Maximum principles in a certain hyperbolic equation

A talk based on some joint works with
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Telegraph equation

$$\left.\begin{array}{l} u_{tt} - u_{xx} + cu_t + \lambda u = f(t,x) \\ \\ u(t,x) \text{ doubly periodic} \\ \text{(same period: } T_t = T_x = 2\pi) \end{array}\right\}$$

Parameter: $c > 0$ (c can be negative: $t \to -t$)

Function: f doubly periodic

Telegraph equation

$$\left.\begin{array}{l} u_{tt} - u_{xx} + cu_t + \lambda u = f(t,x) \\ \\ u(t,x) \text{ doubly periodic} \end{array}\right\}$$

Note: A boundary value problem

Periodicity: $T_t \neq T_x$ or "$T_t = \infty$" $\Rightarrow$ similar results

Dimension in x: Extensions to 2 and 3

Concept of Solution

$$u_{tt} - u_{xx} + cu_t + \lambda u = f, \qquad f \in L^1(\mathbb{T}^2)$$

$u \in L^1(\mathbb{T}^2)$ is a solution if

$$\int_{\mathbb{T}^2} (\varphi_{tt} - \varphi_{xx} - c\varphi_t + \lambda\varphi)u = \int_{\mathbb{T}^2} f\varphi, \qquad \forall \varphi \in \mathfrak{D}(\mathbb{T}^2)$$

Regularity

$$u_{tt} - u_{xx} + cu_t + \lambda u = f$$

$\lambda > 0 \Rightarrow \exists_1\, u$ solution and

i) $\|u\|_{C(\mathbb{T}^2)} \leq C_1 \|f\|_{L^1(\mathbb{T}^2)}$ if $f \in L^1(\mathbb{T}^2)$

ii) $\|u\|_{C^{0,\alpha}(\mathbb{T}^2)} \leq C_p \|f\|_{L^p(\mathbb{T}^2)}$ if $f \in L^p(\mathbb{T}^2)$

$(\alpha = 1 - \frac{1}{p})$

Maximum Principle: idea

$$\left.\begin{array}{l} u_{tt} - u_{xx} + cu_t + \lambda u = f(t,x) \\ \\ u \text{ doubly periodic} \end{array}\right\}$$

MP: $f \geq 0 \;\Rightarrow\; u \geq 0$

SMP: $f \geq 0 \;\Rightarrow\; u > 0$

$$u_{tt} - u_{xx} + cu_t + \lambda u = f(t, x)$$

Track 1.- No maximum principle for $\mathbf{c} = 0$

- $\lambda = 0 \Rightarrow \nexists$ solution if $f \geq 0$

- $\lambda \neq 0 : u(t,x) = 1 - \cos t \cos x \geq 0$
  $u_*(t,x) = u(t,x) + \varepsilon w(t,x)$

$$u_{tt} - u_{xx} + cu_t + \lambda u = f(t, x)$$

Track 2.- Maximum principle for $\mathbf{c} > 0$

- $u = u(t)$: MP $\Leftrightarrow 0 < \lambda \leq \frac{c^2}{4} + \frac{1}{4}$

- $u = u(x)$: MP $\Leftrightarrow \lambda > 0$

$$u_{tt} - u_{xx} + cu_t + \lambda u = f(t,x)$$

Conjecture

$$\text{MP} \Leftrightarrow 0 < \lambda \leq \frac{c^2}{4} + \frac{1}{4}$$

$(c > 0)$

Main results (1)

- $\exists\, \nu(c)$ / MP $\Leftrightarrow \lambda \in (0, \nu(c)]$

- MP $\Rightarrow$ SMP

Main results (2)

- Properties

$$\frac{c^2}{4} < \nu(c) \le \frac{c^2}{4} + \frac{1}{4}$$

$$\nu(c) \to 0 \quad \text{as } c \searrow 0$$

$$\nu(c) - \frac{c^2}{4} \to \frac{j_0^2}{8\pi^2} \quad \text{as } c \nearrow +\infty$$

($j_0 \equiv$ first positive zero of the Bessel function J_0)

$$u_{tt} - u_{xx} + cu_t + \lambda u = f(t, x)$$

Proof

- Green's function: we must "meet" it

Applications to Sine-Gordon Equation

$$u_{tt} - u_{xx} + cu_t + a\sin u = f(t,x) + s \tag{1}$$

with $a > 0$, $f \in L^1(\mathbb{T}^2)$, $\int_{\mathbb{T}^2} f = 0$ and $s \in \mathbb{R}$.

Result 1 (Qualitative)

If $|a| \leq \nu(c)$, there exists $I \subseteq \mathbb{R}$ nonempty closed interval such that (1) has solutions if and only if $s \in I$.

Applications to Sine-Gordon Equation

$$u_{tt} - u_{xx} + cu_t + a \sin u = f(t,x) \tag{2}$$

with $a > 0$, $f \in L^1(\mathbb{T}^2)$, $\int_{\mathbb{T}^2} f = 0$.

Result 2 (Quantitative)

$\|U\|_\infty \leq \frac{\pi}{2}$, $a \leq \nu(c)$ $\quad \Rightarrow$

$\exists u$ doubly periodic solution of (2) such that $\|u - U\|_\infty \leq \dfrac{\pi}{2}$.

$$u_{tt} - u_{xx} + cu_t + a\,\mathsf{sin}\,u = f(t,x) \tag{2}$$

Auxiliar problem

$$u_{tt} - u_{xx} + cu_t = f(t,x),\ \int_{\mathbb{T}^2} u = 0 \Rightarrow \exists_1\, U \in C(\mathbb{T}^2) \text{ solution}$$

$\|U\|_\infty \leq \frac{\pi}{2} \quad \Rightarrow$

$$u_* = U - \frac{\pi}{2},\ u^* = U + \frac{\pi}{2} \quad \text{lower and upper solutions of (2)}$$

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