SCHRÖDINGER OPERATORS ASSOCIATED TO A HOLOMORPHIC MAP

Sebastián Montiel*
Antonio Ros*

Universidad de Granada
Spain

In this work we will expose certain ideas and results concerning a kind of Schrödinger operators which can be considered on a compact Riemann surface. These operators will be constructed by using as a potential the energy density of a holomorphic map from the surface to the two-sphere. Besides the interest that their study has from an analytical point of view, we will see that they appear, in a natural way, in different geometrical situations such as the study of the index of complete minimal surfaces with finite total curvature and the study of the critical points of the Willmore functional.

This paper is, in fact, an expanded version of an invited lecture given by the first author in the Global Differential Geometry and Global Analysis Conference held at the Technische Universität of Berlin in June, 1990.

Introduction and preliminaries

Let Σ be a compact Riemann surface and $\phi : \Sigma \to \mathbb{S}^2$ a holomorphic map from this surface to the unit two–sphere $\mathbb{S}^2$. Consider any metric ds^2 on Σ compatible with the complex structure and let Δ and ∇ be its Laplacian and gradient respectively. Having chosen this metric, one has the following Schrödinger operator

$$L = \Delta + |\nabla\phi|^2. \tag{1-1}$$

Our aim here is to study spectral properties of these operators and relate them to the map ϕ and the surface Σ. Of course, the eigenvalues and eigenfunctions of such an operator L depend strongly on the metric ds^2. However, we want to obtain information from it which only refers to ϕ and Σ. This can be done in two ways: first, by looking for spectral properties of L which are independent on the chosen metric; or, second, by putting on Σ a particular metric especially related to our problem.

First, denote by Q_ϕ the quadratic form corresponding to the self–adjoint operator L, that is

$$Q_\phi(u,u) = \int_\Sigma \left\{ |\nabla u|^2 - |\nabla\phi|^2 u^2 \right\} dA \quad u \in W_1(\Sigma) \tag{1-2}$$

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where dA is the measure associated to the metric ds^2 and $W_1(\Sigma)$ is the corresponding Sobolev space. Because of the conformal invariance of the Dirichlet integral, the form Q_ϕ does not depend on the chosen metric ds^2. Hence, all these Schrödinger operators L in (1-1) have the same number of bounded states, that is, the same number of negative eigenvalues, and the same kernel. So, we may define *the index of the holomorphic map* ϕ as the index of the quadratic form Q_ϕ:

$$\text{Ind } \phi = \text{index } Q_\phi = \# \text{ bounded states of any } L \tag{1-3}$$

and *the nullity space* $N(\phi)$ *of* ϕ as the common kernel of all these operators L:

$$N(\phi) = \text{kernel of any } L = \left\{ u \in C^\infty(\Sigma) \,|\, \Delta u + |\nabla \phi|^2 u = 0 \right\}. \tag{1-4}$$

The dimension of this space will be called *the nullity of the holomorphic map* ϕ and we will represent it by

$$\text{Nul } (\phi) = \dim N(\phi). \tag{1-5}$$

Notice that, as ϕ is holomorphic, then ϕ is harmonic, that is

$$\Delta \phi + |\nabla \phi|^2 \phi = 0.$$

So, the space $L(\phi)$ of the linear functions of the components of ϕ is contained in $N(\phi)$, that is

$$L(\phi) = \left\{ \langle \phi, a \rangle \;|\, a \in \mathbb{R}^3 \right\} \subset N(\phi) \tag{1-6}$$

and, so, Nul $\phi \geq 3$ provided that ϕ is not a constant map. It is also clear that the index of a holomorphic map ϕ, such as we have just define it, vanishes if and only if ϕ is constant.

The second way that we have mentioned above is to choose on the surface Σ a particular metric coming from the situation that we are considering. We have, at once, a natural candidate: the metric ds^2_ϕ induced on Σ by ϕ from the standard metric ds^2_0 of the sphere $\mathbb{S}^2$, that is

$$ds^2_\phi = \phi^* \, ds^2_0 = \frac{|\nabla \phi|^2}{2} ds^2. \tag{1-7}$$

This metric has constant one Gauss curvature, conical singularities at the branching points of ϕ and finite area $4\pi \deg \phi$. The Schrödinger operator L_ϕ that one gets by using that metric is nothing but

$$L_\phi = \Delta_\phi + 2, \tag{1-8}$$

where Δ_ϕ is the Laplacian of the branched metric ds^2_ϕ. Even though the metric ds^2_ϕ is not regular, the eigenvalues and eigenfunctions of this Laplacian are well defined via a variational approach, since the codimension of the singularities set is two (see [Ty]). Hence, if λ is an eigenvalue of L_ϕ, its corresponding eigenspace is given by

$$(1\text{-}9) \qquad V_\lambda(\phi) = \left\{ u \in W_1(\Sigma) \,|\, Q_\phi(u,v) = \lambda \int_\Sigma uv \, dA_\phi, \ \forall v \in W_1(\Sigma) \right\}.$$

It follows, from elliptic regularity, that $V_\lambda(\phi) \subset C^\infty(\Sigma)$.

So, the spectrum of the operator L_ϕ is, up to a constant, the spectrum of the Laplacian Δ_ϕ of the branched metric ds^2_ϕ. In fact, in terms of this metric, the index of a holomorphic map that we defined above can be interpreted as the number of eigenvalues of its Laplacian which are less than two,

$$(1\text{-}10) \qquad \operatorname{Ind} \phi = \# \text{ eigenvalues of } \Delta_\phi < 2.$$

These eigenvalues are especially important because they do not come from the spectrum of the standard sphere $\mathbb{S}^2$, since its first non zero eigenvalue is exactly two. Also, one has that

$$(1\text{-}11) \qquad \operatorname{Nul} \phi = \text{multiplicity of } 2 \text{ as an eigenvalue of } \Delta_\phi.$$

It is important to remark that the spectrum of the metric ds^2_ϕ is a sequence naturally associated to the holomorphic map ϕ. So, an interesting question is: What kind of information about the complex structure of the surface Σ and the map ϕ can be recovered from that sequence?

Another particular metric that one could consider on the surface Σ in order to study the corresponding Schrödinger operator L given by (1-1) is the hyperbolic metric with constant -1 curvature, provided that the genus of the surface is greater than one. In this case, that we have not studied in detail, one can see from [Gui–K] and from Lemma 7 below that, in the most of the cases, the spectrum of the operator L determines the map ϕ up to an isometry of $\mathbb{S}^2$.

This paper is devoted to the study of these spectral invariants associated to a holomorphic map and is organized as follows. Before stating the results, we will deal with three geometrical topics where the invariants that we have described appear: the theory of complete minimal surfaces in $\mathbb{R}^3$, the study of the Willmore surfaces and the study of the determinant of the Laplacian of metrics on compact surfaces. So, in this way, these invariants will get different geometrical meanings.

After that, we will show some results that we have obtained recently on this subject. In fact, we will prove that each function in the nullity space of a holomorphic map ϕ, which is not a linear function of $L(\phi)$, see (1-6), can be represented as the support function of a branched complete minimal surface in $\mathbb{R}^3$ with planar ends and whose extended Gauss map is ϕ. Next, we will obtain some information about the holomorphic maps with the lowest index. Also, we will get lower and upper bounds for the index and the nullity when the branching values of the holomorphic map are in an special position on the sphere. In particular, if all these branching values lie in an equator, we will compute explicitly these invariants.

In the case that our surface Σ has genus zero, we will show that the index and the nullity of $A \circ \phi$ coincide with those of ϕ, for each Möbius transformation A of the sphere.

Also, we will compute the index and the nullity of a generic ϕ and give general bounds for these invariants.

We will finish the paper by making a detailed study of the index and the nullity for a holomorphic map $\phi : \overline{\mathbb{C}} \to \mathbb{S}^2$ with degree three and we will see that its index is five and its nullity is three, except when its four ramification points form an equianharmonic quadruple, that is, they are placed, up to a Möbius transformation, on the vertices of a regular thetrahedron. In this case the index is four and the nullity five.

Some related geometrical problems

In this section, we will see that there are some geometrical problems involving a compact Riemann surface Σ where a holomorphic map ϕ from the surface to the two–sphere appears and where a good knowledge of the quadratic form Q_ϕ defined in (1-2) provides an important tool in order to solve them.

Complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature. Let M be an orientable surface and $f : M \to \mathbb{R}^3$ a minimal immersion into the three–dimensional Euclidean space. Recall that such an immersion is a critical point of the area for all perturbations of M with compact support. Denote by $N : M \to \mathbb{S}^2$ its Gauss map. If $D \subset M$ is a compact domain of M and $f_t : D \to \mathbb{R}^3$ is a variation of f whose variational field is uN, where $u \in C_0^\infty(D)$, then the second derivative of the induced area $A(f_t)$ is given by

$$\left.\frac{d^2}{dt^2}\right|_{t=0} A(f_t) = \int_D \left\{|\nabla u|^2 + 2Ku^2\right\} dA$$

where K is the Gauss curvature function of the surface M. So, *the second variation operator L of the area* is

$$L = \Delta - 2K = \Delta + |\nabla N|^2.$$

From this second variation operator L we have, for each compact domain D of M, a measure of its instability: *the index of the domain D* defined as follows:

$$\text{index } D = \#\text{ negative eigenvalues of } L \text{ with Dirichlet boundary condition.}$$

It is a classical result due to Schwarz [Schw] (see also [Ba–do C] for a stronger formulation) that, if D is small enough, then D is stable, that is the index of D vanishes. Now, for taking arbitrary large domains on the surface, suppose that M is complete. Then, in this case, Fischer-Colbrie and Schoen [FC–Sch], independently do Carmo and Peng [do C–Pe] and, independently also, Pogorelov [Po] showed that a large enough piece of M is unstable, provided that M is not a plane. So, it seems natural to ask: How does the index of a compact domain of a complete minimal surface in $\mathbb{R}^3$ change when its size increases? To answer this, the index of the whole of M is defined in the following way:

$$\text{index } M = \sup_{D \subset M} \text{index } D.$$

This number can become infinite. By the way, the following nice theorem was proved by Fischer-Colbrie [FC] and independently by Gulliver and Lawson [Gu], giving a geometrical consistency to the index of a complete minimal surface defined above:

$$\text{index } M < \infty \Longleftrightarrow \left| \int_M K\, dA \right| < \infty,$$

that is, this index is finite if and only if the total Gaussian curvature of the surface is finite. Moreover, complete minimal surfaces with finite total curvature form a well–known family. In fact, through the work of Osserman, see [Oss], we know that such a surface has a conformal structure equivalent to that of a compact Riemann surface Σ after removing a finite number of its points. On the other hand, the Gauss map N of this surface extends to a holomorphic map $\phi : \Sigma \to \mathbb{S}^2$. A consequence of Fischer-Colbrie's work is that, in this case, the index of M, as a minimal surface, coincides with the index of its extended Gauss map ϕ, the invariant that we have defined in (1-3) and (1-10). Also, it can be shown, by using elliptic regularity, that the space of the bounded Jacobi fields on the surface M is actually the nullity space $N(\phi)$ of ϕ defined in (1-4). So, as a consequence we have got:

Conclusion 1. *If $\phi : \Sigma \to \mathbb{S}^2$ is a holomorphic map from a compact Riemann surface Σ to the two–sphere $\mathbb{S}^2$, the index of ϕ defined in* (1-3) *can be thought of as the index of any complete minimal surface in $\mathbb{R}^3$ with finite total curvature which has ϕ as extended Gauss map, and the nullity space $N(\phi)$ of ϕ is the space consisting of all bounded Jacobi fields on any of these surfaces.*

Willmore surfaces. We will consider now a second geometrical context in which the Schrödinger operator (1-1) associated to a holomorphic map defined on a compact Riemann surface appears again. This is the study of Willmore surfaces, that is, those surfaces immersed into the three–dimensional Euclidean space or into the three–sphere which are critical points of the Willmore functional.

