A Maximum Principle with Applications to the Forced Sine-Gordon Equation

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1 Introduction

In this note I report some work done in collaboration with R. Ortega. The details and a list of references can be seen in [3].

Our objective is to study the forced sine-Gordon equation. In particular we want to study properties of solutions of the problem

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) + cu_t(t,x) + a\sin u(t,x) = f(t,x), \\ u \text{ doubly periodic (same period in } t \text{ and } x). \end{cases} \tag{1.1}$$

As we can see, the sine-Gordon equation is a P.D.E. that is very similar to the O.D.E. called the forced pendulum equation

$$\begin{cases} \ddot{u}(t) + c\dot{u}(t) + a\sin u(t) = f(t), \\ u \text{ periodic.} \end{cases} \tag{1.2}$$

As we all know, there are many results for the forced pendulum equation. For example in [2] Mawhin obtained results for this equation using the method of upper and lower solutions.

So we try to generalize those results to equation (1.1). In order to do this, it is known that it is necessary to have a maximum principle for the associated linear equation. This is our first objective: we are going to see when a maximum principle holds and some properties of it.

At the end of the note we are going to go back to the sine-Gordon equation to see how the method of upper and lower solutions applies to it.

2 Some examples

The linear equation associated to the sine-Gordon equation is the telegraph equation. It is given by the operator

$$\mathfrak{L}_\lambda u = u_{tt} - u_{xx} + cu_t - \lambda u,$$

acting on doubly periodic functions (with the same period in t and x).

Loosely speaking, we understand that the operator $\mathfrak{L}_\lambda$ has a maximum principle if it satisfies the condition

$$\mathfrak{L}_\lambda u = f, \quad f \geq 0 \quad \Rightarrow \quad u \geq 0.$$

Also we can consider this principle as a positivity principle for the inverse operator of $\mathfrak{L}_\lambda$.

It is interesting to begin with some easy examples in order to have an idea about what to expect. First, what happens when we have no friction, that is, when $c = 0$?

2.1 No maximum principle for $\mathbf{c} = 0$

We consider the equation

$$\mathfrak{L}_\lambda u = u_{tt} - u_{xx} - \lambda u = f.$$

If $\lambda = 0$, then we integrate over a fundamental region Ω and we have that $\int_\Omega f = 0$ is a necessary condition for existence of doubly periodic solutions. But this is impossible if $f \geq 0$. So there is no solution if f is positive.

If $\lambda \neq 0$ we can build a counter-example. Let $\lambda < 0$ be a fixed constant (if $\lambda > 0$ then the reasoning is similar). We take $u(t, x) = 1 - \cos t \cos x$ (an eigenfunction of the wave operator). Neither this function nor its image by the operator $\mathfrak{L}_\lambda$ changes sign. The minimum of u is equal to zero. So we try to perturb it with another function w and a small parameter ε such that the new function $u_* = u + \varepsilon w$ changes sign but its image $\mathfrak{L}_\lambda u_*$ does not. In order to do this we can make an easy computation.

Well, we have no maximum principle when $c = 0$. Now what happens when we have friction, that is, if $c \neq 0$?

2.2 Towards the maximum principle for $\mathbf{c} > 0$

For $c > 0$ we are going to consider that u depends only on one variable, x or t. If $u = u(x)$ we have the second order operator

$$\ell_\lambda u = -\frac{d^2 u}{dx^2} - \lambda u.$$

There is a classical maximum principle. Such a principle is

$$\text{Maximum Principle for } \ell_\lambda u \Leftrightarrow -\lambda > 0.$$

If $u = u(t)$ we have the second order operator

$$L_\lambda u = \frac{d^2 u}{dt^2} + c\frac{du}{dt} - \lambda u.$$

Now we cannot apply classical tools. Some authors speak of an anti-maximum principle (see for instance [1]). In this case

$$\text{Maximum Principle for } L_\lambda u \Leftrightarrow 0 < -\lambda \leq \tfrac{c^2}{4} + \tfrac{1}{4}.$$

Moreover if $-\lambda > \frac{c^2}{4} + \frac{1}{4}$, then we can find f positive such that u changes sign.

