Families of numerical semigroups: Frobenius pseudo-varieties and trees associated to them

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Aim of this work

Purpose

- *To define a structure that allows to build and to arrange the elements of certain families of numerical semigroups.*

Procedure (Basic idea)

- *To analyze the family of all numerical semigroups and other outstanding families of numerical semigroups.*

Preliminaries

- $\mathbb{N} = \{0, 1, 2, \ldots\}$
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Definition

- *A* submonoid *of* $\mathbb{N}$ *is a set* $M \subseteq \mathbb{N}$ *that contains the zero element and is closed under addition.*
- *A* numerical semigroup *is a submonoid* S *of* $\mathbb{N}$ *such that* $\mathbb{N} \setminus S$ *is finite.*

- $\mathrm{F}(S) = \max(\mathbb{Z} \setminus S)$ (*Frobenius number*)
- $\mathrm{g}(S) = \sharp(\mathbb{N} \setminus S)$ (*genus*)
- $\mathrm{m}(S) = \min(S \setminus \{0\})$ (*multiplicity*)

- If $A \subseteq \mathbb{N}$ is a nonempty set,

$$\langle A \rangle = \{\lambda_1 a_1 + \ldots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\},\ a_1, \ldots, a_n \in A,\ \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}.$$

- If $M = \langle A \rangle$, then A is a *system of generators* of M.
- In addition, if no proper subset of A generates M, then A is a *minimal system of generators* of M.

Lemma

- *Every* submonoid *of* $\mathbb{N}$ *admits a unique minimal system of generators, which in addition is finite.*
- $S = \langle A \rangle$ *is a numerical semigroup if and only if* $\gcd\{A\} = 1$.

- The cardinality of the minimal system of generators of S is called the *embedding dimension* of S and will be denoted by $\mathrm{e}(S)$.

Example

$$S = \{0, 5, 7, 9, 10, 12, 14, \rightarrow\} = \{0, 5, 7, 9, 10, 12\} \cup \{z \in \mathbb{Z} \mid z \geq 14\}$$

- $\mathbb{N} \setminus S = \{1, 2, 3, 4, 6, 8, 11, 13\}$.
- $\mathrm{F}(S) = 13$.
- $\mathrm{g}(S) = 8$.
- $\mathrm{m}(S) = 5$.
- $\langle 5, 7, 9 \rangle$ is the minimal system of generators of S.
- $\mathrm{e}(S) = 3$.

Results on numerical semigroups

Lemma

- *If S, T are numerical semigroups (with $S \neq \mathbb{N}$), then $S \cup \{\mathrm{F}(S)\}$ and $S \cap T$ are numerical semigroups too.*

Theorem

- *Let S, T be numerical semigroups and let $A = \{a_1, \ldots, a_n\}$ be the minimal system of generators of S.*

  Then $S = T \cup \{\mathrm{F}(T)\}$ if and only if $T = S \setminus \{a_i\}$, where a_i is a minimal generator of S such that $a_i > \mathrm{F}(S)$.

  Moreover, if this is the case, then $\mathrm{F}(S) < \mathrm{F}(T) = a_i$ and $\mathrm{g}(T) = \mathrm{g}(S) + 1$.

Example

$$S_1 = \{0,5,7,9,10,12,14,\rightarrow\} = \{0,5,7,9,10,12\} \cup \{z \in \mathbb{Z} \mid z \geq 14\}$$

- $\langle 5,7,9 \rangle$ is the minimal system of generators of S_1.
- $\mathrm{F}(S) = 13$.
- There does not exist a numerical semigroup T such that $S_1 = T \cup \{\mathrm{F}(T)\}$.

Example

$$S_2 = \{0,3,5,\rightarrow\} = \{0,3\} \cup \{z \in \mathbb{Z} \mid z \geq 5\}$$

- $\langle 3,5,7 \rangle$ is the minimal system of generators of S_2.
- $\mathrm{F}(S) = 4$.
- There exist two numerical semigroups T such that $S_2 = T \cup \{\mathrm{F}(T)\}$.
  - $T_1 = \langle 3,5,7 \rangle \setminus \{5\} = \{0,3,6,\rightarrow\} = \langle 3,7,8 \rangle$.
  - $T_2 = \langle 3,5,7 \rangle \setminus \{7\} = \{0,3,5,6,8,\rightarrow\} = \langle 3,5 \rangle$.

The tree of the set of numerical semigroups

Definition

- *Let $\mathcal{S}$ be the set formed by all numerical semigroups. We denote by $\mathrm{G}(\mathcal{S})$ the tree associated to $\mathcal{S}$. In this tree,*
  - *the vertices are the elements of $\mathcal{S}$,*
  - *(T,S) is an edge if $S = T \cup \{\mathrm{F}(T)\}$,*
  - *and $\mathbb{N}$ is the root.*

Proposition

- *If S is a numerical semigroup, then the unique path connecting S with $\mathbb{N}$ is given by $\mathrm{C}(S) = \{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n\}$ (the* chain of numerical semigroups associated to *S), where $S_0 = S$, $S_{i+1} = S_i \cup \{\mathrm{F}(S_i)\}$, for all $i < n$, and $S_n = \mathbb{N}$.*

Proposition

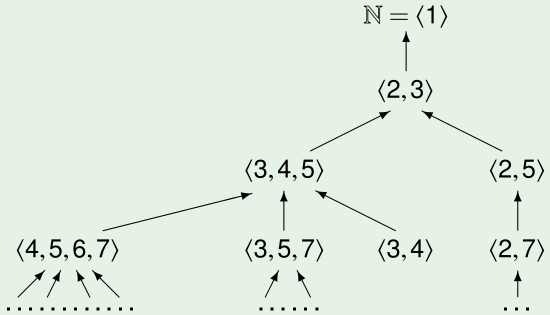
- *The children of $S \in \mathcal{S}$ are $S \setminus \{a_1\}, \ldots, S \setminus \{a_r\}$, where $a_1, \ldots, a_r$ are the minimal generators of S that are greater than $\mathrm{F}(S)$.*

Example

- $\mathrm{C}(\langle 4,5,6,7\rangle) = \{\langle 4,5,6,7\rangle, \langle 3,4,5\rangle, \langle 2,3\rangle, \langle 1\rangle = \mathbb{N}\}$.