In fact, take a branched conformal immersion $F : \Sigma \to \mathbb{S}^3$ from a compact Riemann surface Σ into the three–sphere. Denote by N a unit normal field for F and by H the mean curvature of F with respect to this N. Then, one can consider *the conformal Gauss map of F* that we will represent by ψ and which is given by:

$$\psi = (H, HF + N) : \Sigma \longrightarrow \mathbb{R} \times \mathbb{R}^4 = \mathbb{R}^5.$$

If we endow the space $\mathbb{R}^5$ with the Minkowski metric $\langle , \rangle$ with signature $(-,+,+,+,+)$ and consider on $\mathbb{C}^5$ its $\mathbb{C}$–bilinear extension, that we will also denote by $\langle , \rangle$, one can get by taking into account that $\langle F, F \rangle = 1$ and $\langle F_z, F_z \rangle = 0$, for a local complex coordinate z on Σ, that

$$\text{(2-1)} \qquad \langle \psi, \psi \rangle = 1 \quad \langle \psi_z, \psi_z \rangle = 0 \quad |\psi_z|^2 = (H^2 - G)|F_z|^2,$$

where G is the determinant of the second fundamental form of F. So, if $\mathbb{S}^4_1$ stands for the set of length $+1$ vectors in the Minkowski space $\mathbb{R}^5$, that is, for the four–dimensional De

Sitter space, we have that the conformal Gauss map ψ is actually a weakly conformal map

$$\psi : \Sigma \longrightarrow \mathbb{S}^4_1$$

which is regular away from the umbilical and branching points of F and whose induced area coincides, up to a constant, with the value that the Willmore functional takes on F. This conformal Gauss map had been considered by Blaschke [Bl] and, recently, has been rediscovered by Bryant [Br1] and its importance lies on the following fact, proved by him in [Br1], that one can easily get from (2-1):

$$\psi_{z\bar{z}} + |\psi_z|^2\psi = 0 \Longleftrightarrow H_{z\bar{z}} + |F_z|^2(H^2 - G)H = 0, \tag{2-2}$$

that is,

$$\psi \text{ is harmonic} \Longleftrightarrow F \text{ is a Willmore immersion.}$$

Suppose now that, in fact, F is a critical point of the Willmore functional. Then we can dispose of a map $\psi : \Sigma \to \mathbb{S}^4_1$ which is conformal and harmonic. Remember that, in this type of situation, when the target manifold was an Euclidean four–sphere, a quartic holomorphic form had been constructed by Calabi and Chern (see [Che], for instance). So, a good trick to do here is to forget the 1 in $\mathbb{S}^4_1$ and define a quartic form q_F like in the Euclidean case, as follows:

$$q_F = \langle \psi_{zz}, \psi_{zz} \rangle \, (dz)^4$$

where z means a local complex coordinate on the surface Σ. From (2-2), we can see that q_F is holomorphic provided that F is a Willmore immersion (see [Br1]). But now there is a difference with the Euclidean case because this quartic form q_F vanishing has quite a strong consequence. One can easily prove that q_F is identically zero if and only if either F is umbilical or the image of the conformal Gauss map ψ lies in a degenerate hyperplane of the Minkowski space, that is

$$q_F = 0 \Longleftrightarrow \begin{cases} F \text{ is umbilical} \\ \exists A \in \mathbb{R}^5 - \{0\}, \ \langle A, A \rangle \,, \ \langle \psi, A \rangle = 0. \end{cases}$$

In this case, choose any Lorentz plane in the Minkowski space containing the vector A and call $\mathbb{R}^3$ to its orthogonal complement which is Euclidean. So, we can write ψ in the following way:

$$\psi = uA + \phi, \quad u \in C^\infty(\Sigma), \quad \phi : \Sigma \to \mathbb{R}^3.$$

But now, we have

$$\langle \psi, \psi \rangle = 1 \Longrightarrow \langle \phi, \phi \rangle = 1$$
$$\langle \psi_z, \psi_z \rangle = 0 \Longrightarrow \langle \phi_z, \phi_z \rangle = 0$$
$$\psi_{z\bar{z}} + |\psi_z|^2\psi = 0 \Longrightarrow u_{z\bar{z}} + |\phi_z|^2 u = 0.$$

The first two equations tell us that ϕ maps Σ onto the two–sphere $\mathbb{S}^2$ and that ϕ is holomorphic. The latter can be written in the following way:

$$\Delta_0 u + |\nabla_0 \phi|^2 u = 0$$

where Δ_0 and ∇_0 are respectively the Laplacian and the gradient of the local Euclidean metric $|dz|^2$. Hence, the function u lies in the nullity space $N(\phi)$ of the holomorphic map ϕ defined in (1-4). Furthermore, the reader can see, as a little exercise, that, if $u \in L(\phi)$, see (1-6), then F would be umbilical. Of course, all this process can be inverted and, so, we get the following

Conclusion 2. *Each non–umbilical Willmore immersion with Bryant's quartic form identically zero (for example, each spherical Willmore immersion) can be represented by a pair (ϕ, u) consisting of a holomorphic map ϕ to the two–sphere and a non–linear function u lying in the nullity space $N(\phi)$ of ϕ.*

As a consequence, it seems to be interesting in order, for instance, to finish the study of the moduli space of spherical Willmore surfaces, started by Bryant (see [Br1] and [Br2]), to determine which holomorphic maps from a compact Riemann surface to the two–sphere have nullity greater than three.

Determinant of the Laplacian. There is a third geometrical problem involving the spectral invariants that we have introduced in the first section for a holomorphic map from a compact Riemann surface to the two–sphere. It is the study of the determinant of the Laplacian of metrics on compact surfaces.

This topic was initiated by Polyakov and continued, among others, by Onofri and Virasoro [On–V] from a physical point of view, in connection with quantum string theory. In fact, in this theory, one deals with the space of metrics in a conformal class on a given compact surface and the considered action can be described as follows. Take a metric ds^2 on a compact Riemann surface Σ compatible with its complex structure and consider the spectrum of its Laplacian

$$\text{Spec}\,\Delta = \{\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_k < \cdots\}.$$

So, we can construct a functional on this space by mapping the metric ds^2 on the product of all its non–zero eigenvalues

$$ds^2 \longrightarrow \prod_{k=1}^{\infty} \lambda_k,$$

of course, after making sense out of this expression. This can be done, for instance, by using some elementary complex analysis as one can see in [La] or in [Os–Ph–Sa]. Onofri and Virasoro showed that a metric ds^2 is a solution for the Euler–Lagrange equation of the variational problem corresponding to this functional, when its area is constrained to take a fixed value, if and only if it has constant Gauss curvature.

In this formulation of the Polyakov quantum string theory, due to Onofri and Virasoro, the Gauss–Bonnet theorem imposes a limitation from the physical point of view: if we fix a constant to be the Gauss curvature of a metric on the surface Σ, then its area can take only one possible value. This limitation could be removed by considering metrics with singularities, such as the metric ds^2_ϕ induced on Σ by a holomorphic map ϕ to the two–sphere. All these metrics have constant one Gauss curvature but their areas depend on the degree of the map ϕ. So, this degree could be introduced as a new quantum number in the theory.

Then, if $\phi : \Sigma \to \mathbb{S}^2$ is a holomorphic map from a compact Riemann surface to the two–sphere, we will consider the class C_ϕ of all the compatible metrics on Σ which have the same area and singularities as the metric ds^2_ϕ. By following, for example, [Os–Ph–Sa], one can make a study of the determinant of the Laplacian functional on this set C_ϕ and get:

Conclusion 3. *The metric ds^2_ϕ induced on a compact Riemann surface Σ by means of a holomorphic map $\phi : \Sigma \to \mathbb{S}^2$ is a critical point of the determinant of the Laplacian functional on the class C_ϕ. Moreover, the Hessian of this functional at the point ds^2_ϕ is nothing but the quadratic form Q_ϕ defined in* (1-2)*, restricted to the space of the functions which are orthogonal to the constants. So, the index of the metric ds^2_ϕ as a critical point of the determinant of the Laplacian is* $\operatorname{Ind} \phi - 1$.

Results about index and nullity of holomorphic maps

We think that the interest of the three geometrical situations that we have just refered to is a sufficient motivation to study the index and the nullity of meromorphic functions defined on compact Riemann surfaces. We want now to state some results that we have obtained on this subject.

Nullity space and minimal surfaces with planar ends. We will begin by giving a certain representation for the functions which lie in the nullity space $N(\phi)$ of a holomorphic map ϕ defined on a compact Riemann surface Σ and taking values on the two–sphere. In Conclusion 1 of the second section, we had stated that $N(\phi)$ contains the bounded Jacobi fields on any complete minimal surface M in $\mathbb{R}^3$ with finite total curvature and extended Gauss map ϕ. On the other hand, we know that fields on $\mathbb{R}^3$ whose flow consists of isometries (Killing fields) or dilatations induce on M Jacobi fields. The next result gives conditions on the minimal surface M in order for these Jacobi fields to be bounded. Before ennoncing it, we need some definitions.

