (3) that the family

$$\{|\nabla F|_e : F \in \mathcal{P}(D)\}$$

is uniformly bounded on compact subsets of D , and using Lemma 1 we conclude that $\mathcal{P}(D)$ is compact.

Suppose now that \mathcal{P} is compact. Passing to the universal covering and using I), it is enough to prove iii) for the canonical metrics of $\Sigma = D, \mathbb{C}$ or \mathbf{S}^2 . As the group of direct isometries of these metrics acts transitively on Σ in the three cases, the uniform bound follows from the punctual bound and the axiom I) applied to the above transformations group. \square

Lemma 3 *Let \mathcal{P} be a compact property. Then*

- i) $\mathcal{P}(\mathbb{C})$ contains only constant maps.
- ii) If $F \in \mathcal{P}(D - \{0\})$, then F has a meromorphic extension to the disc D .

Proof. i) Let $F \in \mathcal{P}(\mathbb{C})$ and $z \in \mathbb{C}$. Define the sequence $F_n : \mathbb{C} \rightarrow \mathbf{S}^2$, $n = 1, 2, \dots$ by $F_n(w) = F(z + nw)$, for any $w \in \mathbb{C}$. From I) we see that F_n lies in $\mathcal{P}(\mathbb{C})$. So, $|\nabla F_n|_e(0) = n|\nabla F|_e(z)$ is a bounded sequence, that is $|\nabla F|_e(z) = 0$ for any z . Then F is constant.

ii) The canonical metric on $D - \{0\}$ is given by

$$ds_c^2 = \left(\frac{1}{|z| \log |z|} \right)^2 |dz|^2.$$

Hence $D(1/2) - \{0\}$ has finite area with respect to this metric. Let $F \in \mathcal{P}(D - \{0\})$. From Lemma 2 there is a positive constant C , such that $|\nabla F|_c \leq C$ in $D - \{0\}$. So

$$\int_{D(\frac{1}{2}) - \{0\}} |\nabla F|_c^2 dx dy = \int_{D(\frac{1}{2}) - \{0\}} |\nabla F|_c^2 dA_c \leq \int_{D(\frac{1}{2}) - \{0\}} C^2 dA_c < \infty,$$

where the first equality follows from the conformal invariance of the Dirichlet integral. As F has finite energy in a neighborhood of the origin, then it has a meromorphic extension to the disc (reason for instance as follows: As the energy of F is also the area of its spherical image counted with multiplicity, for small $r > 0$, $F(D(r) - \{0\})$ cannot be all the sphere. So the associated \mathbb{C} -valued map, $f : D(r) - \{0\} \rightarrow \mathbb{C}$, can be assumed holomorphic, and we conclude using the normality of the sequence $f_n(z) = f(z/2^n)$, $n \in \mathbb{N}$, as in ¹⁴ p. 258). \square

Zalcman³⁴ theorem states that a closed property which satisfies $\mathcal{P}(\mathbb{C}) \subset \{\text{constant maps}\}$ must be compact. We will need this result in the following, more precise form:

Theorem 1 ³⁴. *Let \mathcal{P} be a closed property. Then we have the following alternative:*

- i) \mathcal{P} is a compact property, or
- ii) there exists $F \in \mathcal{P}(\mathbb{C})$ with $|\nabla F|_c(0) = 1$ and $|\nabla F|_c(z) \leq 1$ for any $z \in \mathbb{C}$.

Proof. Suppose that \mathcal{P} is not compact. Let ds_c^2 be the canonical metric on D . Using Lemma 2 we find sequences $\{G_n\} \subset \mathcal{P}(D)$ and $\{z_n\} \subset D$ such that $|\nabla G_n|_c(z_n) \rightarrow \infty$. By the homogeneity of the metric ds_c^2 , we can suppose that $z_n = 0$ for any n . From I) we have that the maps $H_n : D \rightarrow \mathbf{S}^2$, defined by $H_n(z) = G_n(z/2)$ for any z in D , are in $\mathcal{P}(D)$. Moreover $|\nabla H_n|_c(0) \rightarrow \infty$ and, using (3), $|\nabla H_n|_c = 0$ on ∂D for each $n \in \mathbb{N}$. So, composing if necessary with conformal transformations of D , we can assume that

$$\max_D |\nabla H_n|_c = |\nabla H_n|_c(0), \text{ for any } n.$$

Put $R_n = 2|\nabla H_n|_c(0)$. In terms of the Euclidean metric $|dz|^2$, the properties of the maps $\{H_n\}$ can be written as

