ALMOST PERIODIC SOLUTIONS OF FORCED SINE-GORDON EQUATIONS

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The forced sine-Gordon equation can be considered as a natural extension to partial differential equations of the forced pendulum equation. It is known that, if f is almost periodic and not too large, the pendulum equation has almost periodic solutions. Our aim is to extend this result to the sine-Gordon equation. A crucial tool in the proofs is a recent maximum principle for the telegraph equation. This maximum principle holds up to space dimension three.

1. Introduction

The aim of this note is to study almost periodic solutions of the sine-Gordon equation

$$u_{tt} - \Delta_x u + cu_t + a \sin u = f(t,x) \quad in \;\; \mathfrak{D}'(\mathbb{R} \times \mathbb{T}^n) \tag{1}$$

when $n = 1$, 2 or 3, $0 < a < \frac{c^2}{4}$, $c > 0$, f is almost periodic and $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

As it is usual, it is convenient to consider the linear equation associated to (1). In our case it is the telegraph equation

$$u_{tt} - \Delta_x u + cu_t + \lambda u = f(t,x) \quad in \;\; \mathfrak{D}'(\mathbb{R} \times \mathbb{T}^n). \tag{2}$$

In dimension one ($n = 1$) it is known that the solutions of (2) gain regularity (see Refs. 3 and 5). This is sufficient to have compactness and we can apply a reasoning due to Amerio (see Ref. 1) in order to get our purpose. When $n = 3$ there is no regularity and we can not use Amerio's ideas.

Nevertheless we are going to give a simpler argument which is based on completeness, more exactly on Banach Contraction Principle. Anyway (with compactness or with completeness) we need a maximum principle

*Work partially supported by BFM2002-01308, MCYT Spain.

and this is impossible when $n \geq 4$ (see Ref. 4). So this is the reason why we take $n = 1,\ 2$ or 3.

On the other hand, the results that we are going to get can be considered as extensions of the equivalent ones for the forced pendulum equation

$$\ddot{u} + c\dot{u} + a \sin u = h(t). \tag{3}$$

In fact, in Ref. 2 it is proved the following

Theorem 1.1. *If $c \geq 0$ and h is almost periodic and such that $\|h\|_{L^\infty} < a$, then there exists $\varepsilon > 0$ such that (3) has a unique solution $u \in C^1(\mathbb{R})$ almost periodic satisfying*

$$\frac{\pi}{2} + \varepsilon \leq u(t) \leq \frac{3\pi}{2} - \varepsilon.$$

Moreover, $\dot{u}$ is almost periodic also.

Again the proof is based on an argument of Amerio's type and the compactness plays a crucial role.

Remark 1.1. In the previous theorem we consider $c \geq 0$ and in equation (1) we take $c > 0$. The reason of this difference is that we are going to use a maximum principle that fails for the wave equation ($c = 0$).

Finally, and before going on to the following section, we must recall the notion of almost periodicity. Since $\mathbb{R} \times \mathbb{T}^n$ is a locally compact topological group we can use Bochner definition. Given a function $f : \mathbb{R} \times \mathbb{T}^n \to \mathbb{R}$ and a vector $\alpha = (\alpha_0, \tilde{\alpha})$ in $\mathbb{R} \times \mathbb{T}^n$, the translate $T_\alpha f$ is defined as

$$(T_\alpha f)(t, x) = f(t + \alpha_0, x + \tilde{\alpha}).$$

A continuous function f is almost periodic if from every sequence $\{\alpha_m\}_{m\in\mathbb{N}}$ in $\mathbb{R} \times \mathbb{T}^n$ it is possible to extract a subsequence $\{\alpha_k\}_{k\in\mathbb{N}}$ such that $T_{\alpha_k} f$ has a uniform limit. The class of almost periodic functions will be denoted by $AP(\mathbb{R} \times \mathbb{T}^n)$, endowed with the L^∞-norm it becomes a Banach space immersed in $L^\infty(\mathbb{R} \times \mathbb{T}^n) \cap C(\mathbb{R} \times \mathbb{T}^n)$.

A similar definition will be considered when we will take a function $h : \mathbb{R} \to \mathbb{R}$. In this case, we will use the spaces $AP^k(\mathbb{R}) = \{h \in AP(\mathbb{R}) \ / \ h \in C^k(\mathbb{R})$ and $h^{(j)} \in AP(\mathbb{R})$ for each $1 \leq j \leq k\},\ k \in \mathbb{Z}$.

2. Ordinary differential equations

Before considering equation (1), we are to going to apply our strategy to obtain a new proof of Theorem 1.1.

First we recall some results on the linear equation

$$\ddot{u} + c\dot{u} - \lambda u = h(t), \quad t \in \mathbb{R}, \tag{4}$$

with $c \geq 0$ and $\lambda > 0$.