Let Σ be a compact Riemann surface and

$$X : M = \Sigma - \{p_1, \dots, p_k\} \longrightarrow \mathbb{R}^3$$

a minimal immersion with finite total curvature. The points $p_1, \dots, p_k$ are called *the ends of* X. Such an end $p_i \in \Sigma$ is *embedded* if X is injective on a punctured neighbourhood of p_i on Σ. It is said to be *planar* if there exists $a_i \in \mathbb{S}^2$ such that the harmonic function $\langle X, a_i \rangle$ is bounded near to p_i and said to be of *catenoid type* if there exists $a_i \in \mathbb{S}^2$ such that $\langle X, a_i \rangle$ has a logarithmic singularity at p_i. Notice that an embedded end is obliged to be planar or of catenoid type (see [Sch]) and that, in the literature, authors use these denominations only for embedded ends.

Proposition 1. *Let Σ be a compact Riemann surface and $X : \Sigma - \{p_1, \dots, p_k\} \to \mathbb{R}^3$ a complete minimal immersion with finite total curvature whose extended Gauss map is the holomorphic map $\phi : \Sigma \to \mathbb{S}^2$. We have that*

a) *If all the ends of X are either planar or of catenoid type and there exists $a \in \mathbb{S}^2$ such that $\phi(p_i) = \pm a$, $i = 1, \dots, k$, then $\langle X \wedge \phi, a\rangle \in N(\phi)$.*
b) *If all the ends of X are planar, then $\langle X, \phi\rangle \in N(\phi)$.*

Proof. As we have pointed out above, we only need to prove that these two functions defined on $\Sigma - \{p_1, \dots, p_k\}$ are bounded near to the ends $p_1, \dots, p_k$ of X. Fix $i \in \{1, \dots, k\}$ and suppose, by rotating $\mathbb{S}^2$, that $\phi(p_i)$ is the north pole. Now, we identify ϕ, via stereographic projection from this north pole, with a meromorphic function $g : \Sigma \to \overline{\mathbb{C}}$. So

$$\phi = \frac{1}{1 + |g|^2}\left(g + \overline{g}, -i(g - \overline{g}), -1 + |g|^2\right) \tag{3-1}$$

and, by using the Weierstrass representation, we have

$$X = \frac{1}{2}\Re \int \left(1 - g^2, i(1 + g^2), 2g\right)\omega \tag{3-2}$$

where ω is a meromorphic differential on Σ which is holomorphic away from the ends $p_1, \dots, p_k$. As $g(p_i) = \infty$, we can choose a local complex coordinate z on a certain neighbourhood D_i of p_i in Σ such that $z(p_i) = 0$ and

$$g(z) = \frac{1}{z^m}, \quad m \geq 1, \quad z \in D_i.$$

With respect to this local coordinate, we have

$$\omega = \left(\frac{a}{z^r} + \frac{b}{z^{r-1}} + \dots + \frac{c}{z} + h(z)\right) dz,\ z \in D_i,\ r \in \mathbb{Z}$$

where $a \in \mathbb{C} - \{0\}$ and h is holomorphic on D_i. Hence, as $r + 2m \geq 2$ because X is complete, if $z \in D_i$

$$X(z) = (1 - r - 2m)\left(\Re\frac{a}{z^{r+2m-1}}, \Im\frac{a}{z^{r+2m-1}}, 0\right) + Y(z) \tag{3-3}$$

where $Y(z)$ contains terms of lower growth.

On the other hand, we know that there exists $a_i \in \mathbb{S}^2$ such that $\langle X, a_i\rangle$ is either bounded or has a logarithmic singularity at p_i. In both two cases, as $r + 2m - 1 \geq 1$ and $a \neq 0$, we have from (3-3) that $a_i = (0, 0, \pm 1)$, that is, the third coordinate of X is bounded in the case b) or has a logarithmic singularity in the case a). We will deal with each case separately.

Case a). We have that

$$\langle X \wedge \phi, a_i\rangle = \det\big(X, \phi, \pm(0, 0, 1)\big) = \pm\det(\tilde{X}, \tilde{\phi})$$

where $\tilde{X}$ and $\tilde{\phi}$ are the orthogonal projections of X and ϕ on the plane $x_3 = 0$ in $\mathbb{R}^3$. So, from (3-1) and (3-3), we get

$$\langle X \wedge \phi, a_i \rangle = (1 - r - 2m)\Re \frac{ia}{z^{r+m-1}} + W(z) \tag{3-4}$$

where W includes only terms of lower growth. As the third coordinate of X has at p_i a logarithmic singularity and

$$X_3 = \Re \int \omega g = \Re \int \frac{1}{z^m} \left(\frac{a}{z^r} + \dots \right) dz,$$

then $m + r \leq 1$, and using (3-4) we conclude.

Case b). In this case the third coordinate of X is bounded at p_i and, so, $m + r \leq 0$. On the other hand, using (3-1) and (3-3), one has

$$\left\langle \tilde{X}, \tilde{\phi} \right\rangle = (1 - r - 2m)\Re \frac{a}{z^{r+m-1}} + W'(z)$$

and $X_3\phi_3$ is bounded from our hypothesis. So $\langle X, \phi \rangle = \left\langle \tilde{X}, \tilde{\phi} \right\rangle + X_3\phi_3$ is bounded.

Remark. The result above is also true for complete *branched* minimal surfaces. Moreover, notice that the hypothesis in a) is automatically satisfied when X is an embedding (see [J–M] for example).

Now, let $\phi : \Sigma \to \mathbb{S}^2$ be a non–constant holomorphic map. In terms of a local complex coordinate z, the fact that ϕ is holomorphic is given by

$$\langle \phi_z, \phi_z \rangle = 0$$

where $\langle , \rangle$ is the usual bilinear complex product on $\mathbb{C}^3$. By using this equality and $\langle \phi, \phi \rangle = 1$, we obtain the following equations:

$$\begin{array}{ll} \langle \phi, \phi_z \rangle = 0 & \phi_{zz} = (\partial \log |\phi_z|^2)\phi_z \\ \phi_{z\overline{z}} + |\phi_z|^2 \phi = 0 & \partial\overline{\partial} \log |\phi_z|^2 = -|\phi_z|^2 \end{array} \tag{3-5}$$

where ∂ and $\overline{\partial}$ denote differentiation with respect to the variables z and $\overline{z}$ respectively. Take a function $u \in N(\phi)$ in the nullity space of ϕ defined in (1-4). By using on Σ the Euclidean local metric $|dz|^2$ we have that

$$u_{z\overline{z}} + |\phi_z|^2 u = 0.$$

Let $B(\phi) = m_1 p_1 + \cdots + m_k p_k$ be the ramification divisor of ϕ. Then we define a map

$$X(u) : \Sigma - \{p_1, \dots, p_k\} \longrightarrow \mathbb{R}^3$$

by

$$X(u) = u\phi + \frac{1}{|\phi_z|^2} \left\{ u_z \phi_{\overline{z}} + u_{\overline{z}} \phi_z \right\}. \tag{3-6}$$

Clearly $X(u)$ is a real vector and the expression on the right side of (3-6) is independent of the choice of the isothermal parameter z. Differentiating with respect to z, one obtains, by using (3-5)

$$X(u)_z = \left(u_{zz} - (\partial \log |\phi_z|^2) u_z\right) \frac{\phi_{\overline{z}}}{|\phi_z|^2}. \tag{3-7}$$

Notice that $X(u)$ is a constant vector if $u \in L(\phi)$ defined in (1-6). Hence

$$\langle X(u)_z, \phi \rangle = 0 \quad \langle X(u)_z, \phi_{\overline{z}} \rangle = 0 \quad \langle X(u)_z, X(u)_z \rangle = 0.$$

In particular, $X(u)$ is a conformal map from $\Sigma - \{p_1, \ldots, p_k\}$ to $\mathbb{R}^3$. Moreover, by using (3-5) again

$$\begin{aligned} 0 &= \langle X(u)_z, \phi \rangle_{\overline{z}} = \langle X(u)_{z\overline{z}}, \phi \rangle \\ 0 &= \langle X(u)_z, \phi_{\overline{z}} \rangle_{\overline{z}} = \langle X(u)_{z\overline{z}}, \phi_{\overline{z}} \rangle . \end{aligned}$$

Since $X(u)_{z\overline{z}}$ is real, we also have that $\langle X(u)_{z\overline{z}}, \phi_z \rangle = 0$ and, so

$$X(u)_{z\overline{z}} = 0,$$

that is, $X(u)$ is harmonic.

As a conclusion, the map $X(u) : \Sigma - \{p_1, \ldots, p_k\} \to \mathbb{R}^3$ is a branched minimal immersion whose Gauss map is ϕ and whose support function is $\langle X(u), \phi \rangle = u$. To study the behaviour of the minimal immersion $X(u)$ at its ends $p_1, \ldots, p_k \in \Sigma$, it is convenient to identify ϕ with a meromorphic function $g : \Sigma \to \overline{\mathbb{C}}$ as in (3-1) and to use the Weierstrass representation (3-2) for $X(u)$. So, we have

$$X(u)_z \, dz = \frac{\omega}{2} \left(1 - g^2, i(1 + g^2), 2g\right).$$

Comparing this equation with (3-7), we can see that the meromorphic differential ω is

$$\omega = \frac{1}{g'} \left(u_{zz} + (\frac{2\overline{g} g'}{1 + |g|^2} - \frac{g''}{g'}) u_z \right) dz.$$

It is easy to prove that, at the point p_i, $i = 1, \ldots, k$, there exists a possible pole of ω with order less than or equal to $m_i + 1$. Then, the meromorphic quadratic differential σ on Σ given by

$$\sigma = \omega \, dg \tag{3-8}$$

has a possible pole with order less than or equal to one at each end p_i. Let z be a local complex coordinate on a certain neighbourhood of the point p_i on Σ such that $z(p_i) = 0$ and suppose that on this neighbourhood we have, up to a rotation of $\mathbb{R}^3$

$$g(z) = z^{m_i + 1}, \quad m_i \geq 1.$$

So, locally, from (3-8)

$$X(u)_z\,dz = \frac{\sigma}{2\,dg}\left(1-g^2, i(1+g^2), 2g\right) =$$
$$\frac{1}{2(m_i+1)}\left(\frac{1}{z^{m_i}} - z^{m_i+2}, i(\frac{1}{z^{m_i}} + z^{m_i+2}), 2z\right)\left(\frac{a}{z} + \sum_{i\geq 0} a_i z^i\right) dz$$

where

$$\left(\frac{a}{z} + \sum_{i\geq 0} a_i z^i\right)(dz)^2$$

is a local expression for σ around the point p_i. Then, the third coordinate of $X(u)_z\,dz$ is holomorphic at p_i and, so, $\langle X, (0,0,1)\rangle$ is bounded near to p_i. Moreover, as

$$X(u) = \Re \int X(u)_z\,dz,$$

we have $\Im\ \mathrm{Res}_{p_i}\ X(u)_z\,dz = 0$, that is $\Im a_{m_i-1}(1,i,0) = 0$. So, $a_{m_i-1} = 0$ and

$$\mathrm{Res}_{p_i} X(u)_z\,dz = 0.$$

As a consequence:

Proposition 2. *Let Σ be a compact Riemann surface and $\phi : \Sigma \to \mathbb{S}^2$ a holomorphic map. If $u \in N(\phi) - L(\phi)$, then the map $X(u)$ defined in (3-6) is a complete branched minimal immersion with finite total curvature and planar ends whose extended Gauss map is ϕ and whose support function is u.*

If we denote by $M(\phi)$ the linear space of all the complete branched minimal immersions (including the constant maps) into $\mathbb{R}^3$ with finite total curvature and planar ends whose extended Gauss map is ϕ, we have by taking into account the two linear maps:

$$\begin{aligned} u \in N(\phi) &\longmapsto X(u) \in M(\phi),\ \text{see (3-6)},\\ X \in M(\phi) &\longmapsto \langle X, \phi\rangle \in N(\phi), \end{aligned} \tag{3-9}$$

the following result:

Theorem 3. *Let Σ be a compact Riemann surface and $\phi : \Sigma \to \mathbb{S}^2$ a holomorphic map. Then, the linear spaces $N(\phi)$ and $M(\phi)$ are isomorphic via the linear maps given in* (3-9)*. Moreover, we have an induced isomorphism*

$$\frac{N(\phi)}{L(\phi)} \cong \frac{M(\phi)}{\textit{constants}}.$$

Proof. It suffices to prove that the linear maps given in (3-9) are inverse to each other and use Proposition 1 and Proposition 2. The second assertion is nothing but a consequence of the fact that the first map in (3-9) applies the linear functions of $L(\phi)$ defined in (1-6) onto the subspace of $M(\phi)$ consisting of the constants.

Remark. Combining Conclusion 2 and this Theorem 3, one can obtain a relation between branched Willmore surfaces in $\mathbb{S}^3$ and complete branched minimal surfaces in $\mathbb{R}^3$ with finite total curvature and planar ends. This relation was obtained by Bryant in [Br1] in the non–ramified case.

By using the Weierstrass representation (3-2), we can identify each element X of $M(\phi)$ with a pair (g,ω), where g is the meromorphic function $g:\Sigma\to\overline{\mathbb{C}}$ associated to ϕ by means of (3-1) and ω is a meromorphic differential on Σ which is holomorphic on $\Sigma-\{p_1,\ldots,p_k\}$, where $B(\phi)=m_1p_1+\cdots+m_kp_k$ is the ramification divisor of ϕ. If $q_1,\ldots,q_d\in\Sigma$ are the poles of g, we define a divisor $D(\phi)$ on Σ by

$$D(\phi)=\sum_{i=1}^{k}(m_i+1)p_i-2\sum_{i=1}^{d}q_i.$$

Then, Theorem 3 above can be reformulated in the following way:

Theorem 4. *The linear space $N(\phi)/L(\phi)$ is isomorphic to*

$$\left\{\omega\in H^0\big(\kappa_\Sigma\otimes[D(\phi)]\big)\,|\,\mathrm{Res}_{p_i}\omega=0,\ \Re\int_\gamma\omega\left(1-g^2,i(1+g^2),2g\right)=0,\ \gamma\in H_1(\Sigma,\mathbb{Z})\right\}$$

where κ_Σ is the canonical line bundle on Σ and $[D(\phi)]$ is the line bundle associated to the divisor $D(\phi)$.

Notice that the space $N(\phi)/L(\phi)$ can also be described in terms of the meromorphic quadratic differential σ defined in (3-8). In fact

Theorem 5. *The linear space $N(\phi)/L(\phi)$ is isomorphic to*

$$\left\{\sigma\in H^0\Big(2\kappa_\Sigma\otimes\Big[\sum_{i=1}^{k}p_i\Big]\Big)\ |\ \begin{cases}Res_{p_i}\dfrac{\sigma}{dg}=0,\\ \Re\displaystyle\int_\gamma\frac{\sigma}{dg}\left(1-g^2,i(1+g^2),2g\right)=0,\ \gamma\in H_1(\Sigma,\mathbb{Z})\end{cases}\right\}.$$

Holomorphic maps of low index. We shall begin by gathering some information concerning the index of the holomorphic maps from a compact Riemann surface to the two–sphere and by trying to answer the following natural question: What can one say about these holomorphic maps which have the least index? Of course, we are excluding the trivial case of index zero. So, we are looking for the properties that such a holomorphic map must satisfy in order to have index one. This case is doubly interesting because it includes, from Conclusions 1 and 3, complete minimal surfaces of finite total curvature with the lowest index and branched metrics which are local maxima of the detrminant of the Laplacian functional. These metrics should provide classical models of vacuum states for the Polyakov quantum string theory.

Theorem 6. *Let Σ be a compact Riemann surface and $\phi : \Sigma \to \mathbb{S}^2$ a holomorphic map with index one. Then, there are no holomorphic maps from Σ to $\mathbb{S}^2$ with degree less than $\deg\phi$. Moreover, if $\psi : \Sigma \to \mathbb{S}^2$ is a holomorphic map with $\deg\psi = \deg\phi$, then $\psi = G \circ \phi$ for a certain conformal transformation G of $\mathbb{S}^2$.*

Proof. From the definition (1-2), we have

$$Q_\phi(1,1) = -\int_\Sigma |\nabla\phi|^2 \, dA = -8\pi \deg\phi < 0,$$

since ϕ cannot be a constant map. As the index of ϕ is one, we obtain, from (1-3), that the form Q_ϕ is positive semidefinite on the hyperplane of $W_1(\Sigma)$ orthogonal to the constants with respect to the bilinear form associated to Q_ϕ. That is,

$$\int_\Sigma \left\{|\nabla u|^2 - |\nabla\phi|^2 u^2\right\} dA \geq 0 \quad \text{if} \quad \int_\Sigma |\nabla\phi|^2 u \, dA = 0$$

and the equality happens if and only if $u \in N(\phi)$.

Let $\psi : \Sigma \to \mathbb{S}^2$ be any holomorphic map. As $|\nabla\phi|^2$ is positive except at a finite set, we can use a known result by Hersch [He] to obtain a conformal transformation C of $\mathbb{S}^2$ such that

$$\int_\Sigma |\nabla\phi|^2 (C \circ \psi) \, dA = 0.$$

Then

$$\begin{aligned} 0 \leq \int_\Sigma & \left\{|\nabla(C\circ\psi)|^2 - |\nabla\phi|^2 |C\circ\psi|^2\right\} dA = \\ & \int_\Sigma \left\{|\nabla\psi|^2 - |\nabla\phi|^2\right\} dA = 8\pi(\deg\psi - \deg\phi). \end{aligned}$$

Hence $\deg\phi \leq \deg\psi$ for each $\psi : \Sigma \to \mathbb{S}^2$ holomorphic. If the equality held, then we would have that $\langle C\circ\psi, a\rangle \in N(\phi)$ for all $a \in \mathbb{R}^3$ and, as $\langle C\circ\psi, a\rangle \in N(C\circ\psi)$, we would obtain that $|\nabla\phi|^2 = |\nabla(C\circ\psi)|^2$. By using Lemma 7 below one would get an isometry A of $\mathbb{S}^2$ such that $\phi = A \circ C \circ \psi$.

Lemma 7. *Let Σ be a compact Riemann surface and $\phi, \psi : \Sigma \to \mathbb{S}^2$ two holomorphic maps such that $|\nabla\phi|^2 = |\nabla\psi|^2$ with respect to any metric compatible with the complex structure. Then, there exists an isometry $A : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\psi = A \circ \phi$.*

Proof. As $|\nabla\phi|^2 = |\nabla\psi|^2$, we have $N(\phi) = N(\psi)$ from the definition (1-4). Fix $a \in \mathbb{S}^2$ and put $u = \langle\psi, a\rangle$. Then $u \in N(\phi)$ from (1-6). Hence, the map $X(u)$ defined in (3-6) is harmonic on $\Sigma - \{p_1, \ldots, p_k\}$, where $p_1, \ldots, p_k$ are the ramification points of ϕ. But, in our case,

$$\begin{aligned} |X| =& u^2 + 2\frac{|u_z|^2}{|\phi_z|^2} = \langle\psi, a\rangle^2 + 2\frac{|\langle\psi_z, a\rangle|^2}{|\phi_z|^2} \\ & \leq 1 + 2\frac{|\psi_z|^2}{|\phi_z|^2} = 3. \end{aligned}$$

So, X is bounded and, hence, is a constant vector map $b \in \mathbb{R}^3$. Then

$$\langle \psi, a \rangle = u = \langle X, \phi \rangle = \langle b, \phi \rangle \,.$$

This occurs for each $a \in \mathbb{S}^2$. Taking into account that $|\psi|^2 = |\phi|^2 = 1$, one can conclude the proof.

From Theorem 6, we are led to consider another question: Which is the least degree that a meromorphic function can have on a given compact Riemann surface? And, how many meromorphic functions, up to a Möbius transformation, attain that degree? This is an old problem ant the Brill–Noether theorem tries to answer it (see [Fa–Kr] or [Gr–Ha]). In fact, this theorem asserts that, for the generic compact Riemann surface of genus greater than two, the least possible degree is

$$1 + \left[\frac{1 + \text{genus}\ \Sigma}{2}\right]$$

and that there is no uniqueness up to Möbius transformations. The cases of genus zero or one are trivial and, so, we have the following result.