$$|\nabla H_n|_c(0) = R_n \rightarrow \infty \quad \text{and} \quad |\nabla H_n|_c(z) \leq \frac{R_n}{1 - |z|^2},$$

for any $z \in D$ and $n \in \mathbb{N}$. We define a new sequence of meromorphic maps $F_n : D(R_n) \rightarrow \mathbf{S}^2$, by $F_n(z) = H_n(z/R_n)$ for each z in $D(R_n)$. Then F_n is in $\mathcal{P}(D(R_n))$,

$$|\nabla F_n|_e(0) = 1 \quad \text{and} \quad |\nabla F_n|_e(z) \leq \left(1 - \frac{|z|^2}{R_n^2}\right)^{-1},$$

for any z in $D(R_n)$ and n in \mathbb{N} . So $|\nabla F_n|_e$ is uniformly bounded on compact subsets of \mathbb{C} and then, taking a subsequence, we can assume that $\{F_n\}$ converges to a certain map $F \in \mathcal{M}(\mathbb{C})$. As \mathcal{P} is closed, using III), we see that $F|_{D(r)} \in \mathcal{P}(D(r))$ for each $r > 0$, and from II) we have that $F \in \mathcal{P}(\mathbb{C})$. Finally, from the properties of $\{F_n\}$ we conclude that

$$|\nabla F_n|_e(0) = 1 \quad \text{and} \quad |\nabla F_n|_e \leq 1 \text{ in } \mathbb{C}.$$

Then F satisfies the conditions of ii). □

Now we will study some concrete compact properties. We start with the most famous one.

EXAMPLE 1. Let X be a subset of \mathbf{S}^2 . We can consider the property \mathcal{P}_X defined for any Riemann surface Σ by

$$\mathcal{P}_X(\Sigma) = \{F \in \mathcal{M}(\Sigma) : F(\Sigma) \cap X = \emptyset\} \cup \{\text{constant maps}\}.$$

\mathcal{P}_X satisfies trivially the axioms I) and II). It also verifies III) thanks to the Hurwitz theorem. So \mathcal{P}_X is closed. If X has three or more points, then \mathcal{P}_X is a compact property. This facts contains the classical Montel-Caratheodory theorem, the little Picard theorem and the great Picard theorem. Using Theorem 1, we will give a new proof of these results.

Theorem 2 *Let $X = \{a, b, c\} \subset \mathbf{S}^2$ be a subset with three distinct points. Then \mathcal{P}_X is a compact property.*

Proof. Suppose that \mathcal{P}_X is not compact. Then, from Theorem 1, there exists a non constant map $F \in \mathcal{P}_X(\mathbb{C})$. Clearly, we can assume the normalization $X = \{(0, 0, -1), (0, 0, 1), (1, 0, 0)\}$. If f is the $\overline{\mathbb{C}}$ -valued map associated to F , then f is an holomorphic map $f : \mathbb{C} \rightarrow \overline{\mathbb{C}}$ with $0, 1 \notin f(\mathbb{C})$. Hence f has a 2^n -th holomorphic root, f_n , for any natural number n . Moreover $f_n \in \mathcal{P}_{X_n}(\mathbb{C})$, where $X_n = \{\theta \in \overline{\mathbb{C}} : \theta^{2^n} = 1\} \subset \overline{\mathbb{C}} = \mathbf{S}^2$. Then we see that \mathcal{P}_{X_n} is not compact, and Theorem 1 says us that, for any n , there is $F_n \in \mathcal{P}_{X_n}(\mathbb{C})$ such that $|\nabla F_n|_e(0) = 1$ and $|\nabla F_n|_e \leq 1$ in \mathbb{C} . Hence, there is a subsequence of $\{F_n\}$ converging in our topology to some meromorphic map $G \in \mathcal{M}(\mathbb{C})$, with $|\nabla G|_e(0) = 1$. So G is non constant. On the other hand, as

$X_n \subset X_{n+1}$ for each n , we conclude using Hurwitz theorem that $G \in \mathcal{P}_Y(\mathbb{C})$, where $Y = \cup_n X_n$. But G is an open map and Y is dense in the horizontal equator of \mathbf{S}^2 . Then $G(\mathbb{C})$ is disjoint with this equator and so it is contained in an open half sphere, which contradicts Liouville theorem. \square

