PROPERLY EMBEDDED SURFACES WITH CONSTANT MEAN CURVATURE

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1. Introduction

In this paper we derive some global properties of properly embedded surfaces in $\mathbb{R}^3$ of non-zero constant mean curvature $H$. We call such a surface an $H$-surface. Our main result is a maximum principle at infinity for these $H$-surfaces.

Theorem 1. Let $M_1$ and $M_2$ be connected disjoint $H$-surfaces in $\mathbb{R}^3$. Then $M_2$ is not on the mean convex side of $M_1$.

The surface $M_1$ separates $\mathbb{R}^3$ into two connected components since $M_1$ is properly embedded. The mean convex side of $M_1$ is the component $W_1$ of $\mathbb{R}^3 - M_1$, towards which points the mean curvature vector of $M_1$.

In the minimal case, the Halfspace Theorem implies that two disjoint proper immersed minimal surfaces must be parallel planes, see [3].

Assume $H \neq 0$ and $M_2 \subset W_1$. If there exist points $x \in M_1$ and $y \in M_2$ whose distance is $\text{dist}(M_1, M_2)$, then the theorem above can be proved directly as follows. There are no focal points of $M_1$ along the interior of the line segment from $x$ to $y$, since the segment minimizes distance between $M_1$ and $M_2$, and after a small translation of $M_2$ we can assume that there are not focal points of $M_1$ in the segment $xy$. So the equidistant surfaces to $M_1$ are non-singular along this segment, starting with a small neighborhood of $x$ on $M_1$. Their mean curvature is strictly increasing when one goes from $x$ to $y$. But the surface at $y$ touches $M_2$ at its mean convex side, a contradiction.

If the above points $x$ and $y$ do not exist, we can take divergent sequences $x_n \in M_1$, $y_n \in M_2$ such that $|x_n - y_n| \to \text{dist}(M_1, M_2)$ and $x_n - y_n$ converge to a vector $v$. So the surface $M_2 + v$ lies in the mean convex side of $M_1$, $M_2 + v \subset W_1$, and touches $M_1$ at infinity, that is, $\text{dist}(M_1, M_2 + v) = 0$. Theorem 1 says that this can not happen, which explains why we call it the maximum principle at infinity.

We will also study $H$-surfaces $M$ in a slab of $\mathbb{R}^3$, between two horizontal planes say. An unsolved problem is whether such a surface admits a horizontal plane of symmetry. It is natural to attack this problem by the technique of Alexandrov.

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reflection [1], starting with horizontal planes coming down from the top of the slab. This method requires that the part of surface above the moving plane be a graph over a horizontal planar domain and that its reflected image with respect to this plane lies in the mean convex side \( W \) of \( M \). If there is a first plane where the symmetry of \( M \) above the plane touches the part of \( M \) below, then the usual maximum principle says \( M \) is symmetric about this plane. In the case \( M \) is compact, we deduce in this way that the surface has a lot of mirror symmetries and so it must be a round sphere, [1]. However in our setting this does not appear to work for several reasons. First, one might not be able to get started. In fact, if we suppose \( M \) has unbounded curvature near the plane \( x_3 = a \), \( a \) being the supremum of \( x_3 \) restricted to \( M \), then moving this plane down till \( x_3 = a - \varepsilon \), for small \( \varepsilon > 0 \), the part of \( M \) above this new plane may not be a vertical graph. If one assumes \( M \) has bounded curvature then this phenomena does not occur, so at least one can begin to do symmetry of the part of \( M \) above the plane. Even then, one quickly encounters another difficulty: it may be the case (and Pascal Collin has constructed \( H \)-surfaces with boundary doing this [2]) that at the last allowed position of the moving plane, the part of \( M \) below this plane and the symmetry of the part above touch for the first time at infinity. Then we can not proceed.

This leads us to study the class \( S \) of (non-necessarily connected) properly embedded \( H \)-surfaces in \( \mathbb{R}^3 \) satisfying the following conditions: \( M \) lies in a horizontal slab, is symmetric about the plane \( P = \{ x_3 = 0 \} \) and \( M_+ = M \cap \{ x_3 > 0 \} \) is a graph over a open set in \( P \). This is the class of surfaces we would obtain if Alexandrov reflection technique could be applied to proper \( H \)-surfaces in a slab. There are many such examples. The Delaunay surfaces are in \( S \) and their width varies from \( 2/H \) (attained for the limit case of a stack of spheres) to \( 1/H \) (for the cylinder). Kapouleas [9] has constructed examples of finite topology in \( S \) which look like a sphere with \( n \) horizontal Delaunay ends, the directions of the ends symmetrically placed, like the \( n \) roots of unity. For further results on surfaces in \( S \) with finite topology see Grosse-Brauckmann, Kusner and Sullivan [8]. Lawson [11] constructed doubly periodic \( H \)-surfaces in \( S \) and Ritoré [17] and Grosse-Brauckmann [7] have given more doubly periodic examples of this type. In particular, Ritoré constructs examples whose width tends to zero (for \( H \) fixed): these surfaces look like two close parallel planes connected by a doubly periodic family of catenoidal necks, the distance between two neighbor necks being small. Notice that each surface in \( S \) has width at most \( 2/H \) by Theorem 6.

We will give a structure result for \( H \)-surfaces in the class \( S \) contained between two parallel planes at distance smaller than \( 1/H \): Assuming that \( M \) has bounded curvature, we prove that the shape of the surface is close to the doubly periodic \( H \)-surfaces constructed by Lawson [11].

