Maximum principles in a certain hyperbolic equation

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Abstract. The maximum principle is a typical tool to be employed with elliptic and parabolic partial differential equations when initial and/or boundary conditions are considered. On the other hand, some of these principles have been proved for hyperbolic equations with initial conditions.

In this note we will see an example of maximum principle for a hyperbolic equation with boundary conditions, namely, the telegraph equation

$$u_{tt} - \Delta_x u + cu_t - \lambda u = f(t, x).$$

In fact, when we look for periodic solutions in space and time, a maximum principle arises in a natural way.

1. Introduction

In this note we consider the hyperbolic equation

$$u_{tt} - \Delta_x u + cu_t + \lambda u = f(t, x)$$

where f and u are periodic functions in each of its variables, that is

$$u(t + T_t, x_1, \ldots, x_n) = u(t, x_1, \ldots, x_n);$$

$$u(t, x_1, \ldots, x_i + T_x, \ldots, x_n) = u(t, x_1, \ldots, x_i, \ldots, x_n),\ 1 \le i \le n.$$

The problem will be to determine the values of the parameters c and λ such that there is a maximum principle. Loosely speaking, the maximum principle means that

$$f \ge 0 \Rightarrow u \ge 0.$$

Maximum principles for elliptical and parabolic equations are well known. For example, Protter and Weinberger's book [8] is a classical reference. For hyperbolic equations the situation is different. In the last chapter of [8] several maximum principles for initial value problems are shown. These principles arose in the study

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of Tricomi's equation, an equation of mixed type. The reference [3] can be consulted as a recent paper in this line.

New results related to boundary problems are presented here. They have been obtained in successive collaborations with J. Mawhin and R. Ortega [4, 5, 7]. Therefore, the proofs and the details can be seen in these references. An application of these results is to show the validity of the method of upper and lower solutions for the study of the sine-Gordon equation

$$u_{tt} - \Delta_x u + cu_t + a \sin u = f(t, x).$$

The rest of this note is organized in three sections. In Section 2 the problem for spatial dimension equal to one is studied. In Section 3 a numerical answer is given to the question of computing c and λ for dimension one also. Finally, in Section 4, we comment some results for higher dimensions and bounded solutions in time.

2. The case of spatial dimension one

We consider the problem

$$\left.\begin{array}{l} u_{tt} - u_{xx} + cu_t + \lambda u = f(t, x) \\ u(t, x) \text{ doubly periodic} \end{array}\right\} \tag{1}$$

where we suppose that f is doubly periodic too. In order to be clearer, the periods in t and x are equal; moreover $T_t = T_x = 2\pi$. By integration over $(0, 2\pi) \times (0, 2\pi)$, it is obvious that a maximum principle is possible only when $\lambda > 0$. The next examples give us a first approximation to the answer of our problem: some relations between c and λ.

Example 2.1. We consider the function $u(t, x) = 1 - \cos t \cos x - \varepsilon \cos nt$, where $n \in \mathbb{Z}$ such that $n^2 > \lambda$ and $\varepsilon > 0$ is small. This function changes sign and if we calculate f when $c = 0$, we obtained

$$f(t, x) = \lambda(l - \cos t \cos x) + \varepsilon(n^2 - \lambda) \cos nt.$$

This function is positive and therefore it is not possible to obtain a maximum principle for $c = 0$. From now on we will suppose that there is friction, $c > 0$.

By the way, we can study the case $c < 0$ with the change of variable $s = -t$, that is, we take the time in the opposite direction.

Example 2.2. Now we suppose that $f = f(x)$. In this case we are considering the problem

$$\left.\begin{array}{l} -u_{xx} + \lambda u = f(x) \\ u(x) \;\; 2\pi\text{−periodic} \end{array}\right\}.$$

With the help of the most basic of the maximum principles [Chapter 1 of [8]], we have that, if $f \geq 0$ and $\int_0^{2\pi} f > 0$, $u > 0$ for any $\lambda > 0$.

Example 2.3. If $f = f(t)$, we have

$$\left.\begin{array}{l} u_{tt} + cu_t + \lambda u = f(t) \\ u(t) \;\; 2\pi\text{−periodic} \end{array}\right\} .$$

We can use the anti-maximum principle to conclude that, if $f \geq 0$ and $\int_0^{2\pi} f > 0$, $u > 0$ if and only if

$$0 < \lambda \leq \frac{c^2}{4} + \frac{1}{4}. \tag{2}$$

Remark 2.1. In examples 2.2 and 2.3, it is possible to compute the respective Green's functions and conclude the same conditions.

