Almost periodic solutions of forced sine-Gordon equations

A note based on some joint works with
J. Mawhin and R. Ortega

Forced dissipative sine-Gordon equation

$$u_{tt} - \Delta_x u + c u_t + a \sin u = f(t, x) \tag{1}$$

Dimensions: $n = 1$, 2 or 3

Parameters: $0 < a < \frac{c^2}{4}\,,\;\; c > 0$

Function: f almost periodic

Question: u almost periodic ?

Almost periodicity (Bochner definition[(*)])

$$\left.\begin{array}{l} f \in C(\mathbb{R}\times\mathbb{T}^n) \\ \\ \alpha = (\alpha_0, \tilde{\alpha}) \in \mathbb{R}\times\mathbb{T}^n \end{array}\right\} \Rightarrow \left\{\begin{array}{l} (T_\alpha f)(t,x) = \\ \\ f(t+\alpha_0, x+\tilde{\alpha}) \end{array}\right.$$

- $f \in AP(\mathbb{R}\times\mathbb{T}^n)$ if,

  "$\forall\ \{\alpha_m\}_{m\in\mathbb{N}} \subset \mathbb{R}\times\mathbb{T}^n \quad \exists\{\alpha_k\}_{k\in\mathbb{N}} \subseteq \{\alpha_m\}_{m\in\mathbb{N}}$ /

  $\{T_{\alpha_k} f\}_{k\in\mathbb{N}}$ has a uniform limit"

- $(AP(\mathbb{R}\times\mathbb{T}^n), \|\cdot\|_{L^\infty})$ is a Banach space immersed in $L^\infty(\mathbb{R}\times\mathbb{T}^n)\cap C(\mathbb{R}\times\mathbb{T}^n)$

(*) $\mathbb{R}\times\mathbb{T}^n$ is a locally compact topological group

Forced dissipative pendulum equation

$$\ddot{u} + c\dot{u} + a \sin u = h(t) \tag{2}$$

Theorem (Mawhin '98)

$$\left.\begin{array}{l} c \geq 0 \\[2ex] h \in AP(\mathbb{R}) \\[2ex] \|h\|_{L^\infty} < a \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \exists_1 \; u \in AP^1(\mathbb{R})^{(*)} \text{ such that} \\[2ex] \quad \bullet \quad u \text{ satisfies (2)} \\[2ex] \quad \bullet \quad \|u - \pi\|_{L^\infty} < \frac{\pi}{2} \end{array}\right.$$

Key: Compactness $\Rightarrow$ Amerio's ideas

(*) $AP^1(\mathbb{R}) = \{u \in AP(\mathbb{R}) \;/\; u \in C^1(\mathbb{R}),\ \dot{u} \in AP(\mathbb{R})\}$

Telegraph equation

$$u_{tt} - \Delta_x u + c u_t + \lambda u = f(t, x) \tag{3}$$

$c > 0,\ \ 0 < \lambda \leq \frac{c^2}{4},\ \ f \in L^\infty(\mathbb{R} \times \mathbb{T}^n)$

- Periodic conditions in x
- $\mathfrak{L}u = u_{tt} - \Delta_x u + c u_t$
- $\mathfrak{L}^* u = u_{tt} - \Delta_x u - c u_t$
- $u \in L^\infty(\mathbb{R} \times \mathbb{T}^n)$ is solution if

$$\int_{\mathbb{R}\times\mathbb{T}^n} (\mathfrak{L}^*\phi + \lambda\phi) u = \int_{\mathbb{R}\times\mathbb{T}^n} f\phi, \quad \forall \phi \in \mathfrak{D}(\mathbb{R} \times \mathbb{T}^n)$$

$$u_{tt} - \Delta_x u + c u_t + \lambda u = f(t,x) \qquad (3)$$

$c > 0,\ \ 0 < \lambda \le \frac{c^2}{4},\ \ f \in L^\infty(\mathbb{R} \times \mathbb{T}^n)$

- $f \in L^\infty(\mathbb{R} \times \mathbb{T}^1) \Rightarrow u \in W^{1,\infty}(\mathbb{R} \times \mathbb{T}^1)$
- $f \in L^\infty(\mathbb{R} \times \mathbb{T}^2) \Rightarrow u \in C(\mathbb{R} \times \mathbb{T}^2)$
- $f \in L^\infty(\mathbb{R} \times \mathbb{T}^3) \Rightarrow u \in L^\infty(\mathbb{R} \times \mathbb{T}^3)$

  ($\exists f \in L^\infty(\mathbb{R} \times \mathbb{T}^3)$ such that $u \notin C(\mathbb{R} \times \mathbb{T}^3)$)

Compactness: only for $n = 1, 2$

- $n = 3$?