In this section we give a priori estimates for $H$-surfaces in different contexts. These results will be used in the proof of our main results in the sections §3 and §4 below. We give a bound, by modifying arguments of Fisher-Colbrie [4], for the radius of a geodesic ball contained in the interior of a stable $H$-surface and we extend Serrin’s height estimate for compact $H$-graphs, [18], to $H$-graphs over arbitrary domains.

Let $M$ be an $H$-surface in $\mathbb{R}^3$. We will consider on $M$ the unit normal vector field $n$ which makes $H > 0$. Equivalently, $n$ will be the normalized mean curvature vector.

An $H$-surface $M$ is said to be stable (in the strong sense) if for any function $u$ with compact support in $M$ we have that
\[
\int_M |\nabla u|^2 - |A|^2 u^2 \geq 0,
\]
where $\nabla u$ and $A$ denote the gradient of $u$ and the shape operator of $M$ respectively. Stability is equivalent to the existence of a positive solution of the equation $\Delta v + |A|^2 v = 0$ on $M$, see [5]. In particular any graph is stable, because the third coordinate of the unit normal vector $n_3$ satisfies the equation above (for $H$-surfaces there is another natural notion of stability, weaker than the one considered in this paper, which is related with the isoperimetric problem: we ask that the integral inequality holds for any $u$ with $\int_M u = 0$).

A fundamental fact about stable $H$-surfaces is that we have an estimate (depending only on $H$) of the largest geodesic ball contained in the interior of the surface. The proof of the next result follows from the ideas of Fischer-Colbrie [4] and it is implicit in López and Ros [12].

**Theorem 2.** Let $M$ be a stable $H$-surface. Then the (intrinsic) distance of any point of $M$ to $\partial M$ is smaller than or equal to $\pi/H$.

**Proof.** The operator $L = \Delta + |A|^2 = \Delta + (4H^2 - 2K)$, $K$ being the Gauss curvature of $M$, has index zero and, so, there is a positive function $u$ on $M$ with $Lu = 0$, (see Proposition 1 of Fisher-Colbrie [4]).

Consider the new metric $d\tilde{s}^2 = u^2 ds^2$ and let $p \in M$ and $R > 0$ be such that the open $d\tilde{s}^2$-geodesic ball $\mathcal{B} = \mathcal{B}(R, p) \subset M$ centered at $p$ is relatively compact in $M$. It is enough to prove that $R \leq \pi/H$.

Let $\gamma$ be a minimizing geodesic for the metric $d\tilde{s}^2$, joining $p$ to $\partial \mathcal{B}$. As the $d\tilde{s}^2$-distance between $p$ and any point of $\partial \mathcal{B}$ is $R$, then if we denote by $a$ the $d\tilde{s}^2$-length of $\gamma$ we have $a \geq R$. Parameterize $\gamma$ by arclength $s$ in the $ds^2$ metric, $0 \leq s \leq a$.

Since $\gamma$ is minimizing for $d\tilde{s}^2$, the second variation of length yields:

\[
0 \leq \int_0^{\tilde{R}} \left( (d\phi)^2 - \tilde{K} \phi^2 \right) d\tilde{s},
\]

for all $\phi$ with $\phi(0) = \phi(\tilde{R}) = 0$, $\tilde{R}$ being the $d\tilde{s}^2$-length of $\gamma$ and $\tilde{K}$ the Gauss curvature of the metric $d\tilde{s}^2$. 
We have that
\[
\frac{d\phi}{ds} = \frac{d\rho}{ds} \frac{ds}{d\bar{s}} = \frac{1}{u} \frac{d\phi}{d\bar{s}}
\]
and
\[
\tilde{K} = \frac{1}{u^2} (K - \Delta \log u).
\]
Moreover, as \(u\) is a Jacobi function on \(B\), we can write
\[
0 = Lu = \Delta u - Ku + \left(2H^2 + \frac{|A|^2}{2}\right)u \geq \Delta u - Ku + 2H^2 u.
\]
Therefore, if \(c = 2H^2\) and \(u'(s) = d(u \circ \gamma)/ds\), we obtain
\[
(2) \quad \Delta \log u = \frac{u\Delta u - |\nabla u|^2}{u^2} \leq K - c - \frac{(u')^2}{u^2}
\]
where we have used that \(|u'| \leq |\nabla u|\).

Then (1) and (2) yield:
\[
(3) \quad \int_0^a \left( c \frac{1}{u} \frac{(u')^2}{u^3} \right) \phi^2 \, ds \leq \int_0^a \frac{K - \Delta \log u}{u} \phi^2 \, ds \leq \int_0^a \frac{1}{u} (\phi')^2 \, ds.
\]
Write \(\phi = u\psi\), where \(\psi(0) = \psi(a) = 0\), so that
\[
\phi' = u'\psi + u\psi' \quad \text{and} \quad \frac{1}{u} (\phi')^2 = \frac{1}{u} (u')^2 \psi^2 + u(\psi')^2 + 2u'\psi\psi'.
\]
Then (3) yields:
\[
\int_0^a \left( c u\psi^2 + \frac{(u')^2}{u} \psi^2 \right) \, ds \leq \int_0^a \left( \frac{(u')^2}{u} \psi^2 + u\psi' + 2u'\psi\psi' \right) \, ds
\]
An integration by parts using \(d(u\psi\psi') = (u'\psi\psi' + u(\psi')^2 + u\psi\psi'')\,ds\) transforms this last inequality to:
\[
(4) \quad \int_0^a (c\psi^2 + (\psi')^2 + 2\psi''\psi) u \, ds \leq 0.
\]
Finally, we take
\[
\psi(s) = \sin \left( \frac{\pi s}{a} \right), \quad 0 \leq s \leq a,
\]
and so (4) becomes:
\[
\int_0^a \left( \frac{\pi^2}{a^2} \cos^2 \left( \frac{\pi s}{a} \right) + \left( c - \frac{2\pi^2}{a^2} \right) \sin^2 \left( \frac{\pi s}{a} \right) \right) u(\gamma(s)) \, ds \leq 0.
\]
Thus \(c < \frac{2\pi^2}{a^2}\) and therefore \(R < a < \frac{\pi}{H}\), which proves the theorem.