After these examples an initial conjecture would be that we will have a maximum principle for (1) if and only if (2) holds. As we shall see later, we are going to need a condition more restrictive than (2).

At this point it is convenient to precise our framework. If we define $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, we can consider the problem

$$u_{tt} - u_{xx} + cu_t + \lambda u = f \quad in \;\; \mathfrak{D}'(\mathbb{T}^2). \tag{3}$$

This problem is equivalent to (1) via the obvious projection from $\mathbb{R}^2$ to $\mathbb{T}^2$ (the 2-dimensional torus). By a solution of (3) we understand a function $u \in L^1(\mathbb{T}^2)$ satisfying

$$\int_{\mathbb{T}^2} u(\phi_{tt} - \phi_{xx} - c\phi_t - \lambda\phi) = \int_{\mathbb{T}^2} f\phi \quad \forall \phi \in \mathfrak{D}(\mathbb{T}^2).$$

As a first result we have the existence and the regularity of the solution.

Proposition 2.4. *If $c > 0$ and $\lambda > 0$, (3) has a unique solution u. Moreover u is continuous.*

Remark 2.2. We can improve this result in two ways.

i) If $f \in L^p(\mathbb{T}^2)$, $1 < p < \infty$, then $u \in C^{0,\alpha}(\mathbb{T}^2)$, $\alpha = 1 - \frac{1}{p}$.
ii) When the function f is replaced by a measure then (3) has a unique solution $u \in L^\infty(\mathbb{T}^2)$. This result is useful in order to compute the Green's function associated to problem (3).

The main result of this section is the following

Theorem 2.5. *Let $c > 0$ be fixed. There exists a function*

$$\nu : (0, \infty) \to (0, \infty), \quad c \mapsto \nu(c)$$

such that there is a maximum principle for (3) if and only if

$$\lambda \in (0, \nu(c)].$$

Moreover, the maximum principle is always strong, that is

$$f \geq 0 \;\; a.e.\mathbb{T}^2, \;\; \int_0^{2\pi} f > 0 \Rightarrow u > 0 \quad \mathbb{T}^2,$$

and the function ν satisfies

i) $\frac{c^2}{4} < \nu(c) \leq \frac{c^2}{4} + \frac{1}{4}$,

ii) $\nu(c) \to 0$ *as* $c \searrow 0$,

iii) $\nu(c) - \frac{c^2}{4} \to \frac{j_0^2}{8\pi^2}$ *as* $c \nearrow +\infty$,

where j_0 *is the first positive zero of the Bessel function* J_0.

We finish this section with an useful remark which will be employed later.

Remark 2.3. Let us consider the Dirac's measure concentrated on $\bar{0} \in \mathbb{T}^2$. In precise terms,

$$< \delta_{\bar{0}}, \phi >= \phi(\bar{0}, \bar{0}) \quad \forall \phi \in C(\mathbb{T}^2).$$

If we replace $f(t,x)$ by the Dirac's measure in (3), the solution is the Green's function associated to problem (3), $G_\lambda(t,x)$. Now, we have an explicit formula for the solution of (3). Namely,

$$u(t,x) = \int_{\mathbb{T}^2} G_\lambda(t-\tau, x-\xi) f(\tau,\xi) \; d\tau d\xi. \tag{4}$$

From this expression, we can prove that u is positive if G_λ and f are positive too. On the other hand, if G_λ is negative in some point then there will not be maximum principle.

3. A numerical approximation to $\nu(c)$

At this moment we can not compute the exact value of $\nu(c)$. But it is possible to obtain a good enough approximation for $c > 1$.

Before exposing the idea for this approximation, we need a theoretical result. We consider the evolution problem

$$\left.\begin{array}{l} u_{tt} - u_{xx} + cu_t + \lambda u = f(t,\cdot), \quad t \in \mathbb{R}, \; x \in \mathbb{T} \\ u_{|t=0} = \phi \in H^1(\mathbb{T}), \;\; \dot{u}_{|t=0} = \psi \in L^2(\mathbb{T}) \end{array}\right\} \tag{5}$$

with $f \in C(\mathbb{T}, L^2(\mathbb{T}))$. Because the equation of (5) is asymptotically stable then all the solutions converge to the doubly periodic solution. Moreover, if f is no negative, and $\int_0^{2\pi} f > 0$, then there exists $T = T(\phi,\psi)$ such that $u(t,x)$ is positive for all $x \in \mathbb{T}$ and $t > T$.

