FAMILIES OF NUMERICAL SEMIGROUPS: FROBENIUS PSEUDO-VARIETIES AND TREES ASSOCIATED TO THEM

AURELIANO M. ROBLES-PÉREZ AND JOSÉ CARLOS ROSALES

Abstract. In order to collect common properties of several families of numerical semigroups, the concept of Frobenius pseudo-variety is introduced. Moreover, we study the tree structure that arise with it.

1. Introduction: some results on numerical semigroups

Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of integers and non-negative integers, respectively. A *submonoid* of $\mathbb{N}$ is a set $M \subseteq \mathbb{N}$ that contains the zero element and is closed under addition. A *numerical semigroup* is a submonoid S of $\mathbb{N}$ such that $\mathbb{N} \setminus S$ is finite. Notable elements of a numerical semigroup S are $\mathrm{F}(S) = \max(\mathbb{Z} \setminus S)$ (*Frobenius number of* S), $\mathrm{g}(S) = \sharp(\mathbb{N} \setminus S)$ (*genus of* S), and $\mathrm{m}(S) = \min(S \setminus \{0\})$ (*multiplicity of* S). (As usual, $\sharp A$ denotes the cardinality of A.)

Remark 1.1. The content of this section can be seen (or easily deduced from the results that appear) in [4] and the references therein.

Given a non-empty set $A \subseteq \mathbb{N}$, the submonoid $M$of $\mathbb{N}$ generated by A is the set

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\},\ a_1, \ldots, a_n \in A,\ \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}.$$

If this is the case, we say that A is a *system of generators* of M. Moreover, if $M \neq \langle B \rangle$ for every $B \subsetneq A$, then A is a *minimal system of generators* of M. It is well known that every submonoid of $\mathbb{N}$ has a unique minimal system of generators, which in addition is finite. On the other hand, $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

If X is the minimal system of generators of a numerical semigroup S, then $\mathrm{e}(S) = \sharp X$ is the *embedding dimension of* S. It is easy to see that $\mathrm{e}(S) \leq \mathrm{m}(S)$. In particular, if $\mathrm{e}(S) = \mathrm{m}(S)$, then we say that S has *maximal embedding dimension* (see [5]).

It is well known that, if S, T are numerical semigroups (with $S \neq \mathbb{N}$), then $S \cup \{\mathrm{F}(S)\}$ and $S \cap T$ are numerical semigroups too. Moreover, we have the following result.

Theorem 1.2. *Let S, T be numerical semigroups and let $A = \{a_1, \ldots, a_n\}$ be the minimal system of generators of S. Then $S = T \cup \{\mathrm{F}(T)\}$ if and only if $T = S \setminus \{a_i\}$, where a_i is a minimal generator of S such that $a_i > \mathrm{F}(S)$. Moreover, if this is the case, then $\mathrm{F}(S) < \mathrm{F}(T) = a_i$ and $\mathrm{g}(T) = \mathrm{g}(S) + 1$.*

A *(directed) graph* G is a pair (V, E), where V is a non-empty set and E is a subset of $\{(v, w) \in V \times V \mid v \neq w\}$. The elements of V are called *vertices* of G and the elements

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of E are called *edges* of G. A *path (of length n)* connecting the vertices x and y of G is a sequence of different edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ such that $v_0 = x$ and $v_n = y$. A graph G is a *tree* if there exists a vertex r (known as the *root* of G) such that, for every other vertex x of G, there exists a unique path connecting x and r. If (x, y) is an edge of the tree, then we say that x is a *child* of y.

Let $\mathcal{S}$ be the set formed by all numerical semigroups. Theorem 1.2 allows us to build the tree associated to $\mathcal{S}$, which we denote by $\mathrm{G}(\mathcal{S})$. In this tree, the vertices are the elements of $\mathcal{S}$ (that is, the numerical semigroups), (T, S) is an edge if $S = T \cup \{\mathrm{F}(T)\}$, and $\mathbb{N}$ is the root. On the other hand, if S is a numerical semigroup, then the unique path connecting S with $\mathbb{N}$ is given by (the *chain of numerical semigroups associated to S*) $\mathrm{C}(S) = \{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n\}$, where $S_0 = S$, $S_{i+1} = S_i \cup \{\mathrm{F}(S_i)\}$, for all $i < n$, and $S_n = \mathbb{N}$.

The next result (see [4, Prop. 7.1]) follows from Theorem 1.2 and the previous paragraph.

Proposition 1.3. *The graph $\mathrm{G}(\mathcal{S})$ is a tree with root equal to $\mathbb{N}$. Moreover, the children of $S \in \mathcal{S}$ are $S \setminus \{a_1\}, \ldots, S \setminus \{a_r\}$, where $a_1, \ldots, a_r$ are the minimal generators of S that are greater than $\mathrm{F}(S)$.*

2. Frobenius varieties

For certain families of numerical semigroups we can observe a similar behaviour to that described in the previous section. For example (see [4]), we have the family of Arf numerical semigroups, the family of saturated numerical semigroups, the family of numerical semigroups having a Toms decomposition, and the family of numerical semigroups defined by a strongly admissible linear homogeneous pattern. This observation led to introduce, in [3], the concept of *(Frobenius) variety.*

Definition 2.1. A *variety* is a non-empty family $\mathcal{V}$ of numerical semigroups that fulfills the following conditions,

(1) if $S, T \in \mathcal{V}$, then $S \cap T \in \mathcal{V}$;
(2) if $S \in \mathcal{V}$ and $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{V}$.