Corollary 8. *Let $\phi : \Sigma \to \mathbb{S}^2$ a holomorphic map defined on a compact Riemann surface. We have:*

a) *If* genus $\Sigma = 0$, *then* Ind $\phi = 1 \Leftrightarrow \deg \phi = 1$.
b) *If* genus $\Sigma = 1$, *then* Ind $\phi = 1$ *is impossible.*
c) *For the generic complex structure with genus greater than two,* Ind $\phi = 1$ *is impossible.*
d) *In all the cases* $\deg \phi \leq 1 + \left[\frac{1+\text{genus}\ \Sigma}{2}\right]$, *provided that Ind $\phi = 1$.*

Suppose now that the holomorphic map $\phi : \Sigma \to \mathbb{S}^2$ that we are considering is the extended Gauss map of a complete minimal surface in $\mathbb{R}^3$ with finite total curvature. Then, we have a non–trivial relation between the degree of ϕ and the genus of the surface Σ. In fact, we know from a result by Osserman [Oss] that

$$\deg \phi \geq 1 + \text{genus}\ \Sigma.$$

So, combining this information with d) in Corollary 8, one has two possible genera for the surface Σ, say zero or one. This latter cannot occur from b). Then, Σ must have genus zero and the map ϕ degree one. Now, we can conclude from another result by Osserman [Oss] the following statement.

Corollary 9. [Lo–R]. *The catenoid and the Enneper surface are the only complete minimal surfaces in $\mathbb{R}^3$ with index one.*

With respect to the determinant of the Laplacian functional, we must remark that we do not know any holomorphic map, in the non–zero genus case, with index one. But, according to Theorem 7, one should look for them, for instance, on a hyperelliptic Riemann surface. In fact, we think that the two–covering map from such a surface to the two–sphere has index one, provided that the images of its Weierstrass points are spread enough on the sphere.

Estimates for the index and the nullity. The case of low index above points out that there is a strong connection between the index of a holomorphic map‘and its degree. In fact, in a final remark of her work about index of minimal surfaces in three–dimensional manifolds [FC], Fischer-Colbrie said that one should be able to get an explicit relation between the index of a holomorphic map from a compact Riemann surface to the two–sphere and its geometry. Even more, Gulliver [Gu] suggested a concrete inequality involving the index and the degree of such a holomorphic map and the genus of the surface. In the last five years, a lot of work have been dedicated to obtain this explicit relation. For arbitrary genus, the first approach was done by Tysk in [Ty]. He obtained the following estimate:

Theorem 10. [Ty]. *If $\phi : \Sigma \to \mathbb{S}^2$ is a holomorphic map defined on a compact Riemann surface, then*

$$\text{Ind}\,\phi < 7.7 \deg \phi.$$

The proof is based on comparing the heat kernel of the standard metric ds_0^2 on $\mathbb{S}^2$ and that of the induced metric ds_ϕ^2 on Σ, see (1-7). This is probably a rough estimate as we will see in the sequel.

Notice that any assertion about the index and the nullity os a holomorphic map $\phi : \Sigma \to \mathbb{S}^2$ can be read as an assertion about the multiplicities of some eigenvalues of the Laplacian Δ_ϕ of the branched metric ds_ϕ^2, as we had remarked in (1-10) and (1-11). From this point of view and by using standard methods such as Courant's nodal theorem [Co–Hi], we will obtain some estimates for the index and the nullity of the holomorphic map ϕ when its branching values are in especial positions on the sphere. In the particular case in which all the branching values lie in an equator, we will get a sharp result about the multiplicities of the eigenvalues of Δ_ϕ which come from the spectrum of the standard two–sphere. As a consequence, we will compute the index and the nullity for such a holomorphic map.

We will start with the following more or less well–known result:

Lemma 11. *Let Ω be a two–dimensional half–sphere of radius one endowed with the standard metric ds_0^2. Then, we have that:*

a) *The eigenvalues of the Laplacian for the Neumann problem on Ω are $\lambda_k = k(k+1)$ with multiplicities $m_k = k+1$, $k = 0, 1, 2, \dots$*
b) *The eigenvalues of the Laplacian for the Dirichlet problem on Ω are $\mu_k = k(k+1)$ with multiplicities $n_k = k$, $k = 1, 2, 3 \dots$*

Sketch of proof. Denote by $A : \mathbb{S}^2 \to \mathbb{S}^2$ the symmetry with respect to the plane where the equator of Ω lies. Then, each eigenfunction for the Neumann problem on Ω can be extended on the whole of $\mathbb{S}^2$ to obtain an A-invariant eigenfunction of the Laplacian of ds_0^2. Also, each eigenfunction for the Dirichlet problem on Ω provides us, after a suitable extension, an A-antiinvariant eigenfunction of the Laplacian of ds_0^2. Now, it suffices to know that $k(k+1)$, $k = 0, 1, 2, \dots$, are the eigenvalues of the standard metric ds_0^2 on $\mathbb{S}^2$ with multiplicities $2k+1$ and to analyse which of the corresponding eigenfunctions are either A-invariant or A-antiinvariant.

Now, if $\phi : \Sigma \to \mathbb{S}^2$ is a holomorphic map defined on a compact Riemann surface, from the definition (1-7) of the metric ds^2_ϕ induced on Σ by ϕ, we have that

$$\text{Spec }\Delta_0 \subset \text{Spec }\Delta_\phi$$

where Δ_0 is the Laplacian of the standard metric ds^2_0 on $\mathbb{S}^2$. Then, for $k = 0, 1, 2, \dots$

$$k(k+1) \in \text{Spec }\Delta_\phi$$

with multiplicity at least $2k+1$. The following computations seek to staudy which is the exact multiplicity of $k(k+1)$ as an eigenvalue of Δ_ϕ and how many eigenvalues of Δ_ϕ are between $k(k+1)$ and $(k+1)(k+2)$.

Lemma 12. *Let $\Omega_1, \dots, \Omega_m$ be a partition of a compact Riemann surface Σ by domains with piecewise smooth boundary. If $\phi : \Sigma \to \mathbb{S}^2$ is a non–constant holomorphic map, then*

$$\#\{\Delta_\phi\text{-eigenvalues } \lambda \text{ on } \Sigma \text{ with } \lambda < k(k+1)\} \geq$$
$$\sum_{i=1}^{m-1} \#\{\Delta_\phi\text{-eigenvalues } \lambda \text{ for the Dirichlet problem on } \Omega_i \text{ with } \lambda \leq k(k+1)\} +$$
$$\#\{\Delta_\phi\text{-eigenvalues } \lambda \text{ for the Dirichlet problem on } \Omega_m \text{ with } \lambda < k(k+1)\}$$

for any $k = 1, 2, 3, \dots$

Proof. Fix $k = 1, 2, 3, \dots$ and denote by V_i, $i = 1, \dots, m-1$, the linear space spanned by the eigenfunctions of Δ_ϕ for the Dirichlet problem on the domain Ω_i whose corresponding eigenvalues are less than or equal to $k(k+1)$. In the same way, let V_m be the linear space spanned by the eigenfunctions of Δ_ϕ for the Dirichlet problem on Ω_m with eigenvalues less than $k(k+1)$. These $V_1, \dots, V_{m-1}, V_m$ can be viewed as subspaces of the Sobolev space $W_1(\Sigma)$ by extending each function that they contain by zero on the whole of Σ. Finally, we denote by V the linear subspace of $W_1(\Sigma)$ spanned by the eigenfunctions of Δ_ϕ on Σ with corresponding eigenvalues less than $k(k+1)$. Afetr this, the assertion that we want to prove in this lemma can be written in the following way:

$$\dim V \geq \sum_{i=1}^{m} \dim V_i. \tag{3-10}$$

But, as a direct consequence of the definitions of these linear spaces, we have that, for any function $u \in W_1(\Sigma)$ which is $L^2(\Sigma)$-orthogonal to V with respect to the metric ds^2_ϕ

$$\int_\Sigma |\nabla u|^2 \, dA_\phi \geq k(k+1) \int_\Sigma u^2 \, dA_\phi \tag{3-11}$$

and the equality holds if and only if u is a $k(k+1)$-eigenfunction of Δ_ϕ on Σ. Also, for any $u \in \bigoplus_{i=1}^m V_i$, we have

$$k(k+1) \int_\Sigma u^2 \, dA\phi \leq \int_\Sigma |\nabla u|^2 \, dA_\phi \tag{3-12}$$

and the equality holds if and only if the V_i-component of u is a $k(k+1)$-eigenfunction of Δ_ϕ for the Dirichlet problem on Ω_i, $i = 1, \ldots, m$. In particular, the V_m-component of u vanishes.

If the inequality (3-10) were not true, then we could find a non–zero function $u \in \bigoplus_{i=1}^m V_i$ and $L^2(\Sigma)$-ortyhogonal to V. This function would satisfy (3-11) and (3-12). Hence, it would be a $k(k+1)$-eigenfunction of Δ_ϕ on Σ identically zero on Ω_m. By using the unique continuation principle [A], u would vanish on Σ. This contradiction proves (3-10).

Lemma 13. *Under the same hypothesis as Lemma 12, we have that*

$$\#\{\Delta_\phi\text{-eigenvalues } \lambda \text{ on } \Sigma \text{ with } \lambda \leq k(k+1)\} \leq$$
$$\sum_{i=1}^{m-1} \#\{\Delta_\phi\text{-eigenvalues } \mu \text{ for the Neumann problem on } \Omega_i \text{ with } \mu < k(k+1)\} +$$
$$\#\{\Delta_\phi\text{-eigenvalues } \mu \text{ for the Neumann problem on } \Omega_m \text{ with } \mu \leq k(k+1)\}$$

for any $k = 0, 1, 2, \ldots$

Proof. For a fixed $k = 0, 1, 2, \ldots$, let V_i be the linear space spanned by the eigenfunctions of Δ_ϕ for the Neumann problem on Ω_i, $i = 1, \ldots, m-1$, with eigenvalues less than $k(k+1)$ and V_m be the space spanned by the eigenfunctions of Δ_ϕ for the Neumann problem on Ω_m with corresponding eigenvalues less than or equal to $k(k+1)$. Then, if $u \in W_1(\Sigma)$ is $L^2(\Sigma)$-orthogonal to each V_i, for $i = 1, \ldots, m$, we have

$$\text{(3-13)} \qquad \begin{aligned} \int_\Sigma |\nabla u|^2 \, dA\phi = \sum_{i=1}^m \int_{\Omega_i} |\nabla u|^2 \, dA\phi \geq \\ k(k+1) \sum_{i=1}^m \int_{\Omega_i} u^2 \, dA_\phi = k(k+1) \int_\Sigma u^2 \, dA_\phi \end{aligned}$$

and the equality holds if and only if $u_{|\Omega_i}$ is a $k(k+1)$-eigenfunction of Δ_ϕ for the Neumann problem on Ω_i, $i = 1, \ldots, m$, and, so, $u_{|\Omega_m} \equiv 0$.