With these results in mind, one could think that for the telegraph equation one should have

$$\text{Maximum Principle for } \mathfrak{L}_\lambda \Leftrightarrow 0 < -\lambda \leq \tfrac{c^2}{4} + \tfrac{1}{4},$$

but this is not true. In fact, we are going to prove that there exists λ^* such that we have a maximum principle if and only if $\lambda^* \leq \lambda < 0$. Moreover $\frac{c^2}{4} < -\lambda^* \leq \frac{c^2}{4} + \frac{1}{4}$ and, in general, $-\lambda^* \neq \frac{c^2}{4} + \frac{1}{4}$.

3 Concept of solution and regularity

Now we have an approximate idea about what we can get. Therefore we can start with our study. But first we make an identification of spaces which will be very useful.

We work with doubly periodic functions. So, if we know a function over a fundamental domain ($[0, 2\pi] \times [0, 2\pi]$ for example) then we know it over the whole plane by periodicity.

We define the torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z})$. We project $\mathbb{R}^2$ onto $\mathbb{T}^2$:

$$\begin{array}{rccc} \Pi: & \mathbb{R}^2 & \longrightarrow & \mathbb{T}^2, \\ & (x,y) & \longmapsto & (\bar{x}, \bar{y}), \\ & \bar{x} & = & x + 2\pi\mathbb{Z}, \\ & \bar{y} & = & y + 2\pi\mathbb{Z}. \end{array}$$

In this way if we consider spaces on the torus, then we speak about functions on the torus or doubly periodic functions on the plane.

$$\begin{array}{ccccc} & \mathbb{R}^2 & \overset{u \text{ doubly periodic}}{\longrightarrow} & \mathbb{R} \\ \Pi & \downarrow & \nearrow u & \\ & \mathbb{T}^2 & & \end{array}$$

For example we have $L^p(\mathbb{T}^2)$, $C(\mathbb{T}^2)$ (continuous functions), $\mathfrak{D}(\mathbb{T}^2) = C^\infty(\mathbb{T}^2)$ (test functions), $\mathfrak{D}'(\mathbb{T}^2)$ (distributions), etc.

We are ready to give rigorous ideas. First the concept of a solution. Consider the telegraph equation with $f \in L^1(\mathbb{T}^2)$,

$$\begin{cases} \mathfrak{L}_\lambda u(t,x) = u_{tt}(t,x) - u_{xx}(t,x) + c u_t(t,x) - \lambda u(t,x) = f(t,x), \\ u \text{ doubly periodic (same period in } t \text{ and } x). \end{cases} \tag{3.1}$$

We say that $u \in L^1(\mathbb{T}^2)$ is a solution of (3.1) if and only if it verifies

$$\int_{\mathbb{T}^2} u(\mathfrak{L}^*\phi - \lambda\phi) = \int_{\mathbb{T}^2} f\phi, \qquad \forall \phi \in \mathfrak{D}(\mathbb{T}^2),$$

where $\mathfrak{L}u = u_{tt} - u_{xx} + cu_t$ and $\mathfrak{L}^*$ is the formal adjoint of $\mathfrak{L}$. Notice that we consider a weak concept of a solution.

We can get as a first result the regularity of the solution.

Theorem 3.1. *If λ is not a real eigenvalue of $\mathfrak{L}$, then* (3.1) *has a unique solution. Moreover*

1. *if $f \in L^1(\mathbb{T}^2)$ then u is continuous;*
2. *if $f \in L^p(\mathbb{T}^2)$ then u is α-Hölder continuous with $\alpha = 1 - \frac{1}{p}$.*

We want to comment on this last result. When we work with parabolic equations we gain regularity, but for hyperbolic ones this is false in general. However the doubly periodic solutions of the telegraph equation gain regularity. In some sense we can consider that our problem is similar to parabolic problems.