EXAMPLE 2. Let a be a positive number, Σ a Riemann surface and ds^2 a metric on Σ compatible with its conformal structure. Let dA and Δ be the canonical measure and the Laplacian of ds^2 , and $|\nabla v|$ the length of the gradient of v with respect to this metric, where v is any smooth function. We define the property \mathcal{P}_a by

$$\mathcal{P}_a(\Sigma) = \left\{ F \in \mathcal{M}(\Sigma) : \int_{\Sigma} (|\nabla v|^2 - a|\nabla F|^2 v^2) dA \geq 0, \text{ for all } v \in C_0^\infty(\Sigma) \right\},$$

where $C_0^\infty(\Sigma)$ denotes the space of compactly supported smooth functions on Σ . Note that $\mathcal{P}_a(\Sigma)$ is independent of the metric ds^2 . We have trivially that \mathcal{P}_a satisfies the hypotheses II) and III). One can see for instance in Fisher-Colbrie and Schoen⁴ that $F \in \mathcal{P}_a(\Sigma)$ if and only if there exists a smooth positive function u on Σ such that $\Delta u + a|\nabla F|^2 u = 0$ (although completeness of the metric is assumed in ⁴, this hypothesis is unnecessary in this concrete result). In particular u is a positive superharmonic function, and from Liouville theorem we have that $\mathcal{P}_a(\mathbb{C}) = \{\text{constant map}\}$. Moreover, if Σ_1 is another Riemann surface and $\phi : \Sigma_1 \rightarrow \Sigma$ is a holomorphic map without branch points, taking on Σ_1 the metric $ds_1^2 = \phi^*(ds^2)$, we have

$$\Delta_1(u \circ \phi) + a|\nabla(F \circ \phi)|_1^2(u \circ \phi) = 0,$$

and so $F \circ \phi \in \mathcal{P}_a(\Sigma_1)$, Δ_1 and $|\nabla v|_1$ being the Laplacian and the length of the gradient of the function v with respect to ds_1^2 . Then \mathcal{P}_a verifies I) and we have proved that \mathcal{P}_a is a compact property.

Remark. Consider the space of harmonic maps, $\mathcal{H}(\Sigma, M^n)$, from the Riemann surface Σ into a fixed compact Riemannian manifold M^n (which can be viewed as an embedded submanifold of the Euclidean space). Of course $\mathcal{M}(\Sigma) \subset \mathcal{H}(\Sigma, \mathbf{S}^2)$. It follows from standard elliptic theory, use for instance Theorems 11.4 and 6.2 of ¹⁰, that in this space of harmonic maps the C^1 -convergence on compact subsets is equivalent to the C^k -convergence on compact subsets of Σ , for any $k \geq 1$. Lemma 1 remain true if we take $\mathcal{H}(D, M)$ instead of $\mathcal{M}(D)$. Lemma 3 is also true for closed properties of harmonic maps (in this case one must invoke Sacks and Uhlenbeck²⁸ in order to conclude that an harmonic map with finite energy on a punctured disc extends, in an harmonic way, through the puncture). Then all the results in this section (except those in Example 1 which depend of the openness, and in

particular of Hurwitz theorem, of meromorphic maps) extend, with the same proof, to harmonic maps from Riemann surfaces into M^n . Example 2 give us compact properties for harmonic maps. Also, Example 1 give us closed properties if we take X a open subset of M^n (some of those may be compact properties). From Theorem 1, in order to prove the compacity of a certain closed property for harmonic maps \mathcal{P} , it is enough to prove a Liouville type Theorem for maps in $\mathcal{P}(\mathbb{C}) \subset \mathcal{H}(\mathbb{C}, M)$ with bounded gradient. Note that, in particular, the holomorphic map associated to such a F ,

$$\left| \frac{\partial F}{\partial z} \right|^2 : \mathbb{C} \rightarrow \mathbb{C},$$

will be constant. A test example to decide the utility of this general point of view in the study of properties of harmonic maps, could be the its application to the solution of the following problem (related to a conjecture of Do Carmo about the Gauss map of constant mean curvature surfaces in \mathbb{R}^3): Are there non-constant harmonic maps $F : \mathbb{C} \rightarrow \mathbf{S}^2$ whose image does not contain any equator of the sphere?

2 Conformal metrics on the disc

Let $ds^2 = e^{2u(z)}|dz|^2$ be a Riemannian metric on the disc and denote by K and Δ its Gauss curvature and its Laplacian. Let $r = |z|$ and ρ be the Euclidean distance and the ds^2 -distance from the points of D to the origin respectively. In this section we will also assume that (D, ds^2) is a ds^2 -disc with center at the origin of D and radius $R > 0$ which is diffeomorphic via the ds^2 -exponential map based at the origin with the corresponding Euclidean disc in the tangent space at $0 \in D$. In particular $\rho \leq R$ and $\rho = R$ on ∂D . Now we study the influence of bounds of the Gauss curvature over the distance function ρ .