Now we consider \(H\)-surfaces \(M\) which are vertical graphs (in short, \(H\)-graphs). The unit normal vector of a graph is never horizontal and so, on each connected component of \(M\), it points either down or up. Recall also that \(H\)-graphs are stable.
Lemma 3. Let $M$ be an $H$-surface given as the graph of a smooth positive function $u$ defined over a domain $\Omega \subset \{x_3 = 0\}$. If $M$ is properly embedded in the halfspace $x_3 > 0$ then,

a) $M$ is contained in the slab $\{0 < x_3 < 2\pi/H\}$,

b) $u$ extends continuously to zero on the boundary of $\Omega$, and

c) the normal vector of $M$ points down.

Proof. As any $H$-graph is stable, it follows from Theorem 2 that $x_3 < 2\pi/H$ on $M$: Otherwise we can find a geodesic ball of radius $\pi/H$ contained in the interior of $M$. This proves a). In particular, we conclude that $u$ extends continuously, with zero boundary values, to the topological boundary of $\Omega$. So it remains to prove c). Take a vertical line $l$ which intersects a connected component $M'$ of $M$ and a sphere $S$ of mean curvature $H$ centered at a high point of $l$ so that the sphere is disjoint from $M$ and $l \cap S$ is above $l \cap M$. Move $S$ down till we have a first contact with $M'$. If the mean curvature vector of $M'$ points up, then the maximum principle would imply that the graph $M$ contains a sphere, which is impossible. □

From the Lemma above we conclude that for an $H$-graph $M \subset \{x_3 > 0\}$, $M$ is properly embedded in the upper halfspace if and only if $u$ extends continuously to zero on the boundary of $\Omega$.

We will consider limits of $H$-graphs. The following propositions justify the existence of such limits. First we prove that an $H$-graph $M$ in the upper halfspace with zero boundary values satisfies an interior curvature estimate in $\{x_3 > 0\}$. This is a known fact but we include a proof for the sake of completeness.

Proposition 4. There is a positive constant $C$, depending only on $H > 0$, such that any $H$-surface $M$ properly embedded in $\{x_3 > 0\}$ given as the graph of a function $u \in C^\infty(\Omega)$, with $\Omega \subset \{x_3 = 0\}$, satisfies

$$|A| \leq \frac{C}{x_3},$$

$|A|$ being the length of the second fundamental form of $M$.

Proof. This can be shown as follows: In case the estimate fails, we can consider a sequence $p_k$, $k = 1, 2, \ldots$, of points in the surfaces $M_k$, satisfying the hypothesis of the assertion such that $|A_k(p_k)|x_3(p_k) > k$, where $A_k$ is the second fundamental form of $M_k$. Let $B(p, r) = \{x \in \mathbb{R}^3/|x - p| < r\}$ and $r_k = x_3(p_k)/2$. It follows that for $q \in M_k \cap B(p_k, r_k)$, the expression $(r_k - |q - p_k|)|A_k(q)|$ attains its maximum at a point $q_k$ (as it vanishes when $q$ approaches to the boundary). Moreover if $r_k' = r_k - |q_k - p_k|$, then we have $S_k = M_k \cap B(q_k, r_k') \subset M_k \cap B(p_k, r_k)$ and $R_k = r_k'|A_k(q_k)| > k/2$. These points $q_k$ are called points of almost maximal curvature and the translated rescaled surfaces $\Sigma_k = |A_k(q_k)|(S_k - q_k)$ converge, up to a subsequence, to a nonflat complete surface $\Sigma_\infty$ in the Euclidean 3-space with mean curvature 0. To see that we observe that the surfaces $\Sigma_k$ pass through the
origin and $\partial \Sigma_k$ is contained in the boundary of the ball $B(0, R_k)$ whose radius $R_k$ converges to $\infty$. Moreover in the ball $B(0, R_k/2)$ the length the second fundamental form of $\Sigma_k$ is bounded by 4, and equals 1 at the origin. The mean curvature of $\Sigma_k$ is equal to $H/|A_k(q_k)|$.