Before going any further, a brief remark about the spectrum of $\mathfrak{L}$. When there is friction, $\mathfrak{L}$ is not selfadjoint and it has a complex spectrum. But we are interested only in real eigenvalues. So we are going to speak about the real spectrum of $\mathfrak{L}$ as the set of real eigenvalues.

In order to calculate it, with Fourier analysis we obtain

$$\lambda_{nm} = m^2 - n^2 + icn \qquad (m \in \mathbb{N}, n \in \mathbb{Z}). \tag{3.2}$$

Observe that if $c = 0$ then $\mathfrak{L}$ is selfadjoint. So we have only real eigenvalues. We can explain this fact if we do "$c \to 0$" in (3.2): the eigenvalues collapse over the real axis. If $\lambda = 0$ we have an infinite-dimensional eigenspace and if $\lambda \neq 0$ we have finite-dimensional eigenspaces.

4 Maximum principle

We use two definitions:

1. $\mathfrak{L}_\lambda$ satisfies the maximum principle if
   - $\lambda \notin \sigma_{\mathbb{R}}(\mathfrak{L})$,
   - $f \in L^1(\mathbb{T}^2)$, $f \geq 0$ a.e. $\mathbb{R}^2 \Rightarrow u(t,x) \geq 0 \quad \forall (t,x) \in \mathbb{R}^2$.
2. $\mathfrak{L}_\lambda$ satisfies the strong maximum principle if
   - $\lambda \notin \sigma_{\mathbb{R}}(\mathfrak{L})$,
   - $f \in L^1(\mathbb{T}^2)$, $f \geq 0$ a.e. $\mathbb{R}^2$, $\int_{\mathbb{T}^2} f > 0 \Rightarrow u(t,x) > 0 \; \forall (t,x) \in \mathbb{R}^2$.

Notice that u is continuous in both cases. So it is defined on the whole plane.

We have the following result.

Theorem 4.1. *There exists a function*

$$\nu : (0,\infty) \to (0,\infty), \quad c \mapsto \nu(c)$$

such that $\mathfrak{L}_\lambda$ satisfies the maximum principle if and only if

$$-\lambda \in (0, \nu(c)].$$

Moreover the maximum principle is always strong and the function ν satisfies

$$\tfrac{c^2}{4} < \nu(c) \le \tfrac{c^2}{4} + \tfrac{1}{4},$$

$$\nu(c) \to 0 \quad \text{as } c \searrow 0,$$

$$\nu(c) - \tfrac{c^2}{4} \to \tfrac{j_0^2}{8\pi^2} \quad \text{as } c \nearrow +\infty,$$

where j_0 is the first positive zero of the Bessel function J_0.

(Observe that j_0 verifies that $\frac{j_0^2}{8\pi^2} < \frac{1}{4}$).

5 Proofs

Now we give a sketch of the proof. It is divided into three steps. In the first and second steps we give the tools. In the third step we see the conclusions.

5.1 Step 1: Green's function

Suppose that u is the solution of our problem. Then we have the integral expression

$$u(t,x) = (G * f)(t,x) = \int_{\mathbb{T}^2} G(t-\tau, x-\xi) f(\tau,\xi)\, d\tau\, d\xi.$$

That is, u is the convolution of G, the Green's function, and f. Then it is clear that we have characterizations for the maximum principle and for the strong maximum principle, which are given in terms of the positivity of G.

Proposition 5.1. *1. The maximum principle holds if and only if $G \ge 0$.*
2. The strong maximum principle holds if and only if $G > 0$ almost everywhere.

So we must determine the sign of G. In order to do this, we calculate an explicit expression of G. If we apply Fourier analysis, then

$$G(t,x) = \frac{1}{4\pi^2} \sum_{m,n=-\infty}^{\infty} \frac{1}{m^2 - n^2 - \lambda + icn}\, e^{i(nt+mx)}.$$

This expression is very useful for regularity results but it is not so useful to determine the sign of G. We must take another way. Let us try with fundamental solutions.