Since the intersection of varieties is another variety, we can define the *variety generated by a family of numerical semigroups $\mathcal{F}$* (denoted by $\mathrm{F}(\mathcal{F})$) as the intersection of all varieties containing $\mathcal{F}$. Let us observe that $\mathrm{F}(\mathcal{F})$ is the smallest, with respect to the inclusion order, variety containing $\mathcal{F}$.

In [3] was shown that several facts, that hold for the above mentioned families (and, as we have said before, that generalize the situation of Section 1), are also satisfied in any variety. Nevertheless, there exist outstanding families that are not varieties, but that preserve such behaviour. For example, the family of numerical semigroups with maximal embedding dimension and multiplicity m is not a variety (see [5]), but satisfies several properties of such families. In order to study this class of numerical semigroups, recently (see [1]) have been introduced the non-homogeneous patterns and, moreover, have been defined the non-homogeneous Frobenius varieties of multiplicity m (or m-varieties for short).

3. Frobenius pseudo-varieties

In order to study families of numerical semigroups that are not varieties, in [2] (see this reference for a detailed development of this section), we introduced the next concept.

Definition 3.1. A *(Frobenius) pseudo-variety* is a non-empty family $\mathcal{P}$ of numerical semigroups that fulfills the following conditions,

(1) $\mathcal{P}$ has a maximum element $\max(\mathcal{P})$ (with respect to the inclusion order);
(2) if $S, T \in \mathcal{P}$, then $S \cap T \in \mathcal{P}$;
(3) if $S \in \mathcal{P}$ and $S \neq \max(\mathcal{P})$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{P}$.

Remark 3.2. Observe that, in [1], the maximum element of an m-variety $\mathcal{M}$ is the numerical semigroup $\{0, m, \rightarrow\} = \{0, m\} \cup \{z \in \mathbb{N} \mid z > m\}$ and the multiplicity of all numerical semigroups, that belong to $\mathcal{M}$, is equal to m. We have removed these restrictions.

3.1. **Pseudo-varieties and varieties.** From the definitions, it is clear that every variety is a pseudo-variety. However, there are pseudo-varieties that are not varieties. For example, if S is a numerical semigroup different from $\mathbb{N}$, then $\{S\}$ is a pseudo-variety but not a variety.

It is clear that, if $\mathcal{P}$ is a pseudo-variety, then $\mathcal{P}$ is a variety if and only if $\mathbb{N} \in \mathcal{P}$. Furthermore, if $\mathcal{P}$ is a family of numerical semigroups with maximum Δ, then $\mathcal{P}$ is a pseudo-variety if and only if $\mathcal{P} \cup \mathrm{C}(\Delta)$ is a variety.

On the other hand, if $\mathcal{P}$ is a pseudo-variety and $S \in \mathcal{P}$, then $\max(\mathcal{P}) \in \mathrm{C}(S)$. Moreover, if S_1, S_2, Δ are numerical semigroups such that $\Delta \in \mathrm{C}(S_1) \cap \mathrm{C}(S_2)$, then $\Delta \in \mathrm{C}(S_1 \cap S_2)$.

The previous comments allow us to establish the following result.

Theorem 3.3. *Let $\mathcal{V}$ be a variety and let Δ be a numerical semigroup such that $\Delta \in \mathcal{V}$. Then $\mathcal{D}(\mathcal{V}, \Delta) = \{S \in \mathcal{V} \mid \Delta \in \mathcal{C}(S)\}$ is a pseudo-variety. Moreover, every pseudo-variety can be obtained in this way.*

Remark 3.4. If we take $\max(\mathcal{P}) = \mathbb{N}$ in Subsections 3.3 and 3.4, then we recover analogous results for varieties that were shown in [3].

3.2. **Examples of pseudo-varieties.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\mathcal{S}(m)$ be the set formed by all numerical semigroups that have multiplicity m. It is clear that the family $\mathcal{S}(m)$ is a pseudo-variety. In the same line, the family of all numerical semigroups with maximal embedding dimension and multiplicity m is another pseudo-variety. We must observe that the family of all numerical semigroups with maximal embedding dimension is not a pseudo-variety.