This permits one to construct a subsequence of $\Sigma_k$ that converges to a complete minimal surfaces $\Sigma_\infty$ passing through the origin and with curvature 1 at the origin. As the Gauss map of $\Sigma_\infty$ lies in the closed lower hemisphere, it follows that the minimal surface $\Sigma_\infty$ is stable and therefore it must be flat, see [5]. This contradiction proves the assertion. □

Proposition 5. Let $\{M_n\}$ be a sequence of $H$-surfaces, $H > 0$, such that $M_n$ is the graph of a positive function $u_n$ over an open subset $\Omega_n \subset \{x_3 = 0\}$ that extends continuously to zero on the boundary of $\Omega_n$. Assume that there are points $p_n \in M_n$ with $p_n \to p$ and $x_3(p) > 0$. Then there exists a subsequence of $\{M_n\}$ which converges on compact subsets of $x_3 > 0$ to the graph of a positive function $u : \Omega \to \mathbb{R}$, over an open subset $\Omega$, which extends continuously to the closure of $\Omega$ with zero boundary values.

Proof. The curvature estimate in the Proposition above implies that, up to a subsequence, the surfaces $M_n$ converge to an $H$-surface $M$ properly immersed in $x_3 > 0$. If $n_3$ denotes the third coordinate of the unit normal vector of $M$, we have that $\Delta n_3 + |A|^2 n_3 = 0$. As $M$ is a limit of graphs we get from item c) in Lemma 3 that $n_3 \leq 0$ and the maximum principle implies that either $n_3 < 0$ in $M$ or $n_3 = 0$ on a connected component of $M$.

In the second case, that component must be a vertical circular cylinder intersected with the upper halfspace. By lemma 3, the height of the surfaces $M_n$ is uniformly bounded, so this case is impossible.

Therefore $n_3 < 0$ and we claim that $M$ is the graph of a positive function $u$ over an open subset $\Omega \subset \{x_3 = 0\}$. To prove this claim note that each point $p \in M$ has a neighborhood $U_p$ which is the graph of a smooth function over an open disc in the plane $x_3 = 0$. If there were two different points $p, q \in M$ lying in the same vertical line, we deduce that the same would be true for the graph $M_n$ for $n$ large enough. This contradiction proves that $M$ meets each vertical line at most once and so $M$ is a graph of a function $u$. From Lemma 3 it follows that $u$ extends continuously to the closure of $\Omega$ with zero boundary values. That proves the proposition. □

If is well-known that compact $H$-graphs with zero boundary values satisfy a height estimate, see Serrin [18]. In the result below we extend this fact to arbitrary $H$-graphs with that boundary condition.

Theorem 6. Let $M$ be an $H$-surface, $H > 0$, given by the graph of a positive function $u$ on a planar domain $\Omega \subset \{x_3 = 0\}$, where $u$ extends continuously to the closure of $\Omega$ with zero boundary values. Then $M$ is contained in the slab $0 < x_3 \leq 1/H$. 

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Proof. First observe that Theorem 2 implies that \( x_3 \) is bounded on \( M \). To prove the upper bound \( x_3 \leq 1/H \) consider on \( M \) the function

\[
\phi_M = \phi = H \, x_3 + n_3,
\]

where \( x_3 \) and \( n_3 \) denote the third coordinate of the position and unit normal vectors on \( M \) (note that \( n_3 < 0 \) on \( M \)). Then \( \Delta \phi = -2(H^2 - K)n_3 \geq 0 \).

If we assume that \( \overline{\Omega} \) is smooth and compact, then the statement is a classical result of Serrin, [18]: to prove it, observe that \( \phi \) is subharmonic and it attains its maximum at the boundary. On \( \partial \Omega \), \( \phi = n_3 \leq 0 \) and therefore \( \phi \leq 0 \) in \( \Omega \). Thus \( H \, x_3 + n_3 \leq 0 \) which implies

\[
x_3 \leq \frac{-n_3}{H} \leq \frac{1}{H}.
\]

Now suppose \( \overline{\Omega} \) is noncompact. If \( \phi \leq 0 \) on \( M \) then we are done, so we can suppose that \( \sup \phi = c > 0 \). Let \( p_n \in M \) be a sequence with \( \phi(p_n) \to c \), \( x_3(p_n) \to x_\infty \), and \( n_3(p_n) \to n_\infty \). Notice that \( x_\infty > 0 \), since if \( x_\infty \leq 0 \), \( c = H \, x_\infty + n_\infty \leq 0 \).

Let \( M_n \) be the horizontal translate of \( M \) which places \( p_n \) on the \( x_3 \)-axis, intersected with the open half space \( \{x_3 > 0\} \). From Proposition 5 we have that a subsequence (that we also denote by \( M_n \)) converges to an \( H \)-graph which is properly embedded in \( \{x_3 > 0\} \). Let \( p_\infty = \lim p_n \) and \( M_\infty \) the connected component of the limit of \( M_n \) which contains the point \( p_\infty \). As \( x_3(p_\infty) = x_\infty > 0 \), we see that \( p_\infty \) is an interior point of \( M_\infty \). As the function \( \phi_\infty = \phi_{M_\infty} \) achieves its maximum at \( p_\infty \) we conclude from the maximum principle that \( \phi_\infty \) is constant on \( M_\infty \), which means that \( M_\infty \) is a spherical cap. In particular \( \phi_\infty \leq 0 \) which contradicts that \( \phi_\infty(p_\infty) = c > 0 \).