On the other hand, if V denotes the subspace of $W_1(\Sigma)$ spanned by the eigenfunctions of Δ_ϕ on Σ with eigenvalues less than or equal to $k(k+1)$, then each function $u \in V$ satisfies

$$\text{(3-14)} \qquad \int_\Sigma |\nabla u|^2 \, dA_\phi \leq k(k+1) \int_\Sigma u^2 \, dA_\phi$$

and the equality holds if and only if u is a $k(k+1)$-eigenfunction of Δ_ϕ on Σ. But the claim of this lemma can be rewritten in this way

$$\dim V \leq \sum_{i=1}^m \dim V_i.$$

If this inequality failed to be true, we could choose $u \in V - \{0\}$ and such that $u_{|\Omega_i}$ would be orthogonal to V_i for each $i = 1, \ldots, m$. Hence, the function u would satisfy

(3-13) and (3-14) and be a $k(k+1)$-eigenfunction of Δ_ϕ on Σ with $u_{|\Omega_m} \equiv 0$. Then, we prove the lemma by using again the unique continuation principle.

These two lemmae can be combined in order to obtain an accurate estimate in the following theorem, where we will find a partition of the surface Σ by domains which have a good behaviour with respect to the Dirichlet and the Neumann problem corresponding to the operator Δ_ϕ.

Theorem 14. *Let Σ be a compact Riemann surface and $\phi : \Sigma \to \mathbb{S}^2$ a non–constant holomorphic map. If all the branching values of ϕ lie in an equator of $\mathbb{S}^2$, then, for $k = 0, 1, 2, \ldots$*

a) *The multiplicity of $k(k+1)$ as an eigenvalue of Δ_ϕ on Σ is $2k+1$.*
b) *The number of eigenvalues λ of Δ_ϕ on Σ such that $k(k+1) < \lambda < (k+1)(k+2)$ is $2(k+1)(\deg \phi - 1)$.*

Proof. We will denote by C the equator of $\mathbb{S}^2$ which contains all the branching values of ϕ and put $d = \deg \phi$. Then $\Sigma - \phi^{-1}(C)$ has exactly $2d$ connected components $\Omega_1, \ldots, \Omega_{2d}$ in such a way that $\phi_{|\Omega_i}$ is an isometry onto a half–sphere for each i. For any $k = 0, 1, 2, \ldots$, it follows from Lemma 11 that

$$\sum_{i=1}^{2d-1} \#\{\Delta_\phi\text{-eigenvalues } \lambda \text{ for the Dirichlet problem on } \Omega_i \text{ with } \lambda \le k(k+1)\} + \\ \#\{\Delta_\phi\text{-eigenvalues } \lambda \text{ for the Dirichlet problem on } \Omega_m \text{ with } \lambda < k(k+1)\} = \\ k(k+1)d - k,$$

and that

$$\sum_{i=1}^{2d-1} \#\{\Delta_\phi\text{-eigenvalues } \mu \text{ for the Neumann problem on } \Omega_i \text{ with } \mu < k(k+1)\} + \\ \#\{\Delta_\phi\text{-eigenvalues } \mu \text{ for the Neumann problem on } \Omega_m \text{ with } \mu \le k(k+1)\} = \\ k(k+1)d + k + 1.$$

From these observations, we obtain by using Lemma 12 and Lemma 13 that

$$\#\{\Delta_p hi\text{-eigenvalues } \lambda \text{ on } \Sigma \text{ with } \lambda < k(k+1)\ \} \ge k(k+1)d - k \\ \#\{\Delta_\phi\text{-eigenvalues } \lambda \text{ on } \Sigma \text{ with } \lambda \le k(k+1)\} \le k(k+1)d + k + 1.$$

As we know that the multiplicity of $k(k+1)$ as an eigenvalues of Δ_ϕ on Σ is at least $2k+1$, we get from the above inequalities that

$$\#\{\Delta_\phi\text{-eigenvalues } \lambda \text{ on } \Sigma \text{ with } \lambda < k(k+1)\ \} = k(k+1)d - k. \\ \text{multiplicity of the } \Delta_\phi\text{-eigenvalue } k(k+1) \text{ on } \Sigma = 2k+1.$$

Now the proof of the theorem follows from a simple arithmetic.

As a direct consequence of Theorem 14, we can compute the index and the nullity defined in (1-3) and (1-5) for a holomorphic map such that all its branching values are mapped on an equator of the two–sphere.

Corollary 15. *If $\phi : \Sigma \to \mathbb{S}^2$ is a non–constant holomorphic map defined on a compact Riemann surface Σ all of whose branching values lie in an equator of $\mathbb{S}^2$, then*

$$\text{Nul } \phi = 3 \qquad \text{Ind } \phi = 2 \deg \phi - 1.$$

Partial versions of this Corollary have been obtained by Li and Tam, see [Li–Tr], Nayatani [N] and Choe [Cho].

This result has a first obvious application, taking into account Conclusion 1, to compute the index of complete minmal surfaces in $\mathbb{R}^3$ with finite total curvature whose extended Gauss map behaves in that especial way. For instance, the index of the Jorge–Meeks $(k+1)$-catenoid described in [J–M] is $2k+1$ because, in this case, $\Sigma = \overline{\mathbb{C}}$ and $\phi(z) = z^k$ and the index of the genus one Chen–Gackstätter surface, recently characterised by López [Lo], is three because $\Sigma = \mathbb{C}/\{1, i\}$ and $\phi \equiv a\wp'/\wp$ where $a \in \mathbb{R}$ and $\wp$ is the Weierstrass function associated to the lattice spanned by $\{1, i\}$ in $\mathbb{C}$.

Another application is the following: let $X : M = \Sigma - \{p_1, \ldots, p_k\} \to \mathbb{R}^3$ be a non–flat complete minimal immersion with finite total curvature and suppose that all the branching values of its extended Gauss map ϕ lie in an equator of the sphere. From Corollary 15 and (1-6), we have that the nullity space $N(\phi)$ of ϕ coincides with $L(\phi) = \{\langle \phi, a \rangle \mid a \in \mathbb{R}^3$. If all the ends of X were planar, then Proposition 1 would assert us that the support function $\langle X, \phi \rangle$ of the surface belongs to $L(\phi)$. But this is impossible from Theorem 3. In a similar way, if all the ends of X were parallel and either planar or of catenoid type, then, from Proposition 1, we would have that the function $\langle X \wedge \phi, a \rangle$ lies in $L(\phi)$. But it is a classical result that this occurs if and only if X is a finite Riemannian covering of the catenoid. Hence, we can state:

Corollary 16. *Let M be a non–flat complete minimal surface in $\mathbb{R}^3$ with finite total cutvature such that all the branching values of its extended Gauss map lie in an equator of $\mathbb{S}^2$. Then*

a) *M has at least a non–planar end.*
b) *If all the ends of M are parallel and either planar or of catenoid type, then M is a finite Riemannian covering of the catenoid.*

For later applications we need to estimate the index and the nullity for holomorphic maps in somewhat more general situations.

Theorem 17. *Let $\phi : \Sigma \to \mathbb{S}^2$ be a non–constant holomorphic map defined on a compact Riemann surface and C be an equator of $\mathbb{S}^2$ such that $\Sigma - \phi^{-1}(C)$ has m connected components $\Omega_1, \ldots, \Omega_m$. If, for each $i = 1, \ldots, m$, the branching values of $\phi_{|\Omega_i}$ lie in another equator C_i of $\mathbb{S}^2$ orthogonal to C, then we have*

$$\text{Ind } \phi \geq m - 1 \quad \text{Ind } \phi + \text{Nul } \phi \leq 4 \deg \phi + 2 - m.$$

Proof. For each connected component Ω_i, $i = 1, \ldots, m$, of $\Sigma - \phi^{-1}(C)$, we will consider the compact Riemann surface Σ_i constructed by glueing two isometric copies of Ω_i along their boundaries and the involutive isometry S_i of Σ_i which permutes both copies. Define a holomorphic map $\phi_i : \Sigma_i \to \mathbb{S}^2$ by extending $\phi_{|\Omega_i}$ by means of the symmetry S_i. So, we have that $\phi_i \circ S_i = S \circ \phi_i$, where S is the symmetry of the two–sphere with respect to the equator C. Moreover, from our hypothesis, all the branching values of ϕ_i lie in the equator C_i. If we denote by d_i the degree of ϕ_i, then $2\deg\phi = d_1 + \cdots + d_m$ and, from Corollary 15,

$$\text{Nul } \phi_i = 3 \qquad \text{Ind } \phi_i = 2d_i - 1.$$

Since each Ω_i was defined as a nodal domain of a linear function of ϕ, the first eigenvalue of the operator L_ϕ, see (1-8), for the Dirichlet problem on Ω_i is equal to zero. Now, let $f \in V_\lambda(\phi_i)$, see (1-9), a λ-eigenfunction of L_{ϕ_i} on Σ_i which is antiinvariant by the involution S_i. Then f must vanish on the boundary of Ω_i. But we have seen that L_ϕ has no negative eigenvalues on Ω_i. So, we conclude that each eigenfunction in $V_\lambda(\phi_i)$ with $\lambda < 0$ is invariant with respect to S_i. Hence $V_\lambda(\phi_i)_{|\Omega_i}$ consists of eigenfunctions for the Neumann problem of L_ϕ on Ω_i. Coversely, each such eigenfunction on Ω_i, extended via S_i on the whole of Σ_i lies in some of the $V_\lambda(\phi_i)$. As a conclusion there are exactly $2d_i - 1$ eigenfunctions of L_ϕ for the Neumann problem on Ω_i associated to negative eigenvalues. An analogous reasoning shows us that the multiplicity of the zero eigenvalue for the Neumann problem of L_ϕ on Ω_i is two. Now, the proof of this theorem follows directly from Lemmae 12 and 13 putting $k = 1$ and taking into account that $L_\phi = \Delta_\phi + 2$.