Lemma 4 i) *If $K \leq 0$, then $\rho \leq Rr$ in D .*

ii) *If $-1 \leq K$, then $c_R(r) \leq \rho$ in D , where $c_R : [0, 1] \rightarrow \mathbb{R}$ is given by*

$$c_R(r) = \log \frac{e^R + 1 + (e^R - 1)r}{e^R + 1 - (e^R - 1)r}.$$

Proof. i) By hypothesis ρ is smooth in $D - \{0\}$. As $K \leq 0$, then a standard comparison theorem gives us $\Delta \rho \geq 1/\rho$, and so, $\Delta \log \rho \geq 0$ in $D - \{0\}$. On the other hand, $\Delta \log r = 0$ in $D - \{0\}$. Hence $\log(\rho/r)$ is a subharmonic function on D with $\log(\rho/r) = \log R$ on ∂D . So the lemma follows from the maximum principle.

ii) As $K \geq -1$, comparing ds^2 with the hyperbolic metric we have $\Delta\rho \geq (\cosh\rho)/(\sinh\rho)$ and by direct computation we obtain

$$\Delta \log \frac{e^\rho - 1}{e^\rho + 1} \leq 0 \text{ in } D - \{0\}.$$

So $\log \frac{1}{r} \frac{e^\rho - 1}{e^\rho + 1}$ is superharmonic in D . Moreover it takes on the boundary of the disc the value $\log \frac{e^R - 1}{e^R + 1}$. From the maximum principle we have

$$\frac{1}{r} \frac{e^\rho - 1}{e^\rho + 1} \geq \frac{e^R - 1}{e^R + 1},$$

or, equivalently, $c_R(r) \leq \rho$ in D . □

Note that c_R is an increasing function with $c_R(0) = 0$ and $c_R(1) = R$.

We consider now a sequence of metrics $ds_n^2 = e^{2u_n}|dz|^2$ on D , each of which satisfies the hypothesis of the beginning of this section with $R = R_n$, where $\{R_n\}$ is an increasing sequence diverging to infinity. Let K_n be the Gauss curvature of ds_n^2 .

Lemma 5 *If $\{K_n\}$ is uniformly bounded and the sequence $\{ds_n^2\}$ converges, uniformly on compact subsets of D , to a smooth metric $ds^2 = e^{2u}|dz|^2$, then ds^2 is complete.*

Proof. We can suppose that $-1 \leq K_n$ in D for each n . Let $\gamma(t)$, $0 \leq t < 1$, be a divergent curve in D with $\gamma(0) = 0$. From Lemma 4 we have

$$c_{R_n}(|\gamma(t)|) \leq \rho_n(\gamma(t)) \leq \int_0^t e^{u_n \circ \gamma} |\gamma'(\xi)| d\xi,$$

where ρ_n denotes the ds_n^2 -distance to the origin. Taking limits when n goes to infinity, as $\gamma([0, t])$ is compact, we obtain

$$\log \frac{1 + |\gamma(t)|}{1 - |\gamma(t)|} \leq \int_0^t e^{u \circ \gamma} |\gamma'(\xi)| d\xi.$$

As γ is a divergent curve we have that $|\gamma(t)|$ goes to 1 when $t \rightarrow 1$. So γ has infinite ds^2 -length and therefore ds^2 is complete. □

3 Minimal surfaces

Let $\psi : \Sigma \rightarrow \mathbb{R}^3$ be a conformal minimal immersion of a Riemann surface Σ in \mathbb{R}^3 . If z is a local complex parameter of Σ , defined on a simply connected relatively compact domain $\Omega \subset \Sigma$, then, from the Weierstrass representation²²,

we know that ψ is determinate by a pair of $\overline{\mathbb{C}}$ -valued meromorphic maps $f, g \in \mathcal{M}(\Omega)$. Moreover f has no poles and, in order to avoid branch points of ψ , its zeroes occur only at the poles of g , and the order of the zero is twice that of the pole. If $N : \Sigma \rightarrow \mathbf{S}^2$ is the Gauss map of ψ , then N is meromorphic and its corresponding $\overline{\mathbb{C}}$ -valued map is precisely g . If we denote by ds^2 and K the metric induced by ψ and its Gauss curvature, then we have

$$\psi = \Re \int \left(\frac{1}{2}(1 - g^2)f, \frac{i}{2}(1 + g^2)f, gf \right) dz, \quad (4)$$

$$ds^2 = \left(\frac{|f|(1 + |g|^2)}{2} \right)^2 |dz|^2 = \frac{|\nabla\psi|_e^2}{2} |dz|^2 \quad \text{and} \quad (5)$$

$$K = - \left(\frac{4|g'|}{|f|(1 + |g|^2)^2} \right)^2 = - \frac{|\nabla N|_e^2}{|\nabla\psi|_e^2}. \quad (6)$$