The lower bound \( 0 < x_3 \) is proved in a similar way using the equation \( \Delta x_3 = 2Hn_3 < 0 \). If \( x_3 \) is negative somewhere on \( M \), then a suitable sequence of horizontal translated images of \( M \) will converge in \( \{x_3 < 0\} \) to an \( H \)-graph \( M'_\infty \) properly embedded in the lower halfspace \( \{x_3 < 0\} \), whose unit normal vector points down and such that \( x_3 \) attains its minimum at the interior, which contradicts the maximum principle. Thus \( x_3 \geq 0 \) on \( M \) and the maximum principle again gives that \( x_3 > 0 \), as we claimed. \( \square \)

3. The Maximum Principle at Infinity

In this section we will prove our main result: Let \( M_1 \) and \( M_2 \) be two connected properly embedded \( H \)-surfaces in \( \mathbb{R}^3 \) \((H > 0)\) such that \( M_2 \) lies in the mean convex side of \( M_1 \). Then we want to show that \( M_2 = M_1 \).

By the comments in the introduction, we can assume that \( M_1 \) is noncompact. We orient \( M_1 \) and \( M_2 \) by unit vector fields \( N_1 \) and \( N_2 \) whose direction is that of the mean curvature. The mean convex side \( W_1 \) of \( M_1 \) is the component of \( \mathbb{R}^3 - M_1 \) such that \( N_1 \) along \( M_1 \) points into \( W_1 \). So \( M_2 \) is contained in the closure of \( W_1 \).

Assuming that \( M_2 \neq M_1 \) we will obtain a contradiction. Clearly \( M_2 \cap M_1 = \emptyset \) by the usual maximum principle. Denote by \( W \) the component of \( W_1 - M_1 \) satisfying \( \partial W = M_1 \cup M_2 \). Thus the boundary of \( W \) is not connected and the mean curvature
vector of $M_1$ points to $W$ (hence $N_1$ as well). If $W$ were mean convex, then the halfspace theorem [3] would imply that $M_1$ and $M_2$ are parallel planes. Therefore the mean curvature vector of $M_2$ (hence $N_2$ as well) points away from $W$.

Let $S$ be a relatively compact domain in $M_1$ with smooth boundary $\Gamma$. Assume also that $S$ is unstable. We will show that there is a surface $\Sigma \subset W$ bounded by $\Gamma$ with constant mean curvature $H$ that is stable. Then by taking the domain $S$ in $M_1$ to be larger and larger, we will obtain a contradiction, using the fact that the distance between a point of a stable $H$-surface and its boundary is bounded.

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Let $\mathcal{F}$ be the family of regions $Q \subset W$ enclosed by $S$ and surfaces $\Sigma \subset W$ with $\partial \Sigma = \Gamma$ and define the functional

$$F(Q) = A(\Sigma) + 2H V(Q),$$

where $A(\Sigma)$ is the area of $\Sigma$ and $V(Q)$ is the volume of $Q$. Our goal is to show that there is $Q \in \mathcal{F}$ minimizing the functional $F$ and such that the interior of $\Sigma$ is contained in the interior of $W$. This will imply that $\Sigma$ is a smooth surface with $\partial \Sigma = \Gamma$ and constant mean curvature $H$. Moreover the mean curvature vector of $\Sigma$ points outside $Q$ and $\Sigma$ is stable in the sense of §2.

We will need certain auxiliary surfaces and regions and some preliminary remarks. Consider the least area surface $S_{\text{min}}$ in $W_1$ spanning the curve $\Gamma$ and homologous to $S$. $S_{\text{min}}$ is a compact smooth minimal surface and by the maximum principle we have $M_1 \cap S_{\text{min}} = \Gamma$. Denote by $Q_{\text{min}}$ the region in $W_1$ enclosed by $S \cup S_{\text{min}}$.

Let $S_0 = M_2 \cap B(\rho)$ be a relatively compact open set in $M_2$, where $\rho$ is a large positive radius and $B(\rho) = \{x \in \mathbb{R}^3 / |x| < \rho\}$. There is an $\varepsilon > 0$ such that the sets $S_t = \{x - tN_2 | x \in S_0\}$, $0 \leq t \leq \varepsilon$, are smooth surfaces parallel to $M_2$ foliating a neighborhood $Q_{\text{par}}$ of $S_0$ in $W$. We put $S_{\text{par}} = S_t$. Denote by $Y$ the unit vector field, normal to the foliation $S_t$, and oriented by $N_2$ and let $H_t$ be the mean curvature of $S_t$. A calculation shows $H_t > H$ for $0 < t$. At any point of $S_t$ we have $\text{div}Y = -2H_t$, where $\text{div}$ is the divergence operator in $\mathbb{R}^3$. Therefore

$$\text{(5)} \quad \text{div}Y > -2H, \quad \text{for} \quad t > 0.$$

Finally, we consider $\varphi$ the first eigenfunction of the Jacobi operator of $S$. So $\varphi$ vanishes at $\Gamma$, is positive at the interior of $S$ and satisfies $\Delta \varphi + |A|^2 \varphi + \lambda_1 \varphi = 0$, where $\lambda_1$ is a negative constant (as $S$ is unstable). After a perturbation of $S$ we can assume that $0$ is not an eigenvalue of $\Delta + |A|^2$. Hence there is a smooth function $v$ on $S$, vanishing at $\Gamma$ and such that $\Delta v + |A|^2 v = 1$ in $S$. The boundary maximum principle implies that the derivative of $\varphi$ with respect to the outwards pointing normal vector is negative along $\Gamma$. Therefore we conclude that, for $a > 0$ small enough, $u = \varphi + av$ is positive at the interior of $S$.