Completeness: YES

"Banach Contraction Principle"

$$\ddot{u} + c\dot{u} + a \sin u = h(t) \tag{2}$$

Theorem (Mawhin '98)

$$\left.\begin{array}{l} c \geq 0 \\ \\ h \in AP(\mathbb{R}) \\ \\ \|h\|_{L^\infty} < a \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \exists_1\, u \in AP^1(\mathbb{R}) \text{ such that} \\ \\ \quad \bullet \quad u \text{ satisfies (2)} \\ \\ \quad \bullet \quad \|u - \pi\|_{L^\infty} < \frac{\pi}{2} \end{array}\right.$$

$$\ddot{u} + c\dot{u} - \lambda u = h(t) \tag{4}$$

$c \geq 0, \ \ \lambda > 0$

$$\left.\begin{array}{l} h \in C(\mathbb{R}) \\ \\ \|h\|_{L^\infty} \text{ bounded} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \exists_1 \ u \in C^1(\mathbb{R}) \text{ such that} \\ \\ \quad \bullet \ u \text{ satisfies (4)} \\ \\ \quad \bullet \ \|u\|_{L^\infty} \leq \frac{1}{\lambda} \|h\|_{L^\infty} \\ \\ \quad \bullet \ \|\dot{u}\|_{L^\infty} \leq \frac{1}{\sqrt{\lambda + \frac{c^2}{4}}} \|h\|_{L^\infty} \end{array}\right.$$

Moreover, $h \geq 0 \ \Rightarrow \ u \leq 0$

Ortega and Tarallo '03

- $u(t) = (G * H)(t) = \int_{-\infty}^{+\infty} G(t,s)h(s)\,ds$
- $G(t,s) = -\frac{1}{2\nu}\, e^{c(s-t)/2} e^{-\nu|t-s|}, \quad \nu = \sqrt{\lambda + \frac{c^2}{4}}$

$$\ddot{u} + c\dot{u} - \lambda u = h(t) \tag{4}$$

$c \geq 0, \;\; \lambda > 0$

$h \in AP(\mathbb{R}) \;\Rightarrow\; u \in AP^1(\mathbb{R})$

- $\|T_\alpha u - T_\beta u\|_{L^\infty} \leq \frac{1}{\lambda}\, \|T_\alpha h - T_\beta h\|_{L^\infty}$
- $\|T_\alpha \dot{u} - T_\beta \dot{u}\|_{L^\infty} \leq \frac{1}{\nu}\, \|T_\alpha h - T_\beta h\|_{L^\infty} \quad (\nu = \sqrt{\lambda + \frac{c^2}{4}}\,)$

$$\ddot{u} + c\dot{u} + a\sin u = h(t) \tag{2}$$

$c \geq 0,\ \ h \in AP(\mathbb{R}),\ \ \|h\|_{L^\infty} < a$

- $A,\ U$ such that

  $$\|h\|_{L^\infty} \leq A < a,\ \ 0 < U < \frac{\pi}{2},\ \ a\sin U > A$$

- $\Omega = \{u \in AP(\mathbb{R})\ /\ \|u - \pi\|_{L^\infty} \leq U\}$

(Complete metric space)

- $\mathcal{F}u = v\ /\ \ddot{v} + c\dot{v} - av = -au - a\sin u + h(t)$
  - $\mathcal{F}(\Omega) \subset AP(\mathbb{R})$
  - $\mathcal{F}(u) = u \Leftrightarrow \ddot{u} + c\dot{u} + a\sin u = h(t),\ \ u \in \Omega$
  - $\mathcal{F}(\Omega) \subset \Omega$
  - $\mathcal{F}$ is a contraction

- $\ddot{u} + c\dot{u} - au = g(t)\ /\ g(t) = -au - a\sin u + h(t)$
  - $g \in AP(\mathbb{R}) \Rightarrow u \in AP^1(\mathbb{R})$

$$\ddot{u} + c\dot{u} + a \sin u = h(t) \tag{2}$$

$c \geq 0, \;\; h \in AP(\mathbb{R}), \;\; \|h\|_{L^\infty} < a$

$$\mathcal{F}u = v \;/\; \ddot{v} + c\dot{v} - av = -au - a\sin u + h(t)$$

- $\mathcal{F}(\Omega) \subset \Omega$?
  - $\triangleright$ $u \in \Omega \;\Rightarrow\; -U \leq u - \pi \leq U$
  - $\triangleright$ $\varphi(\xi) = -a\xi - a\sin\xi$ is decreasing
  - $\triangleright$ $w \equiv \pi + U$ is solution of
    $$\ddot{w} + c\dot{w} - aw = -a(\pi + U)$$
  - $\triangleright$ $z = v - w$ is solution of
    $$\ddot{z} + c\dot{z} - az > 0$$
  - $\triangleright$ maximum principle $\Rightarrow$ $z \leq 0$

$$\ddot{u} + c\dot{u} + a \sin u = h(t) \tag{2}$$

$c \geq 0, \;\; h \in AP(\mathbb{R}), \;\; \|h\|_{L^\infty} < a$

$$\mathcal{F}u = v \; / \; \ddot{v} + c\dot{v} - av = -au - a \sin u + h(t)$$

- $\mathcal{F}$ contraction ?
  - ▷ $v_1 = \mathcal{F}(u_1), \quad v_2 = \mathcal{F}(u_2)$
  - ▷ $d = v_1 - v_2$ is solution of

    $$\ddot{d} + c\dot{d} - ad = \varphi(u_1) - \varphi(u_2)$$

    $(\varphi(\xi) = -a\xi - a \sin \xi)$
  - ▷ $\|d\|_{L^\infty} \leq k \, \|u_1 - u_2\|_{L^\infty}, \quad k < 1$