In general, the intersection of pseudo-varieties is not a pseudo-variety. For example, if S_1, S_2 are different numerical semigroups, then $\mathcal{P}_1 = \{S_1\}$ and $\mathcal{P}_2 = \{S_2\}$ are pseudo-varieties, but not $\mathcal{P}_1 \cap \mathcal{P}_2 = \{\emptyset\}$. In spite of all, we can define the *pseudo-variety generated by a family of numerical semigroups* $\mathcal{F}$ (denoted by $\mathfrak{p}(\mathcal{F})$) as the intersection of all pseudo-varieties containing $\mathcal{F}$. In fact, $\mathfrak{p}(\mathcal{F})$ is the intersection of all pseudo-varieties that contain $\mathcal{F}$ and whose maximum is $\Theta(\mathcal{F})$, where $\Theta(\mathcal{F})$ is the numerical semigroup $\min\left(\bigcap_{S \in \mathcal{F}} \mathrm{C}(S)\right)$.

3.3. **$\mathcal{P}$-monoids.** Let $\mathcal{P}$ be a pseudo-variety. We say that a submonoid M of $\mathbb{N}$ is a $\mathcal{P}$-monoid if it can be expressed as an intersection of elements of $\mathcal{P}$. It is obvious that the intersection of $\mathcal{P}$-monoids is a $\mathcal{P}$-monoid.

If $A \subseteq \max(\mathcal{P})$, we define the *$\mathcal{P}$-monoid generated by A*, denoted by $\mathcal{P}(A)$, as the intersection of all the $\mathcal{P}$-monoids containing A (or, equivalently, the intersection of all elements of $\mathcal{P}$ containing A). We have that $\mathcal{P}(A)$ is the smallest (with respect to the inclusion order) $\mathcal{P}$-monoid containing A.

If $M = \mathcal{P}(A)$, then we say that A is a *$\mathcal{P}$-system of generators* of M. In addition, if $M \neq \mathcal{P}(B)$ for every $B \subsetneq A$, then A is a *minimal $\mathcal{P}$-system of generators* of M.

Theorem 3.5. *Every $\mathcal{P}$-monoid has a unique minimal $\mathcal{P}$-system of generators. Additionally, such a $\mathcal{P}$-system is finite.*

3.4. **The tree associated to a pseudo-variety.** For a pseudo-variety $\mathcal{P}$, we define the graph $\mathrm{G}(\mathcal{P})$ in the following way,

- the set of vertices of $\mathrm{G}(\mathcal{P})$ is $\mathcal{P}$;
- $(S, S') \in \mathcal{P} \times \mathcal{P}$ is an edge of $\mathrm{G}(\mathcal{P})$ if and only if $S' = S \cup \{\mathrm{F}(S)\}$.

If $S \in \mathcal{P}$, then we can define recursively the sequence,

- $S_0 = S$,
- if $S_i \neq \max(\mathcal{P})$, then $S_{i+1} = S_i \cup \{\mathrm{F}(S_i)\}$.

We have that $\mathrm{C}_{\mathcal{P}}(S) = \{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n\}$ (where $S_n = \max(\mathcal{P})$) is precisely the unique path connecting S with $\max(\mathcal{P})$.

Observe that we can construct recursively $\mathrm{G}(\mathcal{P})$ from $\max(\mathcal{P})$. Indeed, it is sufficient to know how to compute the children of each vertex S. Let T a child of S in $\mathrm{G}(\mathcal{P})$, that is, $S = T \cup \{\mathrm{F}(T)\}$ or, equivalently, $T = S \setminus \{\mathrm{F}(T)\}$. Thereby, there exists an integer $x > \mathrm{F}(S)$ such that $T = S \setminus \{x\}$. On the other hand, if M is a $\mathcal{P}$-monoid and $x \in M$, then $M \setminus \{x\}$ is a $\mathcal{P}$-monoid if and only if x belongs to the minimal $\mathcal{P}$-system of generators of M. Thus, we can show the next result (that it is analogue to Proposition 1.3).

Theorem 3.6. *The graph $\mathrm{G}(\mathcal{P})$ is a tree with root equal to $\max(\mathcal{P})$. Moreover, the children of a vertex $S \in \mathcal{P}$ are $S \setminus \{x_1\}, \ldots, S \setminus \{x_r\}$, where $x_1, \ldots, x_r$ are the elements of the minimal $\mathcal{P}$-system of generators of S that are greater than $\mathrm{F}(S)$.*

Remark 3.7. Theorem 3.6 can be used to describe and enumerate the members of families of numerical semigroups having certain properties (that lead to the pseudo-variety structure). For example, we have that, if S' is a child of S in $\mathrm{G}(\mathcal{P})$, then $\mathrm{F}(S') > \mathrm{F}(S)$ and $\mathrm{g}(S') = \mathrm{g}(S) + 1$. Therefore, we get numerical semigroups with greater Frobenius number and genus when we go on along the branches of the tree. Thus, we can use this construction in order to obtain all the numerical semigroups, in the pseudo-variety $\mathcal{P}$, with a given Frobenius number or genus.

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A.M. Robles-Pérez, Department of Applied Mathematics, University of Granada
E-mail address: `arobles@ugr.es`

J.C. Rosales, Department of Algebra, University of Granada
E-mail address: `jrosales@ugr.es`