The genus zero case. From now on, we will assume that our surface Σ has genus zero, that is, Σ is conformally equivalent to $\overline{\mathbb{C}}$. Let $\phi : \overline{\mathbb{C}} \to \mathbb{S}^2$ be a non–constant holomorphic map and put $d = \deg\phi \geq 1$. If $B(\phi) = m_1p_1 + \cdots + m_kp_k$, $m_1 + \cdots + m_k = 2d - 2$, is the ramification divisor of ϕ, we have, from Theorem 5 and the fact that $H_1(\overline{\mathbb{C}}, \mathbb{Z}) = 0$, that

$$\frac{N(\phi)}{L(\phi)} \cong \Big\{\sigma \in H^0\big(2\kappa_{\overline{\mathbb{C}}} \otimes [\sum_{i=1}^{k} p_i]\big) \,|\, \text{Res}_{p_i}\frac{\sigma}{dg} = 0\Big\}, \tag{3-15}$$

where $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is the meromorphic function associated to ϕ in (3-1). The following results are consequences of this isomorphism.

Proposition 18. *For each holomorphic map $\phi : \overline{\mathbb{C}} \to \mathbb{S}^2$, we have* Nul $(\phi) = 3 + 2k$ *for some integer k.*

Proof. It is sufficient to observe that $N(\phi)/L(\phi)$ is a complex linear space according to (3-15).

Theorem 19. *Let $\phi : \overline{\mathbb{C}} \to \mathbb{S}^2$ be a holomorphic map. Then*

$$\text{Nul } (A \circ \phi) = \text{Nul } \phi \quad \text{Ind } (A \circ \phi) = \text{Ind } \phi$$

for each conformal transformation A of $\mathbb{S}^2$.

Proof. If g is the meromorphic function associated to ϕ, then the corresponding meromorphic function h associated to $A \circ \phi$ is given by

$$h = \frac{\alpha g + \beta}{\gamma g + \delta} \text{ with } \alpha, \beta, \gamma, \delta \in \mathbb{C} \text{ and } \alpha\delta - \beta\gamma = 1.$$

The ramification divisors of ϕ and $A\circ\phi$ coincide. So, if $\sigma \in H^0\big(2\kappa_{\overline{\mathbb{C}}}\otimes[\sum_{i=1}^k p_i]\big)$ with $\mathrm{Res}_{p_i}(\sigma/dg)=0$, we have

$$\mathrm{Res}_{p_i}\frac{\sigma}{dh}=\mathrm{Res}_{p_i}\frac{\sigma(\gamma g+\delta)^2}{dg}=0$$

because $\mathrm{Res}_{p_i}(\sigma g^j/dg)=0$, $j=1,\ldots,m_i$, since $g^{j)}(p_i)=0$. From (3-15), we conclude that

$$\frac{N(\phi)}{L(\phi)}\cong\frac{N(A\circ\phi)}{L(A\circ\phi)}$$

and, so, Nul $\phi=$Nul$(A\circ\phi)$. With respect to the index, as the Lie group $\mathcal{C}(\mathbb{S}^2)$ of the conformal transformations of $\mathbb{S}^2$ is connected, we can take a continuous curve $A_t\in\mathcal{C}(\mathbb{S}^2$, $t\in[0,1]$, with $A_0=I$ and $A_1=A$. Consider now the continuous family $Q_t=Q_{A_t\circ\phi}$ of quadratic forms on $W_1(\overline{\mathbb{C}})$ defined in (1-2). As we already know that the nullity of Q_t has constant dimension and since the eigenvalues of Q_t depend continuously on t, we get that index Q_t is constant. In particular, Ind $\phi=$ index $Q_0=$ index $Q_1=$ Ind $(A\circ\phi)$.

From this last invariance theorem, we will be able to obtain a lower bound for the index of a holomorphic map defined on $\overline{\mathbb{C}}$. Firstly, we need to prove the following lemma.

Lemma 20. *Given an integer $d\geq 1$ and a point $a\in\mathbb{S}^2$, there exists a real number $\varepsilon>0$ such that, if $\phi:\Sigma\to\mathbb{S}^2$ is a holomorphic map on a compact Riemann surface with* $\deg\phi=d$ *satisfying:*

a) $\#\phi^{-1}(a)=k$,

b) *the branching values of ϕ different from a are at distance less than ε from $-a$,*

then

$$\mathrm{Ind}\ \phi\geq 2d-k.$$

Proof. Let $d_1,\ldots,d_k$ be the multiplicities of ϕ at the points of $\phi^{-1}(a)$. Take a number $0<r<\pi$ and denote by $\Omega_r\subset\Sigma$ the inverse image by ϕ of a geodesic disk of $\mathbb{S}^2$ with radius $\pi-r$ and center a. Then, the open set Ω_r is isometric to the disjoint union of geodesic disks of radius $\pi-r$ and center at the origin of $(\overline{\mathbb{C}},ds_{\phi_i^2})$, where $\phi_i:\overline{\mathbb{C}}\to\mathbb{S}^2$ is given by $z\mapsto z^{d_i}$, . From standard properties of eigenvalues, we have that Ind ϕ is greater than or equal to the number of negative eigenvalues of L_ϕ for the Dirichlet problem on Ω_r. When r goes to zero this number gets close to

$$\sum_{i=1}^k\mathrm{Ind}\ \phi_i=\sum_{i=1}^k(2d_i-1)=2d-k,$$

from Corollary 15.

Theorem 21. *Let $\phi : \overline{\mathbb{C}} \to \mathbb{S}^2$ be a non–constant holomorphic map. If m is the greatest ramification order of ϕ, then*

$$\text{Ind } \phi \geq m + \deg \phi.$$

So, if $\deg \phi > 1$, we have that

$$\text{Ind } \phi \geq 1 + \deg \phi$$

and, if ϕ has a ramification point with order $\deg \phi - 1$, then

$$\text{Ind } \phi \geq 2 \deg \phi - 1.$$

Proof. Given a ramification point $p \in \overline{\mathbb{C}}$ with order m, we choose an 1-parameter subgroup F_t, $t \in \mathbb{R}$, of conformal transformations of the two–sphere such that

$$F_t\big(\phi(p)\big) = \phi(p) \quad \text{for all } t \in \mathbb{R}$$

and

$$\lim_{t \to \infty} F_t(q) = -\phi(p) \quad \text{for all } q \in \mathbb{S}^2 - \{\phi(p)\}.$$

Then, for great enough $t \in \mathbb{R}$, $F_t \circ \phi : \overline{\mathbb{C}} \to \mathbb{S}^2$ is a holomorphic map with $\deg(F_t \circ \phi) = \deg \phi$ and such that all its branching values, except $\phi(p)$, are as close as one wants to $-\phi(p)$. So, by using Lemma 20 above and Theorem 19, we obtain

$$m + \deg \phi \leq 2 \deg \phi - \#\phi^{-1}\big(\phi(p)\big) \leq \text{Ind } (F_t \circ \phi) = \text{Ind } \phi.$$

Corollary 22. *Let $\phi : \overline{\mathbb{C}} \to \mathbb{S}^2$ be a holomorphic map. Then*

a) $\text{Ind } \phi = 0$ *if and only if ϕ is constant.*
b) $\text{Ind } \phi = 1$ *if and only if* $\deg \phi = 1$.
c) *The case* $\text{Ind } \phi = 2$ *is not possible.*
d) $\text{Ind } \phi = 3$ *if and only if* $\deg \phi = 2$.

Proof. The statements a), b) and c) are already known, see [Cho] and [N], and follow easily from our Theorem 21. If we have $\text{Ind } \phi = 3$, then $1 + \deg \phi \leq 3$ and, so, $\deg \phi = 2$. Conversely, if $\deg \phi = 2$, then ϕ has only two branching values and we can apply Corollary 15.

Now we will return to consider the isomorphism (3-15) which will allow us to achieve a satisfactory description of the nullity and, as a consequence, of the index of holomorphic maps defined on the completed complex plane. Let $g : \mathbb{C} \to \mathbb{C}$ be a meromorphic function with degree $d > 1$ and $z_1, \ldots, z_{2d-2}$ its ramification points which we will suppose all with order one. Moreover, without lost of generality, we may assume that $z_i \neq \infty$ and $g(z_i) \neq \infty$, $i = 1, \ldots, 2d - 2$. If $\phi_g : \overline{\mathbb{C}} \to \mathbb{S}^2$ is the holomorphic map corresponding to g, from the aforementioned isomorphism (3-15), we have that the linear space $N(\phi_g)/L(\phi_g)$ can be described in terms of the space of meromorphic quadratic differentials on $\overline{\mathbb{C}}$ with simple poles at the points $z_1, \ldots, z_{2d-2}$. Such a differential σ can be written as follows

$$\sigma = \sum_{i=1}^{2d-2} \frac{A_i}{z - z_i}(dz)^2 = \frac{R(z)}{(z-z_1)\cdots(z-z_{2d-2})}(dz)^2 \tag{3-16}$$

where $A_i \in \mathbb{C}$ and R is a polynomial with degree less than or equal to $2d-6$ (in particular, notice that $\mathrm{Nul}\,\phi = 3$ if $\deg \phi = 1, 2$), that is

$$\begin{aligned} &\sum_{i=1}^{2d-2} A_i = 0 \\ &\sum_{i=1}^{2d-2} S_1(z_1, \ldots, \widehat{z_i}, \ldots, z_{2d-2}) A_i = 0 \\ &\sum_{i=1}^{2d-2} S_2(z_1, \ldots, \widehat{z_i}, \ldots, z_{2d-2}) A_i = 0 \end{aligned} \tag{3-17}$$

where S_1 and S_2 are the first and second elemental symmetric polynomials in the corresponding variables. Now, we will compute the residues of the meromorphic differential σ/dg at its two–order poles z_i:

$$\mathrm{Res}_{z=z_k} \frac{\sigma}{dg} = \frac{d}{dz}\Big|_{z=z_k} \left\{ \frac{(z-z_k)^2}{g'(z)} \sum_{i=1}^{2d-2} \frac{A_i}{z-z_i} \right\} = \\ \frac{1}{g''(z_k)} \sum_{i \neq k} \frac{A_i}{z_k - z_i} - \frac{1}{2} \frac{g'''(z_k)}{g''(z_k)} A_k.$$

Hence, the diffrential σ/dg has no residues if and only if

$$\sum_{i \neq k} \frac{A_i}{z_k - z_i} - \frac{1}{2}\frac{g'''(z_k)}{g''(z_k)} A_k = 0, \quad k = 1, \ldots, 2d-2. \tag{3-18}$$

Notice that the system of equations (S_g) given by (3-17) and (3-18) with $A_1, \ldots, A_{2d-2}$ as unknowns determine $\mathrm{Nul}\,\phi_g$ in the following way:

$$\mathrm{Nul}\,\phi_g = 3 + 2(2d-2) - 2r(g) \tag{3-19}$$

where $r(g)$ is the rank of the complex $(2d+1) \times (2d-2)$-matrix $Z(g)$ of the coefficoents of the system (S_g).