Lemma 6 *Let $\psi_n : \Sigma \rightarrow \mathbb{R}^3$ be a sequence of conformal minimal immersions, $\{N_n\} \subset \mathcal{M}(\Sigma)$ the sequence of their Gauss maps and K_n the Gauss curvature of ψ_n . Suppose that $\{N_n\}$ converges to a meromorphic map $N \in \mathcal{M}(\Sigma)$, that the sequence $\{K_n\}$ is uniformly bounded and that $\{\psi_n(p_0)\}$ converges for some point $p_0 \in \Sigma$. Then we have the following possibilities:*

- i) N is a constant map, or
- ii) a subsequence $\{K_{n'}\}$ of $\{K_n\}$ converges to zero, or
- iii) a subsequence $\{\psi_{n'}\}$ of $\{\psi_n\}$ converges to a conformal minimal immersion $\psi : \Sigma \rightarrow \mathbb{R}^3$ whose Gauss map is N .

Remark. Of course, the above cases are not incompatible. The topology in i) and ii) is the uniform convergence on compact subsets.

Proof. Suppose that N is non constant and that $-1 \leq K_n$ in Σ for each $n \in \mathbb{N}$. Let $p \in \Sigma$ be a point and (Ω_p, z) a complex local coordinate centered at p . Let g_n and f_n be the pair of maps given by the Weierstrass representation of ψ_n and g the $\overline{\mathbb{C}}$ -valued map corresponding to N .

Take first a point where N is unbranched and with $N(p) \neq (0, 0, 1)$. So $g(0) \in \mathbb{C}$ and $g'(0) \neq 0$. By choosing Ω_p and $\varepsilon > 0$ sufficiently small, we have that g is holomorphic and without branch points on Ω_p , and that

$$\left(\frac{4|g'|}{(1 + |g|^2)^2} \right)^2 \geq 2\varepsilon^2, \quad \text{in } \Omega_p.$$

As $g_n \rightarrow g$, for n large enough, we have

$$\left(\frac{4|g'_n|}{(1 + |g_n|^2)^2} \right)^2 \geq \varepsilon^2, \quad \text{in } \Omega_p,$$

and so

$$\frac{\varepsilon^2}{|f_n|^2} \leq \left(\frac{4|g'_n|}{|f_n|(1+|g_n|^2)} \right)^2 = |K_n| \leq 1 \text{ in } \Omega_p.$$

Finally we obtain $\varepsilon \leq |f_n|$ in Ω_p , for large n , and then $\{f_n\}$ is relatively compact in $\mathcal{M}(\Omega_p)$. Therefore the sequence of globally defined holomorphic 1-forms $\{f_n dz\}$ is relatively compact on

$$\Sigma_1 = \{p \in \Sigma : N(p) \neq (0, 0, 1) \text{ and } p \text{ is not a branch point of } N\},$$

because it is relatively compact in a neighborhood of each of their points. Observe that $\Sigma - \Sigma_1$ is a discrete set. Taking a subsequence we can assume that $\{f_n dz\}$ converges on Σ_1 either to a nonzero holomorphic 1-form $f dz$ or to infinity. We discuss each case separately.

a) $\{f_n dz\}$ converges to infinity on Σ_1 . Let p be a branch point of N with $N(p) \neq (0, 0, 1)$. Hence in a small disc $D(2\varepsilon)$ of Ω_p , g is holomorphic and so the same holds, from Hurwitz theorem, for g_n with n large. Hence f_n has not zeroes on $D(2\varepsilon)$ and converges to infinity on $\partial D(\varepsilon)$. From the maximum modulus principle we conclude that $\{f_n\}$ converges to infinity on $D(\varepsilon)$.

Suppose now that $N(p) = (0, 0, 1)$, that is $g(0) = \infty$. On a small disc $D(2\varepsilon) \subset \Omega_p$ the map g has neither zeroes nor poles other than the origin. So $g_n^2 f_n$ is an holomorphic map without zeroes, for n large. As $g_n^2 f_n$ converges uniformly to infinity on $\partial D(\varepsilon)$, the maximum principle implies that $g_n^2 f_n$ converges to infinity on $D(\varepsilon)$.