Lemma 2.1. *If $h \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ then (4) has a unique solution $u \in C^1(\mathbb{R})$ satisfying*

$$\|u\|_{L^\infty} \leq \frac{1}{\lambda} \|h\|_{L^\infty} \tag{5}$$

and

$$\|\dot{u}\|_{L^\infty} \leq \frac{1}{\nu} \|h\|_{L^\infty},$$

with $\nu = \sqrt{\lambda + \frac{c^2}{4}}$. Moreover, if $h(t) \geq 0$, $\forall t \in \mathbb{R}$, then $u(t) \leq 0$, $\forall t \in \mathbb{R}$.

In Ref. 6, Sec. 6, there is an explicit formula for u which uses a Green's function. The proof of the lemma is trivial with it.

From (5), it is easy to verify that

$$\|T_\alpha u - T_\beta u\|_{L^\infty} \leq \frac{1}{\lambda} \|T_\alpha h - T_\beta h\|_{L^\infty}$$

for arbitrary numbers α, β in $\mathbb{R}$. A similar inequality is valid for $\dot{u}$. So, we have proved the following

Lemma 2.2. *If h is in $AP(\mathbb{R})$ then $u \in AP^1(\mathbb{R})$.*

Remark 2.1. It is possible to improve the previous two lemmas. In fact, $u \in C^2(\mathbb{R})$ and a L^∞-estimate exists for $\ddot{u}$. Therefore, u belongs to $AP^2(\mathbb{R})$.

Now we are ready to give the different proof of Theorem 1.1. We fix constants A and U satisfying

$$\|h\|_{L^\infty} \leq A < a, \quad 0 < U < \frac{\pi}{2}, \quad a \sin U > A.$$

We consider the complete metric space

$$\Omega = \{u \in AP(\mathbb{R}) \;/\; \|u - \pi\|_{L^\infty} \leq U\}$$

and the mapping $\mathcal{F}u = v$, where v is the almost periodic solution of

$$\ddot{v} + c\dot{v} - av = -au - a\sin u + h(t).$$

From its definition we can say that $\mathcal{F}$ maps Ω into $AP(\mathbb{R})$ and the fixed points of $\mathcal{F}$ correspond to the almost periodic solutions of (3) satisfying $\|u - \pi\|_{L^\infty} \leq U$.

Next we prove that $\mathcal{F}$ maps Ω into itself. Given $u \in \Omega$, we know that $-U \le u - \pi \le U$ and we observe that the function $\varphi(\xi) = -a\xi - a\sin\xi$ is decreasing. Hence

$$a\pi - aU + a\sin U \le au + a\sin u \le a\pi + aU - a\sin U.$$

Constants $\pi + U$ and $\pi - U$ are solutions in $AP(\mathbb{R})$ of

$$\ddot{w}_1 + c\dot{w}_1 - aw_1 = -a(\pi + U) \quad \text{and} \quad \ddot{w}_2 + c\dot{w}_2 - aw_2 = -a(\pi - U)$$

respectively. We can apply Lemma 2.1 to compare $w_2 = \pi - U$, v and $w_1 = \pi + U$. In fact, $\pi - U \le v(t) \le \pi + U$ everywhere.

Once we know that $\mathcal{F}(\Omega) \subset \Omega$ we must prove that $\mathcal{F}$ is a contraction. To do this we consider $u_1, u_2 \in \Omega$ with $v_1 = \mathcal{F}u_1$, $v_2 = \mathcal{F}u_2$. The difference $d = v_1 - v_2$ is a solution of

$$\ddot{d} + c\dot{d} - ad = -a(u_1 - u_2) - a(\sin u_1 - \sin u_2).$$

Since

$$\|u_1 + \sin u_1 - u_2 - \sin u_2\|_{L^\infty} \le (1 + \cos(\pi - U))\,\|u_1 - u_2\|_{L^\infty},$$

we can apply (5) to conclude that

$$\|v_1 - v_2\|_{L^\infty} \le \frac{1}{a}\, a\, \|u_1 + \sin u_1 - u_2 - \sin u_2\|_{L^\infty} \le k\, \|u_1 - u_2\|_{L^\infty}$$

with $k = 1 - \cos U$. Since $k < 1$ the fixed point of $\mathcal{F}$ will be the searched almost periodic solution. Letting A to tend to a and U to $\frac{\pi}{2}$, the uniqueness of the fixed point shows that there are no other almost periodic solutions in the ball $\|u - \pi\|_{L^\infty} < \frac{\pi}{2}$. Finally, u satisfies $\ddot{u} + c\dot{u} - au = g(t)$ with $g(t) = -au - a\sin u + h(t)$. Because $g \in AP(\mathbb{R})$, we conclude that $u \in AP^1(\mathbb{R})$.

Remark 2.2. Having in mind Remark 2.1, we prove that $u \in AP^2(\mathbb{R})$.

In the previous proof we have used a classical maximum principle in o.d.e.'s. We can use an anti-maximum principle (see Ref. 5) to obtain a new result in the ball $\|u\|_{L^\infty} < \frac{\pi}{2}$ (for $c > 0$ and $0 < a \le \frac{c^2}{4}$). Now we take

$$\ddot{u} + c\dot{u} + \lambda u = h(t), \tag{6}$$

with $c > 0$ and $0 < \lambda \le \frac{c^2}{4}$. We sum up in the following lemmas the results for (6) which correspond to Lemmas 2.1 and 2.2 for (4).