On the other hand, let M_d be the space of all the meromorphic functions with degree d defined on $\overline{\mathbb{C}}$. There exists a natural action of the group $\mathcal{C}(\mathbb{S}^2)$ of the conformal transformations of $\mathbb{S}^2 \equiv \overline{\mathbb{C}}$ on M_d given by

$$(A, g) \in \mathcal{C}(\mathbb{S}^2) \times M_d \longmapsto A \circ g \in M_d.$$

The corresponding quotient space $\mathcal{M}_d = M_d/\mathcal{C}(\mathbb{S}^2)$ was identified in [Loo] with the complement of an algebraic hypersurface in the complex Grassmannian $G(2, d+1)$ by

$$[g = \frac{P}{Q}] \in \mathcal{M}_d \longmapsto \operatorname{span}\{P, Q\}$$

where P, Q are polynomials of degree d without common roots. Moreover the ramification map $\psi_d : \mathcal{M}_d \to \mathbb{P}^{2d-2}$ given by

$$\psi_d\left(\left[\frac{P}{Q}\right]\right) = \left[\operatorname{coeff}(P'Q - Q'P)\right]$$

is shown in [Loo] to be a branched covering with $(2d-2)!/(d-1)!d!$ sheets. Consider another branched covering $\rho_d : \mathbb{C}^{2d-2} \to \mathbb{P}^{2d-2}$ defined by

$$\rho_d(z_1, \ldots, z_{2d-2}) = \left[\operatorname{coeff}(z - z_1)\cdots(z - z_{2d-2})\right].$$

So, except on an algebraic hypersurface $\mathcal{H}$, we may introduce on $\mathcal{M}_d$ local complex coordinates by means of the multivalued map $\psi_d^{-1} \circ \rho_d$. Observe that, from Theorem 19, we may consider Nul and Ind as integer valued functions on the space $\mathcal{M}_d$ and that, if z is a ramification point of a meromorphic function g and $A \in \mathcal{C}(\mathbb{S}^2)$, then

$$\frac{(A \circ g)'''(z)}{(A \circ g)''(z)} = \frac{g'''(z)}{g''(z)}.$$

Hence, the map given by

$$[g] \in \mathcal{M}_d - \mathcal{H} \longmapsto Z(g)$$

is analytic. Now, we can state

Theorem 23. *The set $\mathcal{N}_d$ consisting of the equivalence classes of meromorphic functions of degree d in $\mathcal{M}_d$ where* Nul > 3 *is an analytic subset. Then, we have* Ind $= 2d-1$ *on $\mathcal{M}_d - \mathcal{N}_d$ and* Ind $\leq 2d-1$ *everywhere on $\mathcal{M}_d$.*

Proof. We know that the map $[g] \mapsto Z(g)$ is analytic and, from (3-19), we have

$$\mathcal{N}_d \subset \mathcal{R} = \{[g] \in \mathcal{M}_d - \mathcal{H} \,|\, r(g) < 2d-2\} \cup \mathcal{H}.$$

So, $\mathcal{N}_d$ will be analytic if we prove that the analytic set $\mathcal{R}$ does not coincide with $\mathcal{M}_d$. In fact, we can take a holomorphic map $g_0 : \overline{\mathbb{C}} \to \mathbb{S}^2$ with degree d and such that all its branching values lie in an equator of $\mathbb{S}^2$. From Corollary 15 and (3-19) and by remembering that the eigenvalues of L_{ϕ_g} depend continuously on g, we have that there are points in a open neighbourhood of $[g_0]$ which lie in $\mathcal{M}_d - \mathcal{R}$.

Hence, the set $\mathcal{N}_d$ does not disconnect the space $\mathcal{M}_d$ and, so, all the points in its complement must have the same index. In fact, one can join any two of them by a continuous curve and when one moves along it the nullity is unchanged. Then, from the continuity of the eigenvalues, the index does not change as well. But, from Corollary 15 again, there are points in that complement whose index is exactly $2d-1$. As a conclusion, Ind is identically $2d-1$ on $\mathcal{M}_d - \mathcal{N}_d$. If one uses again the continuity of the eigenvalues, one has that Ind $\leq 2d-1$ on the whole of $\mathcal{M}_d$.

So, in the genus zero case, we have obtained an upper bound for the index which cannot be improved because it is attained by almost all meromorphic functions. This last theorem has been obtained independently by Ejiri and Kotani [E–Ko]. Combining this upper bound with Theorem 21, we have

Corollary 24. *Let $\phi : \overline{\mathbb{C}} \to \mathbb{S}^2$ be anon–constant holomorphic map which has a ramification point with order* $\deg \phi - 1$ *(for instance, $\phi(z) = a_r z^r + \cdots + a_1 z + a_0$). Then*

$$\text{Ind } \phi = 2 \deg \phi - 1.$$

Finally, we will do a detailed study of the degree three case because there one already gets some anomalies which point out that the explicit formula asked for by Fischer-Colbrie and Gulliver does not exist probably. In fact, we will show that all the meromorphic functions of degree three on $\overline{\mathbb{C}}$ behave in the generic way described in Theorem 23, except in a particular situation: when its four ramification points form an equianharmonic quadruple.

Let $\phi : \overline{\mathbb{C}} \to \mathbb{S}^2$ be a degree three holomorphic map and $z_1, z_2, z_3, z_4 \in \overline{\mathbb{C}}$ its ramification points. If any two of them coincide, then their images (at most three) can be placed on an equator of $\mathbb{S}^2$ by using a conformal transformation. So, in this case, from Corollary 15, we have Nul $\phi = 3$ and Ind $\phi = 5$. Suppose now that z_1, z_2, z_3, z_4 are different points of $\overline{\mathbb{C}}$. In that case, the space

$$H^0\big(2\kappa_{\overline{\mathbb{C}}} \otimes [z_1 + z_2 + z_3 + z_4]\big)$$

of the meromorphic quadratic differentials on $\overline{\mathbb{C}}$ with simple poles at z_1, z_2, z_3, z_4 is a complex 1-dimensional space. In fact, the equation (3-16) says us in this case that these differentials are of the form

$$\sigma = \frac{\lambda}{P'Q - Q'P}(dz)^2, \quad \lambda \in \mathbb{C}$$

if P/Q represents the meromorphic function associated to ϕ with $\max\{\deg p, \deg Q\} = 3$. Then, taking into account (3-15), Nul $\phi > 3$ if and only if Nul $\phi = 5$ and this occurs if and only if

$$\text{Res}_{z=z_i} \frac{Q^2}{(P'Q - Q'P)^2}\, dz = 0, \quad i = 1, 2, 3, 4.$$

This condition can be easily rewritten as

$$P'''Q - PQ''' = P''Q' - P'Q'',$$

that is, the cross ratio of the four roots z_1, z_2, z_3, z_4 of $P'Q - PQ'$ satisfies

$$(z_1, z_2, z_3, z_4) = \rho \quad \text{with } \rho^2 - \rho + 1 = 0.$$

This is equivalent to the fact that, up to a Möbius transformation, $z_1 = 0$, $z_2 = 1$, $z_3 = \varepsilon$, $z_4 = \varepsilon^2$ where ε is a primitive cubic root of the unity. It is easy to see that, up to conformal transformations, the meromorphic function associated to such a ϕ is given by $g(z) = z/(z^3 + 2)$.

So, we have proved that a holomorphic map $\phi : \overline{\mathbb{C}} \to \mathbb{S}^2$ with degree three has Nul $\phi > 3$ if and only if its ramification points form an equianharmonic quadruple and, in this case, Nul $\phi = 5$. As conclusions, we can ennonce the following results.

Theorem 25. *Let* $\phi : \overline{\mathbb{C}} \to \mathbb{S}^2$ *be a degree three holomorphic map. If its four ramification points form an equianharmonic quadruple, then* Nul $\phi = 5$ *and* Ind $\phi = 4$. *Otherwise* Nul $\phi = 3$ *and* Ind $\phi = 5$.

Proof. The assertions about the nullity have been already proved. With respect to the index, notice that, if Nul $\phi = 3$, then, from Theorem 23, Ind $\phi = 5$. In the case Nul $\phi = 5$, we can apply Theorem 17 with $m = 5$ to obtain Ind $\phi \geq 4$ and Ind $\phi +$ Nul $\phi \leq 9$. So, Ind $\phi = 4$.

Corollary 26. *Let* $\phi : \overline{\mathbb{C}} \to \mathbb{S}^2$ *be a holomorphic map. Then* Ind $\phi = 4$ *if and only if* $\deg \phi = 3$ *and its ramification points form an equianharmonic quadruple.*

Proof. If Ind $\phi = 4$ we obtain that ϕ had degree three from Theorem 21 and Theorem 23. So, we conclude from Theorem 25.

Remark. The planar ends complete minimal surfaces associated to the meromorphic function $g(z) = z/(z^3 + 2)$ by means of Theorem 3 are unbranched and have embedded ends. These surfaces have been described by Bryant [Br1] and Rosenberg and Toubiana [Ro–To].

Some problems

Finally, we want to deal with some problems about this topic which can be interesting from our point of view.

The first one asks if the upper bound that we have obtained for the index of holomorphic maps on the genus zero surface in Theorem 23 is valid for all genera. With respect to this, we think that Ejiri and Micallef have made some progress, but we do not know exactly their results.

The second one is an inverse spectral question but strongly related to complex analysis, beacuse we have a branched metric ds^2_ϕ for each holomorphic map ϕ from a compact Riemann surface and we want to obtain from the spectrum of its Laplacian all the possible information, see (1-7).

A third problem is to look for properties that a holomorphic map must satisfy in order to have nullity greater than three. The Gauss map of complete minimal embedded surfaces with finite total curvature are in this situation, see remark after Proposition 1. In a earlier version of this work, we had conjectured that these holomorphic maps, in the genus zero case, were precisely the branching points of the ramification map ψ_d used in the proof of Theorem 23. In fact Ejiri [E] has solved this conjecture in the affirmative. He shows the following equivalent fact: a holomorphic map on the sphere has nullity greater than three if and only if there is a curve of spherical minimal immersions into the four–sphere which collapses in this map.

The last problem asks if each bounded Jacobi field on a complete minimal surface with finite total curvature comes from a deformation of this surface by means of minimal surfaces of the same type, see Conclusion 1. In the examples that we know this is true.

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