For small $\varepsilon > 0$ and $0 < t \leq \varepsilon$, $u$ defines a normal deformation of $S$,

$$S'_t = \{x + tuN_1 | x \in S\} \subset W.$$
The surfaces $S'_t$, $0 < t < \varepsilon$, foliate an open set $Q_{uns} \subset W$. Putting $S_{uns} = S'_\varepsilon$, we have $\partial Q_{uns} = S \cup S_{uns}$, see Figure 3.

If $X$ is the unit normal vector field of the foliation $S'_t$ oriented by $N_1$ and $H'_t$ is the mean curvature of $S'_t$, we have $\text{div} X = -2H'_t$ as above. Moreover

$$
\frac{d}{dt}|_{t=0} 2H'_t = \Delta u + |A|^2 u = -\lambda_1 \varphi + a > 0,
$$

which implies, choosing $\varepsilon$ small enough, that $H'_t > H$. Therefore

(6) \hspace{1cm} \text{div} X < -2H, \text{ for } 0 < t < \varepsilon.

**Figure 1.** If $\Sigma$ meets the exterior of the least area surface $S_{min}$, then $F(Q \cap Q_{min}) < F(Q)$.

**Lemma 7.** Let $Q \in F$ be enclosed by $S$ and $\Sigma$. Assume that $\Sigma$ is smooth at the interior of $W$.

i) If $Q \not\subset Q_{min}$, then $F(Q \cap Q_{min}) < F(Q)$.

ii) If $Q \cap Q_{par} \neq \emptyset$, then $F(Q - Q_{par}) < F(Q)$.

iii) If $Q_{uns} \not\subset Q$, then $F(Q \cup Q_{uns}) < F(Q)$.

**Proof.** The assertion in i) follows because $Q' = Q \cap Q_{min} \in F$ and satisfies $V(Q') < V(Q)$ and $A(\Sigma') \leq A(\Sigma)$, where $\Sigma' = \partial Q' - S$, see Figure 1.

**Figure 2.** If $Q$ cuts the tubular neighborhood $Q_{par}$ of $M_2$ (in gray), then $F(Q - Q_{par}) < F(Q)$. 
The claim in ii) is a consequence of the inequality \( \text{div } Y > -2H \) on \( Q_{\text{par}} \), see Figure 2. From the divergence theorem we obtain

\[ -2HV(Q \cap Q_{\text{par}}) < \int_{Q\cap Q_{\text{par}}} \text{div } Y = \int_{\partial (Q\cap Q_{\text{par}})} \langle Y, \nu \rangle = \int_{Q\cap S_{\text{par}}} \langle Y, \nu \rangle + \int_{\Sigma \cap Q_{\text{par}}} \langle Y, \nu \rangle, \]

where \( \nu \) is the outer pointing unit normal to the boundary of \( Q \cap Q_{\text{par}} \). Using that \( \nu = -Y \) along \( Q \cap S_{\text{par}} \) and \( \langle Y, \nu \rangle \leq 1 \) at the points of \( \Sigma \cap Q_{\text{par}} \), we deduce, after rearrangement, that

\[ -2HV(Q \cap Q_{\text{par}}) + A(Q \cap S_{\text{par}}) < A(\Sigma \cap Q_{\text{par}}), \]

and, therefore

\[ F(Q - Q_{\text{par}}) = 2H (V(Q) - V(Q \cap Q_{\text{par}})) + A(\Sigma - Q_{\text{par}}) + A(Q \cap S_{\text{par}}) < 2HV(Q) + A(\Sigma \cap Q_{\text{par}}) + A(\Sigma - Q_{\text{par}}) = F(Q), \]

as we claimed.

![Figure 3](image.png)

**Figure 3.** If \( Q \) does not contain the shadowy region \( Q_{\text{uns}} \), then \( F(Q \cup Q_{\text{uns}}) < F(Q) \).

To prove iii) we argue as in the proof of ii), but here we use that \( \text{div } X < -2H \) on \( Q_{\text{uns}} \), see Figure 3.

\[ -2HV(Q_{\text{uns}} - Q) > \int_{Q_{\text{uns}} - Q} \text{div } X = \int_{\partial (Q_{\text{uns}} - Q)} \langle X, \nu \rangle = \int_{S_{\text{uns}} - Q} \langle X, \nu \rangle + \int_{\Sigma \cap Q_{\text{uns}}} \langle X, \nu \rangle. \]

As \( \nu = X \) on \( S_{\text{uns}} - Q \) and \( \langle X, \nu \rangle \geq -1 \) at the other points of the boundary, we have

\[ 2HV(Q_{\text{uns}} - Q) + A(S_{\text{uns}} - Q) < A(\Sigma \cap Q_{\text{uns}}). \]

Hence

\[ F(Q \cup Q_{\text{uns}}) = 2H (V(Q) + V(Q_{\text{uns}} - Q)) + A(S_{\text{uns}} - Q) + A(\Sigma - Q_{\text{uns}}) < 2HV(Q) + A(\Sigma \cap Q_{\text{uns}}) + A(\Sigma - Q_{\text{uns}}) = F(Q), \]

as desired. \( \square \)
Proof of Theorem 1. Assume 0 lies in $M_1$ and let $S(r)$ be the connected component of $M_1 \cap B(r)$, where $r > 0$, and $B(r)$ is the euclidean ball of radius $r$ centered at 0. Since $M_1$ is properly embedded, the boundary of $S(r)$ is contained in $\partial B(r)$. Let $\gamma$ be a path in $W$ joining 0 to a point $y$ in $M_2$. Choose $r$ large enough so that $\gamma$ is contained in $B(r)$ and $\text{dist}(\gamma, \partial B(r)) > 2\pi/H$.