Therefore it follows from (5) that $|\nabla \psi_n|_e$ converges to infinity on Ω_p for each $p \in \Sigma$. As $|\nabla N_n|_e$ converges to $|\nabla N|_e$ on Ω_p , we conclude finally, using (6), that $\{K_n\}$ converges to the zero function on Σ .

b) $\{f_n dz\}$ converges to an holomorphic 1-form $f dz$ on Σ_1 . Let $p \in \Sigma - \Sigma_1$. If $D(\varepsilon)$ is a small disc contained in Ω_p , as $f_n \rightarrow f$ on $\partial D(\varepsilon)$, we see that $\{f_n\}$ is uniformly bounded on $\partial D(\varepsilon)$, and then, using the maximum modulus principle, it is also bounded on $D(\varepsilon)$. So $\{f_n\}$ is relatively compact on $D(\varepsilon)$. We conclude easily that $f dz$ extends in a holomorphic way to Σ and that the global 1-forms $f_n dz$ converge to $f dz$ on Σ . Moreover, from the Hurwitz theorem, we have that the zeroes of $f dz$ occur precisely at the poles of g , and that the order of the zero is twice the order of the pole. Then $g f dz$ and $g^2 f dz$ are holomorphic in Σ , $g_n f_n dz \rightarrow g f dz$ and $g_n^2 f_n dz \rightarrow g^2 f dz$. As $\{\psi_n(p_0)\}$ converges for some point p_0 of Σ , we conclude finally, from (4), that the sequence of harmonic maps $\{\psi_n\}$ converges uniformly on compact subsets of Σ to an *unbranched* conformal minimal immersion $\psi : \Sigma \rightarrow \mathbb{R}^3$, whose Weierstrass representation is given by the pair $(g, f dz)$. In particular N is the Gauss map of ψ . \square

Let \mathcal{P} be a compact property. We say that \mathcal{P} satisfies a curvature estimate if there exists a positive constant $C = C(\mathcal{P})$ such that any conformal minimal immersion $\psi : \Sigma \rightarrow \mathbb{R}^3$ whose Gauss map lies in $\mathcal{P}(\Sigma)$ verifies the curvature bound

$$|K|d^2 \leq C \quad (7)$$

in Σ where K is the Gauss curvature of ψ and d is its geodesic distance to the boundary of Σ . Note that we are assuming that C does not depend on Σ .

Theorem 3 *Let \mathcal{P} be a compact property for meromorphic maps. Then the following alternative holds:*

- i) \mathcal{P} satisfies a curvature estimate, or
- ii) there is a conformal complete minimal immersion $\psi : D \rightarrow \mathbb{R}^3$, whose Gauss map lies in $\mathcal{P}(D)$ and whose Gauss curvature K satisfies $|K(0)| = 1$ and $|K| \leq 1$ in D .

Proof. Suppose the assertion in i) does not hold. We will construct a minimal surface satisfying ii). There exists a sequence of (non complete) minimal surfaces $\psi_n : \Sigma_n \rightarrow \mathbb{R}^3$ and points $p_n \in \Sigma_n$ such that

$$K_n(p_n) = -\frac{1}{2}, \quad \text{for each } n \quad \text{and} \quad d_n(p_n) \rightarrow \infty, \quad (8)$$

where K_n is the Gauss curvature of ψ_n and d_n its geodesic distance to the boundary. Now we will make several normalizations. In each step we will translate to the new surfaces and new points the notation ψ_n , Σ_n , p_n , K_n and d_n .

a) We change the surface Σ_n by its geodesic disc of center at p_n and radius $d_n(p_n) - 1$. So the new surfaces satisfy (8) and, for each n , the new function $|K_n|d_n^2$ is bounded and vanishes at the boundary.

b) We change p_n by the point of Σ_n where the function $|K_n|d_n^2$ attains its maximum and we change Σ_n by the geodesic disc of center at the new p_n and radius $d_n(p_n)$. Note that the new sequence $|K_n(p_n)|d_n(p_n)^2$ diverges to infinity.

c) Using homotheties of \mathbb{R}^3 we expand conveniently our surfaces in order to obtain $\psi_n(p_n) = 0$ and $K_n(p_n) = -1/2$, for each n . So the surfaces satisfy again (8). Moreover, using the maximizing property of p_n , we obtain

$$|K_n(p)| \leq \frac{1}{2} \left(\frac{d_n(p_n)}{d_n(p)} \right)^2, \quad (9)$$

for each $p \in \Sigma_n$ and for each $n \in \mathbb{N}$.