Consider $d = -\lambda - \frac{c^2}{4} \geq 0$. Then the function U, given by

$$U(t,x) = \begin{cases} \frac{1}{2} e^{-\frac{c}{2}t} J_0\left(\sqrt{d(t^2-x^2)}\right), & |x| < t, \\ 0\,, & \text{otherwise} \end{cases}$$

is the fundamental solution of $\mathfrak{L}_\lambda u = \delta_0$ in $\mathfrak{D}'(\mathbb{R}^2)$ ($\delta_0 \in M(\mathbb{R}^2)$ such that $< \phi, \delta_0 >= \phi(0,0), \ \ \forall \phi \in \mathfrak{D}(\mathbb{R}^2)$).

We are looking for periodic solutions. So we make copies of U, we translate them and we add them. In this way we find the Green's function for our problem, arriving at the expression

$$G(t,x) = \sum_{(n,m)\in\mathbb{Z}^2} U(t + 2\pi n, x + 2\pi m) \quad \text{in } \mathbb{T}^2. \tag{5.1}$$

This double sum converges and so G is well defined. Moreover, G is continuous except in the characteristic lines (the family of lines $\mathcal{C} = \{x \pm t = 2\pi N,\ N \in \mathbb{Z}\}$).

With the expression (5.1) of G, we can study the sign in a much better way than with the previous Fourier series. There is one special case in which we can improve the situation.

Indeed, if $d = 0$ then $J_0(0) = 1$ and U and G have easy expressions. In particular,

$$G(t,x) = \begin{cases} \frac{1}{2} \frac{1+e^{-c\pi}}{(1-e^{-c\pi})^2}\, e^{-\frac{c}{2}t}, & \text{if } (t,x) \in \mathcal{D}_{10}, \\ \frac{e^{-c\pi}}{(1-e^{-c\pi})^2}\, e^{-\frac{c}{2}t}, & \text{if } (t,x) \in \mathcal{D}_{01}. \end{cases}$$

(We let $\mathcal{D}_{ij}$ denote the connected component of $\mathcal{D} = \mathbb{R}^2 - \mathcal{C}$ with center at the point $(i\pi, j\pi)$, where $i + j$ is an odd number).

With this explicit expression we easily deduce the following properties.

Proposition 5.2. *1. Jumps in discontinuities are known (and independent of d).*

2. G is analytic in $\overline{\mathcal{D}_{10}}$ and $\overline{\mathcal{D}_{01}}$.

3. G is not identically zero in $\overline{\mathcal{D}_{10}}$ and $\overline{\mathcal{D}_{01}}$.

4. $Z = \{(t,x) \in \mathbb{R}^2 \ / \ G(t,x) = 0\}$ has measure zero. So whenever there is a maximum principle it is strong.

All these properties can also be proved for $d > 0$.

We must point out that we can get G in another way. If we consider $\lambda = -\frac{c^2}{4}$ and make the change $u = e^{-ct/2}v$, then we arrive at the wave equation and we know its explicit solution (D'Alembert formula). Then we make the inverse change and we have an explicit integral expression of the solution of the telegraph equation. Straightforward computations lead us to G.

5.2 Step 2: Linear positive operators

In this step we are going to review some facts about linear positive operators. Let us consider a Banach space X and a closed cone C in X. Then X is an ordered Banach space with the ordering

$$x, y \in X, \quad x \geq y \Leftrightarrow x - y \in C.$$

Given an operator A over X, we say that

- A is positive if $A(\mathrm{C}) \subseteq \mathrm{C}$,
- A is strongly positive if $A(\mathrm{C} - \{0\}) \subseteq \mathring{\mathrm{C}} \quad (\mathring{\mathrm{C}} \neq \emptyset)$.

If A is compact and strongly positive, then we can apply Krein-Rutman's theory and we obtain the following result.

Theorem 5.3. *There exists a unique positive λ_0 which is the spectral radius of A and such that its eigenfunction u is strictly positive.*

As a corollary we have a sufficient condition in order to find positive solutions of certain linear equations.