From Theorem 2, and our choice of $r$, we know that $S = S(r)$ is unstable.

Let $K$ be the compact connected domain contained in $Q_{\text{min}}$, bounded by $S$ and $M_2 \cap Q_{\text{min}}$, see Figure 4.

![Figure 4](image)

Let $a$ be the infimum of $F$ on $F$. If $(Q_n, \Sigma_n)$ is a minimizing sequence for $F$ (i.e. $\lim_{n \to \infty} F(Q_n, \Sigma_n) = a$) then by Lemma 7, we can cut and paste the $Q_n$ to form a new minimizing sequence $(\tilde{Q}_n, \tilde{\Sigma}_n)$ so that $\tilde{Q}_n \subset K$ and $\text{int} \tilde{\Sigma}_n \subset \text{int} K$.

By compactness [15, 5.5], there is a minimum $Q$ of $F$ in $K$ bounded by a rectifiable current $\Sigma$ with the support of $\Sigma$ contained in $W$, $\partial \Sigma = \partial S$, and $\Sigma$ disjoint from $M_2$. Regularity [14, Corollary 3.7] implies the part of $\Sigma$ in the interior of $K$ is a smooth surface of mean curvature $H$. This stable surface $\Sigma$ intersects $\gamma$ (since the union of $\Sigma$ and $S$ bounds $Q$). However, for $z$ in the intersection of $\gamma$ and $\Sigma$, we have $\text{dist}(z, \partial \Sigma) > 2\pi/H$. This contradicts Theorem 2 and completes the proof of Theorem 1.

$\square$


We now consider properly embedded $H$-surfaces $M$ with $\partial M = \emptyset$ and $H > 0$, which fit between two parallel planes $P_0$ and $P_1$. The width of $M$ is the infimum of the distance between such planes. The only compact connected $M$ is the sphere of width $2/H$, so the surfaces we consider are non compact. We know very little about these surfaces, even assuming bounded curvature. Assuming $M$ is connected, some questions we can not answer are:

If $M$ has bounded width, is it at most $2/H$?
If $M$ is between the planes $P_0$ and $P_1$, is there a parallel plane $P$ between $P_0$ and $P_1$ with $M$ symmetric by $P$?

We will study a special class of surfaces of bounded width. Let $S$ be those (non-necessarily connected) properly embedded $H$-surfaces $M$, $H > 0$, which are symmetric with respect to the plane $P = \{x_3 = 0\}$ and such that $M_+ = M \cap \{x_3 > 0\}$ is a graph over an open subset $\Omega$ in $P$. Theorem 6 implies that each surface in $S$ has width at most $2/H$. Among the surfaces in $S$, the sphere, the Delaunay surfaces, and Kapouleas [9] finite topology examples have width at least $1/H$. Lawson doubly periodic $H$-surfaces [11] and the related ones constructed by Ritoré [17] and Grosse-Brauckmann [7] may have arbitrarily small width. For some of these doubly periodic surfaces the domain $\Omega$ consists of the plane $P$ where we have removed infinitely many pairwise disjoint compact convex disks. In this section we will prove that any surface in $S$ with width smaller than $1/H$ (= the width of a cylinder of mean curvature $H$) looks like these doubly periodic surfaces.

**Theorem 8.** Suppose $M$ in $S$ has width less than $1/H$ and $M_+$ is a graph over the open subset $\Omega \subset P$. Then the components of $P - \Omega$ are strictly convex. In particular $M$ is connected. If moreover $M$ has bounded curvature, then $P - \Omega$ is a countable disjoint union of strictly convex compact disks.

**Proof.** First we observe that if $M$ were contained in two non parallel slabs, then the slabs intersect and $M$ is cylindrically bounded. A theorem of Kusner, Korevaar and Solomon [10] would imply that the components of $M$ are spheres or Delaunay surfaces, which contradicts our width assumption. Therefore, $M$ is contained in the slab perpendicular to $P$.

The proof uses an operator $L$ that has its origins in a paper by Payne and Philippin [16] defined on any surface $M \in S$. The operator is of the form

$$Lf = \Delta f + \langle \nabla f, X \rangle,$$

where $X$ is a tangent vector field to $M$, singular where $M$ is horizontal, and $\nabla f$ denote the gradient of a function $f$ on $M$.

As we have oriented $M$ so that $H > 0$, the maximum principle implies that $n_3 < 0$ in $M_+$. Consider

$$\psi = 2Hx_3 + n_3 \quad \text{on } M_+,$$

Clearly $\psi = 0$ on $\Gamma = \partial M_+ = P \cap M$ since $M$ is vertical along $\Gamma$. A simple calculation shows $\Delta \psi = 2Kn_3$, where $K$ is the Gauss curvature of $M$. Now if we look for a vector field $X$ such that $L\psi = 0$, then it suffices to find $X$ satisfying $\langle X, \nabla \psi \rangle = -2Kn_3$.