d) If we take $p \in \Sigma_n$ such that its geodesic distance to the point p_n is smaller than $R_n = \sqrt{d_n(p_n)}$ then $d_n(p) \geq R_n^2 - R_n$ and, using (9), we have

$$|K_n(p)| \leq \frac{1}{2} \left(\frac{R_n^2}{R_n^2 - R_n} \right)^2, \text{ for each } n. \quad (10)$$

So, as $R_n \rightarrow \infty$, if we change Σ_n by its geodesic disc centered at p_n and radius R_n , we have that $|K_n| \leq 1$ in Σ_n for each n large enough, $d_n(p_n) = R_n$ and our surfaces satisfy (8).

e) The exponential map of the surfaces Σ_n with base point p_n is defined on a Euclidean disc of radius R_n in the tangent space at p_n . Moreover, as the Gauss curvature of a minimal surface is non positive, this map is a local diffeomorphism. Hence, taking pull-back with these exponential maps, we obtain a new sequence of minimal surfaces $\psi_n : \Sigma_n \rightarrow \mathbb{R}^3$, where the surfaces Σ_n are topological discs. From the uniformization theorem Σ_n (viewed as a Riemann surface) is the disc D (the case $\Sigma_n = \mathbb{C}$ is impossible because the surface is non flat and \mathcal{P} is compact): Moreover we can suppose that $p_n = 0$ for each n . By construction, the surfaces $\psi_n : D \rightarrow \mathbb{R}^3$ are geodesics discs with center at the origin of D and radius R_n and their exponential maps are diffeomorphisms from the corresponding Euclidean discs in the tangent space at 0 onto the disc D .

f) As \mathcal{P} is a compact property we can assume, passing to a subsequence if necessary, that the Gauss maps $N_n \in \mathcal{P}(D)$ converge to a map $N \in \mathcal{M}(D)$.

As resume we have constructed a sequence of conformal minimal immersions $\psi_n : D \rightarrow \mathbb{R}^3$ satisfying the hypothesis of Lemma 6. If the option iii) of this lemma holds, then Lemma 5 says that the metric induced by the limit conformal minimal immersion $\psi : D \rightarrow \mathbb{R}^3$ is complete. Moreover we obtain from (10) that the Gauss curvature of ψ satisfies $K(0) = -1/2$ and $K \geq -1/2$ in D and we conclude easily by scaling. So, in order to finish the proof of the theorem, it is enough to show that options i) and ii) of Lemma 6 cannot hold.

The case ii) is impossible because $K_n(0) = -1/2$ for each n . Suppose, reasoning by contradiction, that the Gauss map N is constant. Fix $r, 0 < r < 1$, and put $d_n(r) = \psi_n$ -geodesic distance from 0 to $\partial D(r)$. Then there is a natural number $n(r)$ such that for $n \geq n(r)$ we have that $N_n(D(r))$ is contained in the half sphere centered at N . So by the result of Osserman²⁰, there is a positive constant C such that, for $n \geq n(r)$,

$$|K_n(0)|d_n(r)^2 \leq C. \quad (11)$$

As $|K_n(0)| = 1/2$ for each n and, from Lemma 4, $c_{R_n}(r) \leq d_n(r)$, taking r close to 1 and n large enough, we can made $c_{R_n}(r)$ as large as you want. This

fact contradicts (11) and we conclude that the option i) of Lemma 6 cannot hold. \square

Corollary 1 *Let \mathcal{P} be a compact property. Then the following assertions are equivalents:*

- i) \mathcal{P} satisfies a curvature estimate.
- ii) The plane is the only complete minimal surface in \mathbb{R}^3 whose Gauss map lies in \mathcal{P} .
- iii) There are not complete minimal surfaces conformally equivalent to the disc with bounded Gauss curvature and whose Gauss map lies in $\mathcal{P}(D)$.

Proof. It is clear that i) \Rightarrow ii) \Rightarrow iii), and from Theorem 3 we conclude that iii) \Rightarrow i). \square

Now we apply the above results to the concrete examples we are interested to: minimal surfaces whose Gauss map omits some points of the sphere and stable minimal surfaces.

EXAMPLE 3. We follow the notations of Example 1 above. Fujimoto⁶ proved that if $X \subset \mathbf{S}^2$ has more than four points then \mathcal{P}_X satisfies a curvature estimate. This result is the best possible, see ²² p. 72. Using our results above we can give a simplification of the arguments of ⁶.