Remark 3.1. If $c > 1$ and $\lambda < c^2$, we can use the Liapunov's function

$$V(\phi,\psi) = \int_{\mathbb{T}} \{\psi^2 + (\phi')^2 + \frac{c^2}{2}\phi^2 + c\phi\psi\}$$

for prove that (5) is asymptotically stable.

Now we are ready. The idea is to solve (5) with f equal to the Dirac's measure. The solution in this case is G_λ for each value of λ. Finally, we must observe the sign of G_λ.

In order to develop a numerical program, we consider a spatial discretization of the telegraph equation (Figure 1). That is, we consider a finite set of points

$\{X_1, X_2, \ldots, X_N\}$ that move along parallel straight lines. Sometimes this procedure is called the method of lines [2].

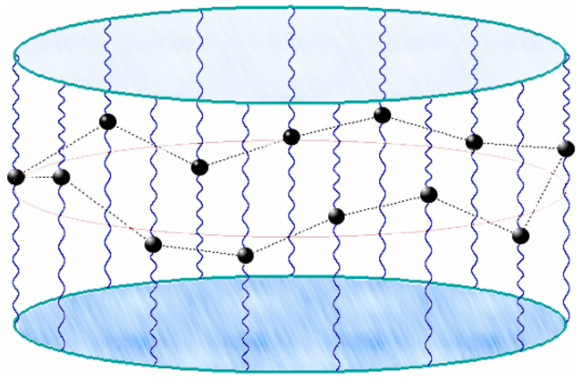

FIGURE 1. Discretization of the telegraph equation

In Figure 1 we have represented the term λu of (3) as the action of two springs on each point X_i. Moreover, as it is usual, cu_t is a term of friction proportional to the velocity of X_i.

Let us consider the function $S_i(t) = (y_i(t), v_i(t))$, the position and the velocity of the point X_i at time t. We have the system of differential equations with impulses

$$(6)\quad \begin{cases} \begin{pmatrix} y_i(t) \\ v_i(t) \end{pmatrix}' = \begin{pmatrix} v_i(t) \\ \frac{y_{i+1}(t)-2y_i(t)+y_{i-1}(t)}{(\Delta x)^2} - cv_i(t) - \lambda y_i(t) \end{pmatrix}, \ t \in (2n\pi, 2(n+1)\pi) \\ (y_1(2n\pi+), v_1(2n\pi+)) = (y_1(2n\pi), v_1(2n\pi)+1) \\ (y_i(2n\pi+), v_i(2n\pi+)) = (y_i(2n\pi), v_i(2n\pi)+1), \quad i = 2, \ldots, N \end{cases}$$

with $n \in \mathbb{N}$ and $\Delta x = \frac{2\pi}{N}$. Because we look for periodic solutions, we must observe in (6) that $y_0(t) = y_N(t)$ and $y_{N+1}(t) = y_1(t)$. On the other hand, the Dirac's measure is represented by a periodic impulse on the first point.

Remark 3.2. Intuitively, the Dirac's measure can be represented as a periodic impulsive force on the point X_1. Therefore, y_1 is a continuous function but v_1 changes instantaneously with each impulse. For a rigorous result about this fact, see [Section 2, [6]].

To solve (6), we use the fourth-order Runge-Kutta's method with $\Delta t = \Delta x$ [2]. The initial values are $S_i(0) = (0, 0)$, $1 \leq i \leq N$. Finally, having in mind the asymptotic stability of (5), we get G_λ when

$$(7)\qquad |S_i((j+1)N\Delta t) - S_i(jN\Delta t)| \approx 0.$$

Remark 3.3. We are looking for the Green's function as the periodic solution of (5) for a certain f. Because we have asymptotic stability for the periodic solutions of (5), it is clear that (7) has to be the stop condition. In fact, we want that the numerical solution does not change after one period of time, that is, $2\pi = N\Delta t$.

Maybe we can think that (7) is restrictive but is better to simplify the numerical program. In fact, we could have used

$$|S_i((j+N)\Delta t) - S_i(j\Delta t)| \approx 0.$$

In the following picture (Figure 2) we see a comparative among the graphics of the functions

i) $f_1(c) = \frac{c^2}{4} + \frac{1}{4}$,

ii) $f_2(c) = \frac{c^2}{4} + \frac{1}{4}\frac{j_0^2}{2\pi^2}$,

iii) $f_3(c) = \nu(c)$ (estimated values),

iv) $f_4(c) = \frac{c^2}{4}$.

from top to bottom. In view of it, we can think that the asymptotic estimation of $\nu(c)$, given by Theorem 2.5, is a good approximation to the exact value if $c > 1$.