Corollary 5.4. *Consider the system*

$$\lambda\varphi = A\varphi + f, \quad f \geq 0,$$

with A in the same conditions as before. If $\lambda > \lambda_0 = \varrho(A)$, then there exists a unique solution φ that is positive.

In order to apply all these results we use the notation $X = C(\mathbb{T}^2)$, $\mathrm{C} = \{u \in X : \ u \geq 0 \text{ in } \mathbb{R}^2\}$ and $A_\lambda = \mathfrak{L}_\lambda^{-1}$ (where λ is not a real eigenvalue of $\mathfrak{L}$ and λ is such that the strong maximum principle holds for $\mathfrak{L}_\lambda$).

5.3 Step 3: Conclusions

1. (a) For $\lambda = \frac{-c^2}{4}$, $\mathfrak{L}_\lambda$ has a maximum principle (remember the case $d = 0$).

   (b) By the theory of linear positive operators, we know that if $\mathfrak{L}_{\lambda_*}$ satisfies a maximum principle, then $\mathfrak{L}_\lambda$ satisfies a maximum principle for each $\lambda \in [\lambda_*, 0)$. So, taking $\lambda_* = -\frac{c^2}{4}$, we have a maximum principle in $[-\frac{c^2}{4}, 0)$.

(c) If $\lambda \notin \sigma_{\mathbb{R}}(\mathfrak{L})$ and $G_\lambda \geq \delta > 0$ (this is equivalent to $\operatorname{ess\,inf}_{\mathbb{T}^2} G_\lambda > 0$) then there exists ε_0 such that $\mathfrak{L}_{\lambda+\varepsilon}$ verifies the strong maximum principle if $|\varepsilon| \leq \varepsilon_0$. Since $\operatorname{ess\,inf}_{\mathbb{T}^2} G_\lambda > 0$ for $\lambda = -\frac{c^2}{4}$, then we have a maximum principle for some $\lambda < \frac{-c^2}{4}$.

(d) We define $\nu = -\inf\{\lambda \in (-\infty, 0) : \ \mathfrak{L}_\lambda$ satisfies the maximum principle$\}$. We can prove that this infimum is a minimum. Moreover, we know that ν depends on c ($\nu = \nu(c)$) and that $\nu(c) < \frac{c^2+1}{4}$. So we have a maximum principle for $\mathfrak{L}_\lambda$ when $\lambda \in [-\nu, 0)$. We must remark that $\operatorname{ess\,inf}_{\mathbb{T}^2} G = 0$ for $-\nu(c)$.

2. In order to prove the asymptotic results:

(a) When c tends to zero we take $\bar{\lambda} < 0$ fixed and $u_*(t,x) = 1 - \cos t \cos x + \varepsilon w(t,x)$ (remember that we have used this function before). This function changes sign and $f = (u_*)_{tt} - (u_*)_{xx} - \bar{\lambda} u_*$ is positive. If c is small, then $g = \mathfrak{L}_{\bar{\lambda}} u_* = f + c(u_*)_t$ is positive too. Therefore the maximum principle does not hold for that $\bar{\lambda}$.

(b) When c tends to infinity we use the estimates

$$|e^{\frac{c}{2}t} G(t,x) - \frac{1}{2} J_0\left(\sqrt{d(t^2 - x^2)}\right)| \leq k_1 e^{-c\pi} \quad \text{if } (t,x) \in \mathcal{D}_{10},$$

$$\begin{aligned} |e^{\frac{c}{2}(t+2\pi)} G(t,x) - \frac{1}{2}\{J_0\left(\sqrt{d[(t+2\pi)^2 - (x-2\pi)^2]}\right) \\ + J_0\left(\sqrt{d[(t+2\pi)^2 - x^2]}\right)\}| \leq k_2 e^{-c\pi} \quad \text{if } (t,x) \in \mathcal{D}_{01}. \end{aligned}$$

And with this we finish the "proof". We go on with the last part of this note.