Theorem 4 ⁶. *If X has more than four point, then \mathcal{P}_X satisfies a curvature estimate.*

Proof. We can assume that the points $a_1, \dots, a_4 \in \mathbb{C}$ and ∞ lie in $X \subset \overline{D}$. Suppose, reasoning by contradiction, that there exists a complete minimal immersion $\psi : D \rightarrow \mathbb{R}^3$ whose Gauss curvature satisfies $|K| \leq 1$ and such that its Gauss map lies in $\mathcal{P}_X(D)$. Let g and f be the pair of holomorphic maps determinate by the Weierstrass representation of ψ . From the boundedness of the curvature we conclude, using Yau³³ version of the Schwarz lemma and (5), that

$$\frac{2}{1 - |z|^2} \leq \frac{1}{2}(1 + |g|^2) \quad \text{in } D. \quad (12)$$

On the other hand, from the Fujimoto⁶ main lemma in , see also Mo and Osserman¹⁷, given ε and ε' satisfying $0 < 4\varepsilon' < \varepsilon < 1$, there is a positive constant C depending only of a_j , $j = 1, \dots, 4$, such that

$$\frac{(1 + |g|^2)^{(3-\varepsilon)/2} |g'|}{\prod_{j=1}^4 |g - a_j|^{1-\varepsilon'}} \leq \frac{C}{1 - |z|^2}. \quad (13)$$

Solving for $1 + |g|^2$ in (12) and substituting its value in (13), we obtain

$$\frac{B}{1 - |z|^2} \leq \lambda, \quad (14)$$

where B is a positive constant and λ is given by

$$\lambda = \left(\frac{|f|^{(3-\varepsilon)/2} \prod_{j=1}^4 |g - a_j|^{1-\varepsilon'}}{|g'|} \right)^{2/(1-\varepsilon)}. \quad (15)$$

If we consider the metric $ds_0^2 = \lambda^2 |dz|^2$ on $D' = \{z \in D : g'(z) \neq 0\}$, we have easily that ds_0^2 is flat and complete at the punctures. Moreover from (14) we conclude that ds_0^2 is complete at the boundary of the disc. So the universal cover of D' must be \mathbb{C} and this contradiction proves the theorem. \square

EXAMPLE 4. With the notation of Example 2 above, a minimal surface is stable if and only if its Gauss map lies in \mathcal{P}_1 . After the results mentioned in the introduction about stability of minimal surfaces, Kawai¹⁵ proved that if $a > 1/8$, then the only complete minimal surface whose Gauss map lies in \mathcal{P}_a is the plane and so, from Theorem 3, \mathcal{P}_a satisfies the curvature estimate (7). We do not know if this result is the best possible. In particular we can reprove in a simple way the curvature estimate for stable minimal surfaces obtained by Schoen³⁰ in the case of minimal surfaces in \mathbb{R}^3 , compare also with Nitsche¹⁸ p. 450.

Theorem 5³⁰. *There exists a positive constant C such that for any stable minimal surface $\psi : \Sigma \rightarrow \mathbb{R}^3$ we have*

$$|K(p)|d(p)^2 \leq C, \quad \text{for all } p \in \Sigma,$$

where K is the Gauss curvature of the surface and d is the geodesic distance to the boundary.

Proof. Do Carmo and Peng¹, Fisher-Colbrie and Schoen⁴ and Pogorelov²⁷ have proved independently that the plane is the only complete orientable stable minimal surface. Then the theorem follows directly from Corollary 1. \square

Remark. There is another relevant similitude between stable minimal surfaces and minimal surfaces whose Gauss map omits some points of the sphere which involves complete minimal surfaces with finite total curvature. These surfaces represent, in the context of relations between complex variable and minimal surfaces, the part which correspond to the compact Riemann

surface theory, see ²². It is proved by Mo and Osserman¹⁷ that a complete minimal surface which takes five prescribed values of \mathbf{S}^2 only a finite number of times, has finite total curvature. On the other hand, it was proved by Fisher-Colbrie³ and Gulliver and Lawson^{11,12} that, if the index of a complete minimal surface is finite (this means that the index of the second variation formula on compact subdomains of the surface is bounded uniformly) then the surface has finite total curvature. In both cases the finiteness assumption can be rewritten as *outside of a compact subset, the Gauss map of the surface omits five points of the sphere (resp. is stable)*. It seems natural to hope that if \mathcal{P} satisfies a curvature estimate, then we can also conclude that a complete minimal surface whose Gauss map verifies the property outside a compact subset must have finite total curvature. However we have been unable to prove it.

Addendum

This paper was written some years ago and has circulated in preprint form. It has been cited by Fujimoto⁹, Osserman^{23,24,25}, Osserman & Ru²⁶, Zalcman³⁵ and in the books by Fujimoto⁸ and Schiff²⁹.

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