Lemma 2.3. *If $h \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ then (6) has a unique solution $u \in C^1(\mathbb{R})$ satisfying (5) and*

$$\|\dot{u}\|_{L^\infty} \le \frac{1}{\nu}\, \|h\|_{L^\infty},$$

with $\nu = \sqrt{\frac{c^2}{4} - \lambda}$ *if* $0 < \lambda < \frac{c^2}{4}$ *and* $\nu = \frac{ec}{4}$ *if* $\lambda = \frac{c^2}{4}$. *Moreover, if* $h(t) \geq 0$, $\forall t \in \mathbb{R}$, *then* $u(t) \geq 0$, $\forall t \in \mathbb{R}$.

Lemma 2.4. *If* h *is in* $AP(\mathbb{R})$ *then* u *belongs to* $AP^1(\mathbb{R})$.

Remark 2.3. Remarks 2.1 and 2.2 also can be applied in this case.

3. Partial differential equations

In this section we are going to see results of almost periodicity for (1) that were exposed in Refs. 3 and 4. First we will state some results for bounded solutions of the telegraph equation (2). We recall in a precise manner the concept of solution when $n = 3$. Cases $n = 1$ and $n = 2$ are similar.

Definition 3.1. Let $c > 0$ and $f \in L^\infty(\mathbb{R} \times \mathbb{T}^3)$. A *bounded solution* of the problem

$$\mathfrak{L}u + \lambda u := u_{tt} - \Delta_x u + c u_t + \lambda u = f(t,x) \quad \text{in } \mathbb{R} \times \mathbb{R}^3$$
$$u(t, x_1 + 2\pi, x_2, x_3) = u(t, x_1, x_2 + 2\pi, x_3) = u(t, x_1, x_2, x_3 + 2\pi) = u(t,x)$$

is a function $u \in L^\infty(\mathbb{R} \times \mathbb{T}^3)$ satisfying

$$\int_{\mathbb{R}\times\mathbb{T}^3} (\mathfrak{L}^*\phi + \lambda\phi)u = \int_{\mathbb{R}\times\mathbb{T}^3} f\phi,$$

for all $\phi \in \mathfrak{D}(\mathbb{R} \times \mathbb{T}^3)$, where $\mathfrak{L}^*\phi = \phi_{tt} - \Delta_x\phi - c\phi_t$, i.e.

$$\mathfrak{L}u + \lambda u = f \quad \text{in } \mathfrak{D}'(\mathbb{R} \times \mathbb{T}^3), \quad u \in L^\infty(\mathbb{R} \times \mathbb{T}^3). \tag{7}$$

The key results is the following one, valid for $n = 1$, 2 or 3.

Lemma 3.1. *For each* $\lambda \in \left(0, \frac{c^2}{4}\right]$ *and each* $f \in L^\infty(\mathbb{R} \times \mathbb{T}^n)$, *the problem (7) has a unique solution* u *such that*

- *(i) if* $n = 1$ *then* $u \in W^{1,\infty}(\mathbb{R} \times \mathbb{T})$.
- *(ii) if* $n = 2$ *then* u *is continuous.*

Moreover, if $f \geq 0$ *a.e. in* $\mathbb{R} \times \mathbb{T}^n$, *then* $u \geq 0$ *a.e. in* $\mathbb{R} \times \mathbb{T}^n$.

Remark 3.1. $W^{1,\infty}(\mathbb{R} \times \mathbb{T})$ denotes the Banach space of functions $u \in L^\infty(\mathbb{R} \times \mathbb{T})$ which are Lipschitz-continuous, with the norm

$$\|u\|_{W^{1,\infty}} = \|u\|_{L^\infty} + [u]_{Lip},$$

where $[u]_{Lip}$ is the best Lipschitz constant of u.

Remark 3.2. When $n = 3$, the solution u can be discontinuous. An example is shown in Ref. 4.

Remark 3.3. When $n = 4$ there is not maximum principle. An example for $\lambda = \frac{c^2}{4}$ is shown in Ref. 4 too.

Remark 3.4. The bounded solution of equation (7) satisfies the estimate

$$\|u\|_{L^\infty} \le \frac{1}{\lambda}\|f\|_{L^\infty}.$$

Our final result is the following

Theorem 3.1. *Assume that*

$$0 < a \le \frac{c^2}{4}, \quad f \in AP(\mathbb{R} \times \mathbb{T}^n) \ \ \textit{and} \ \ \|f\|_{L^\infty} < a.$$

Then the equation (1) has a solution u *in* $AP(\mathbb{R}\times\mathbb{T}^n)$. *Moreover it satisfies* $\|u\|_{L^\infty} < \frac{\pi}{2}$ *and it is unique among the almost periodic solutions having this property.*

The proof is similar to the o.d.e. case.

Remark 3.5. If $n = 1$ then u is more regular, namely, $u \in W^{1,\infty}(\mathbb{R} \times \mathbb{T})$.

Acknowledgments

The author is grateful with R. Ortega and J. Mawhin for the collaboration in several papers that have motivated this note.

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