Denote by $a$ the tangent part of $e_3 = (0,0,1)$. Then $\nabla x_3 = a$, $\nabla n_3 = -Aa$, where $A$ is the shape operator of $M$, and

$$\nabla \psi = 2H\nabla x_3 + \nabla n_3 = (2HI - A)a,$$
$I$ being the identity tensor. Using the basic matrix equality $A^2 - 2HA + KI = 0,$ one has

$$A\nabla \psi = (2HA - A^2)a = Ka.$$ 

Consequently, if one defines

$$X = -\frac{2n_3}{|a|^2}Aa, \quad \text{where} \quad a \neq 0,$$

we conclude

$$\langle X, \nabla \psi \rangle = -\frac{2n_3}{|a|^2} \langle Aa, \nabla \psi \rangle = -\frac{2n_3}{|a|^2} \langle a, A\nabla \psi \rangle = -\frac{2n_3}{|a|^2} \langle a, Ka \rangle = -2Kn_3$$

and $\psi$ is a solution of $L\psi = 0$ where $a \neq 0.$ If $\nu$ denotes the outer conormal to $M_+$ along $\Gamma,$ then by direct computation one has

$$\frac{\partial \psi}{\partial \nu} = k_{\Gamma},$$

where $k_{\Gamma}$ is the curvature of $\Gamma$ in $P$ with respect to the plane unit normal pointing towards $P - \Omega.$ In particular if $\frac{\partial \psi}{\partial \nu} > 0,$ then $\Gamma$ is strictly convex towards $P - \Omega.$

The maximum principle applied to $L\psi = 0$ yields directly the following properties:

(i) If $\psi$ assumes an extremum at $q \in M_+, \text{ then } q \text{ is on } \Gamma \text{ or the unit normal vector of } M \text{ at } q \text{ is vertical (i.e., } q \text{ is a singular point of } X).$

(ii) If $\psi$ has a local maximum at $q \in \Gamma,$ then either $k_{\Gamma}(q) > 0$ or the component of $M$ passing through $q$ is a cylinder.

As $M$ is vertical along $\Gamma$ we have that $\psi = 0$ on $\Gamma.$ We claim that $\psi \leq 0$ on $M_+.$ Suppose first that $\psi$ attains its maximum at $q.$ If $q \notin \Gamma,$ then by (i) we get that the normal vector at $q$ is vertical. So $n_3(q) = -1,$ and $\psi(q) = 2Hx_3(q) - 1.$ Since the width of $M$ is at most $1/H,$ we have $x_3(q) < 1/(2H),$ and $\psi(q) < 0,$ which is impossible. So, $q \in \Gamma$ and then $\psi$ is nonpositive.

If $\psi$ does not attain its maximum, let $q_n \in M_+, \text{ } \psi(q_n) \to \sup \psi.$ If $x_3(q_n) \to 0,$ then $\psi(q_n) = 2Hx_3(q_n) + n_3(q_n) \leq 2Hx_3(q_n) \to 0,$ so $\sup \psi \leq 0$ and $\psi \leq 0$ on $M_+.$

So we can assume $x(q_n)$ converges to a number $c > 0.$ Translate $M$ horizontally so that $q_n$ is over the origin, and let $M_\infty$ be a limit $H$-surface of the translated surfaces in the open halfspace $x_3 > 0.$ This surface is non empty because the point $q_\infty$ obtained as limit of the translated images of $q_n$ lies on $M_\infty.$ Moreover the function $\psi_\infty$ constructed as in (7) on $M_\infty$ attains its maximum at $q_\infty$ and $\psi_\infty(q_\infty) > 0.$ As $M_\infty$ is contained in the slab $0 < x_3 < 1/(2H),$ reasoning as in the first case we obtain a contradiction. Hence $\psi \leq 0$ on $M_+.$

Now we can apply property (ii) to conclude that either $M$ has a cylindrical component or $k_{\Gamma}(q) > 0$ for all $q \in \Gamma.$ Since we are assuming the width is strictly less than $1/H,$ we have $k_{\Gamma} > 0$ at each point of $\Gamma.$ So the connected components of $P - \Omega$ are strictly convex.
Next we prove that, if $M$ has bounded curvature, then the planar curvature $k_T$ is bounded away of zero. Suppose this were not the case; Then there is a sequence $q_n \in \Gamma$, with $k_T(q_n) \to 0$. Translate $M$ horizontally so that $q_n$ transforms to a fixed point $\sigma$. The curvature bounds allows us to take a limit surface $M_\infty$ in the whole $\mathbb{R}^3$ (not only in the half space $\{x_3 > 0\}$) of the translated surfaces. This surface $M_\infty$ is not necessarily embedded (it could have tangential selfintersections at the level $x_3 = 0$) but retains any other properties of surfaces in $\mathcal{S}$. In particular, assertion $(ii)$ applies to $M_\infty$. Moreover its width is smaller than $1/H$ and the function $\psi_\infty$ constructed as in (7) on $(M_\infty)_+$ is nonpositive (because it is a limit on nonpositive functions) and vanishes at $\sigma$. As the curvature of the curve $M_\infty \cap P$ at $\sigma$ is zero, we conclude from $(ii)$ that one of the components of $M_\infty$ must be a cylinder. This contradiction proves that $k_T < 1/\rho$ for some positive constant $\rho$. This implies that any connected component of $\Gamma$ is a closed Jordan curve contained in a disk of radius $\rho$. Therefore, if the number of components of $P - \Omega$ where finite, then $\Omega$ would contain arbitrarily large round disks, which is clearly impossible (use for instance Theorem 2). This completes the proof of the theorem. □

References

PROPERLY EMBEDDED SURFACES WITH CONSTANT MEAN CURVATURE


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