    $(\|d\|_{L^\infty} \leq \frac{1}{a} \, \|\varphi(u_1) - \varphi(u_2)\|_{L^\infty})$

    $(\|\varphi(u_1) - \varphi(u_2)\|_{L^\infty} \leq a(1 - \cos U)\|u_1 - u_2\|_{L^\infty})$

$$\ddot{u} + c\dot{u} + a \sin u = h(t) \tag{2}$$

Theorem

$$\left.\begin{array}{l} c > 0, \quad a \le \frac{c^2}{4} \\[1em] h \in AP(\mathbb{R}) \\[1em] \|h\|_{L^\infty} < a \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \exists_1\ u \in AP^1(\mathbb{R}) \text{ such that} \\[1em] \quad \bullet \quad u \text{ satisfies (2)} \\[1em] \quad \bullet \quad \|u\|_{L^\infty} < \frac{\pi}{2} \end{array}\right.$$

Lemma

$$\left.\begin{array}{l} c > 0, \quad 0 < \lambda \le \frac{c^2}{4} \\[1em] h \in C(\mathbb{R}) \\[1em] \|h\|_{L^\infty} \text{ bounded} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \exists_1\ u \in C^1(\mathbb{R}) \text{ such that} \\[1em] \quad \bullet \ \ddot{u} + c\dot{u} + \lambda u = h(t) \\[1em] \quad \bullet \ \|u\|_{L^\infty} \le \frac{1}{\lambda} \|h\|_{L^\infty} \\[1em] \quad \bullet \ \|\dot{u}\|_{L^\infty} \le \frac{1}{\nu(c,\lambda)} \|h\|_{L^\infty} \end{array}\right.$$

Moreover, $h \ge 0 \ \Rightarrow \ u \ge 0$

$$u_{tt} - \Delta_x u + c u_t + a \sin u = f(t, x) \tag{1}$$

Theorem

$$\left.\begin{array}{l} c > 0, \quad a \leq \frac{c^2}{4} \\[2ex] f \in AP(\mathbb{R} \times \mathbb{T}^n) \\[2ex] \|f\|_{L^\infty} < a \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \exists_1\, u \in AP(\mathbb{R} \times \mathbb{T}^n) \text{ s.t.} \\[2ex] \quad \bullet \quad u \text{ satisfies (1)} \\[2ex] \quad \bullet \quad \|u\|_{L^\infty} < \frac{\pi}{2} \end{array}\right.$$

$$u_{tt} - \Delta_x u + c u_t + \lambda u = f(t,x) \tag{3}$$

Theorem $(n = 1,\ 2,\ 3)$

$$\left.\begin{array}{l} c > 0, \quad 0 < \lambda \leq \frac{c^2}{4} \\ \\ f \in L^\infty(\mathbb{R} \times \mathbb{T}^n) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \exists_1\ u \in L^\infty(\mathbb{R} \times \mathbb{T}^n) \\ \\ \text{such that } u \text{ satisfies (3)} \end{array}\right.$$

Moreover,

- $\|u\|_{L^\infty} \leq \frac{1}{\lambda} \|f\|_{L^\infty}$
- $n = 1 \ \Rightarrow \ u \in W^{1,\infty}(\mathbb{R} \times \mathbb{T})$
- $n = 2 \ \Rightarrow \ u \in C(\mathbb{R} \times \mathbb{T}^2)$
- $f \in C(\mathbb{R} \times \mathbb{T}^n) \ \Rightarrow \ u \in C(\mathbb{R} \times \mathbb{T}^n)$
- $f \geq 0$ a.e. $\mathbb{R} \times \mathbb{T}^n \ \Rightarrow \ u(t,x) \geq 0$ a.e. $\mathbb{R} \times \mathbb{T}^n$

$$u_{tt} - \Delta_x u + c u_t + \lambda u = f(t,x) \tag{3}$$

- $\exists f \in L^\infty(\mathbb{R} \times \mathbb{T}^3)$ such that $u \notin C(\mathbb{R} \times \mathbb{T}^3)$
- There is not maximum principle for $n = 4$ (at least for $\lambda = \frac{c^2}{4}$)

Almost periodic solutions of forced pendulum and forced sine-Gordon equations

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