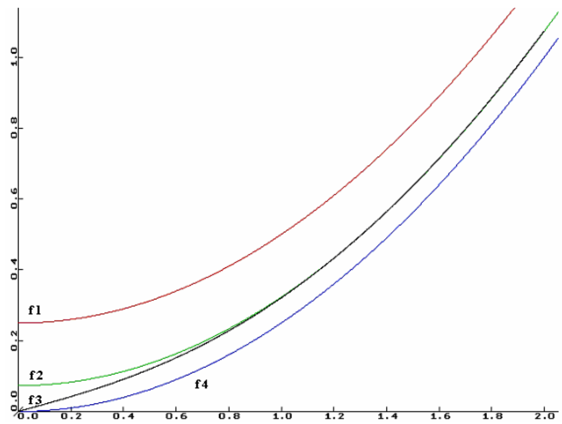

FIGURE 2. Estimation of $\nu(c)$

In order to state the accuracy of Figure 2, we finish this section with the following table.

c	1	1.25	1.5	1.75	2
$f_2(c)$	0.3232	0.46387	0.6357	0.8389	1.0732
$f_3(c)$	0.3211	0.46384	0.6367	0.8404	1.0750

c	3	4	5	6	7
$f_2(c)$	2.3232	4.0732	6.3232	9.0732	12.3232
$f_3(c)$	2.3255	4.0758	6.3260	9.0763	12.3267

4. Extensions to bounded solutions and higher dimensions

Another type of interesting solutions of the telegraph equation are the solutions which are periodic in space and bounded in time [4], $u \in L^\infty(\mathbb{R} \times \mathbb{T})$. Of course, in this case we suppose that $f \in L^\infty(\mathbb{R} \times \mathbb{T})$. Before giving the theorem in this case, we are going to consider the telegraph equation up to spatial dimension three [5],

$$u_{tt} - \Delta_x u + cu_t + \lambda u = f(t,x) \quad in \ \mathfrak{D}'(\mathbb{R} \times \mathbb{T}^d) \tag{8}$$

with $d = 1, 2, 3$.

We sum up in the following theorem the results of [4] and [5].

Theorem 4.1. *Let f be a function in $L^\infty(\mathbb{R} \times \mathbb{T}^d)$. Moreover, we consider $c > 0$ and $0 < \lambda \le \frac{c^2}{4}$. Then*

i) *equation (8) has a unique solution $u \in L^\infty(\mathbb{R} \times \mathbb{T}^d)$;*

ii) *if $d = 1$, $u \in W^{1,\infty}(\mathbb{R} \times \mathbb{T})$;*

iii) *if $d = 2$, u is continuous;*

iv) *there is a maximum principle, that is*

$$f \ge 0 \ a.e. \ in \ \mathbb{R}^{1+d}, \ then \ u(t,x) \ge 0 \ a.e. \ in \ \mathbb{R}^{1+d}.$$

Remark 4.1. For bounded in time solutions we have the exact value for $\nu(c)$. Indeed, assume $f = f(t)$ is T-periodic in Example 2.3. Then there is a maximum principle if and only if

$$0 < \lambda \le \frac{c^2}{4} + \left(\frac{\pi}{T}\right)^2.$$

Since T is arbitrary, (8) has no maximum principle if $\lambda > \frac{c^2}{4}$.

Remark 4.2. The maximum principle is not strong. Indeed, if we define

$$u(t) = \left(t^+\right)^3 e^{-at},$$

then $f = u'' + cu' + \lambda u$ is bounded and nonnegative if $a = \frac{c - \sqrt{c^2 - 4\lambda}}{2}$, $\lambda \in \left(0, \frac{c^2}{4}\right]$.

The situation changes completely for $d = 4$. It is possible to give an example in order to prove that there is no maximum principle for (8) if $\lambda = \frac{c^2}{4}$ and $d = 4$ [[5], Section 5]. Therefore, the maximum principle is a peculiar property of dimensions 1, 2 and 3.

We return to the case of periodic solutions in time and space. We consider the problem

$$u_{tt} - \Delta_x u + cu_t + \lambda u = f(t,x) \quad in \ \mathfrak{D}'(\mathbb{T}^d) \tag{9}$$

with $d \ge 2$. From theorem 4.1 it is easy to give another one that partially generalizes Proposition 2.4 and Theorem 2.5 for $d = 2, 3$. Of course, following the ideas of [7], there will be a result about $\nu(c) = \nu(c, d)$, $d = 2, 3$. Finally, we expect that there will exist an example of non existence of maximum principle for periodic solutions and d=4. It will be a suitable modification of the example in the case of bounded solutions.

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