6 Upper and lower solutions

We return to the nonlinear equation

$$\mathfrak{L}u = u_{tt} - u_{xx} + c u_t = F(t,x,u) \quad \text{ in } \mathfrak{D}'(\mathbb{T}^2), \tag{6.1}$$

where $F : \mathbb{T}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Carathéodory conditions. We say that u_* is a *lower solution* if and only if

$$u_* \in L^\infty(\mathbb{T}^2) \ \text{ and } \ \mathfrak{L}u_* \leq F(t,x,u_*) \ \text{ in } \mathfrak{D}'(\mathbb{T}^2).$$

We say that u^* is an *upper solution* if and only if it verifies the reversed inequality. Again we are using this concept in a weak sense, but in this case we must take positive test functions.

We have a result of classical style about the existence of solutions.

Theorem 6.1. *Let u^*, u_* be upper and lower solutions of* (6.1) *satisfying*

$$u_* \leq u^* \quad a.e.\ \mathbb{R}^2.$$

In addition, assume

$$F(t,x,u_2) - F(t,x,u_1) \geq -\nu(u_2 - u_1) \tag{6.2}$$

for a.e. $(t,x) \in \mathbb{R}^2$ and every $u_1,\ u_2$, with

$$u_*(t,x) \leq u_1 \leq u_2 \leq u^*(t,x).$$

(The constant $\nu = \nu(c)$ was defined by Theorem 4.1). Then (6.1) *has a solution $u \in C(\mathbb{T}^2)$ satisfying*

$$u_* \leq u \leq u^* \quad a.e.\ \mathbb{R}^2.$$

We use the method of upper and lower solutions but we must observe two facts:

1. We want to apply the maximum principle. So we need that the associated linear equation satisfies $F(t,x,u) \geq -\nu(c)$. For this we impose the condition (6.2).

2. In this case the upper and lower solutions are weak. So we need a maximum principle for measures.

Lemma 6.2. *If $\lambda \in [-\nu, 0)$ and $\mathfrak{L}_\lambda u \geq 0$ in $\mathfrak{D}'(\mathbb{T}^2)$ for $u \in L^1(\mathbb{T}^2)$ then $u \geq 0$ a.e. in $\mathbb{R}^2$.*

7 Applications to sine-Gordon equation

At last we will see two results for the sine-Gordon equation, one qualitative and another one quantitative.

In the first one, we consider a free parameter s, but the mean value of f is equal to zero,

$$u_{tt} - u_{xx} + cu_t + a\sin u = f(t,x) + s. \tag{7.1}$$

The problem is equivalent to (1.1). Remember that $f = \tilde{f} + \bar{f}$ with $\int_{\mathbb{T}^2} \tilde{f} = 0$ and $\frac{1}{4\pi^2}\int_{\mathbb{T}^2} f = \bar{f}$.

Following Mawhin's ideas in his result for the forced pendulum (see [2]) we have a necessary and sufficient condition for existence of solutions.

Theorem 7.1. *If $|a| \leq \nu(c)$, there exists an $I \subseteq \mathbb{R}$ nonempty closed interval such that* (7.1) *has solutions if and only if $s \in I$.*

For the quantitative result we forget the parameter s and suppose that $\int_{\mathbb{T}^2} f = 0$:

$$u_{tt} - u_{xx} + cu_t + a \sin u = f(t,x). \tag{7.2}$$

We need to solve the auxiliary problem

$$\mathfrak{L}u = f(t,x), \ \int_{\mathbb{T}^2} u = 0 \quad (\text{with } f \in L^1(\mathbb{T}^2)).$$

For this problem we have that there exists a unique solution $U \in C(\mathbb{T}^2)$. Moreover if $\|U\|_\infty \leq \frac{\pi}{2}$, then $u_* = U - \frac{\pi}{2}$ and $u^* = U + \frac{\pi}{2}$ are lower and upper solutions of (7.2).

Theorem 7.2. *If $\|U\|_\infty \leq \frac{\pi}{2}$ and $0 < a \leq \nu$, then* (7.2) *has a doubly periodic solution u such that $\|u - U\|_\infty \leq \frac{\pi}{2}$.*

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