Example

- The first levels (with respect the genus) of $\mathrm{G}(\mathcal{S})$.



Observe that the vertex $\langle 3,4\rangle$ has not got children. We say that it is a *leaf*.

(Frobenius) varieties

Definition (Rosales, 2008)

- *A* variety *is a non-empty family $\mathcal{V}$ of numerical semigroups that fulfills the following conditions,*
  - *if $S, T \in \mathcal{V}$, then $S \cap T \in \mathcal{V}$;*
  - *if $S \in \mathcal{V}$ and $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{V}$.*

Families that are varieties

- Arf numerical semigroups.
- Saturated numerical semigroups.
- Numerical semigroups having a Toms decomposition.
- Numerical semigroups defined by strongly admissible linear patterns.

Families that are not varieties

- Numerical semigroups with maximal embedding dimension and multiplicity m.
- Numerical semigroups defined by non-homogeneous patterns.

(Frobenius) pseudo-varieties

Definition (R.-P. and Rosales, 2013)

- *A* pseudo-variety *is a non-empty family* $\mathcal{P}$ *of numerical semigroups that fulfills the following conditions,*
  - $\mathcal{P}$ *has a maximum element* $\max(\mathcal{P})$ *(with respect to the inclusion order);*
  - *if* $S, T \in \mathcal{P}$, *then* $S \cap T \in \mathcal{P}$;
  - *if* $S \in \mathcal{P}$ *and* $S \neq \max(\mathcal{P})$, *then* $S \cup \{\mathrm{F}(S)\} \in \mathcal{P}$.

Remark

- *If we take* $\max(\mathcal{P}) = \mathbb{N}$, *then we recover known results for varieties.*

Pseudo-varieties and varieties

Lemma

- *If $\mathcal{P}$ is a pseudo-variety, then $\mathcal{P}$ is a variety if and only if $\mathbb{N} \in \mathcal{P}$.*
- *If $\mathcal{P}$ is a family of numerical semigroups with maximum Δ, then $\mathcal{P}$ is a pseudo-variety if and only if $\mathcal{P} \cup \mathrm{C}(\Delta)$ is a variety.*

Lemma

- *If $\mathcal{P}$ is a pseudo-variety and $S \in \mathcal{P}$, then $\max(\mathcal{P}) \in \mathrm{C}(S)$.*
- *If S_1, S_2, Δ are numerical semigroups such that $\Delta \in \mathrm{C}(S_1) \cap \mathrm{C}(S_2)$, then $\Delta \in \mathrm{C}(S_1 \cap S_2)$.*

Theorem

- *Let $\mathcal{V}$ be a variety and let Δ be a numerical semigroup such that $\Delta \in \mathcal{V}$.*
  Then $\mathcal{D}(\mathcal{V}, \Delta) = \{S \in \mathcal{V} \mid \Delta \in \mathcal{C}(S)\}$ is a pseudo-variety.
  Moreover, every pseudo-variety can be obtained in this way.

Examples of pseudo-varieties

Families that are pseudo-varieties

- Numerical semigroups that have multiplicity m.
- Numerical semigroups with maximal embedding dimension and multiplicity m.
- Numerical semigroups defined by a non-homogeneous pattern.
  (Bras-Amorós, García-Sánchez and Vico-Oton, 2013.)

Pseudo-variety $\mathfrak{p}(\mathcal{F})$ generated by a family of numerical semigroups $\mathcal{F}$

- Intersection of all pseudo-varieties containing $\mathcal{F}$.

Remark

- $\mathfrak{p}(\mathcal{F})$ *is the intersection of all pseudo-varieties that contain* $\mathcal{F}$ *and whose maximum is the numerical semigroup* $\Theta(\mathcal{F}) = \min(\bigcap_{S\in\mathcal{F}} \mathrm{C}(S))$.

Family that is not pseudo-variety

- Numerical semigroups with maximal embedding dimension.

Example

- $\mathcal{F} = \{S_1 = \langle 5,7,9\rangle, S_2 = \langle 4,6,7\rangle\}$.
- $\mathrm{C}(S_1) = \{S_{10} = \langle 5,7,9\rangle, S_{11} = \langle 5,7,9,13\rangle, S_{12} = \langle 5,7,9,11,13\rangle,$
  $S_{13} = \langle 5,7,8,9,11\rangle, S_{14} = \langle 5,6,7,8,9\rangle, S_{15} = \langle 4,5,6,7\rangle,$
  $S_{16} = \langle 3,4,5\rangle, S_{17} = \langle 2,3\rangle, S_{18} = \langle 1\rangle = \mathbb{N}\}$.
- $\mathrm{C}(S_2) = \{S_{20} = \langle 4,6,7\rangle, S_{21} = \langle 4,6,7,9\rangle, S_{22} = \langle 4,5,6,7\rangle, S_{23} = \langle 3,4,5\rangle,$
  $S_{24} = \langle 2,3\rangle, S_{25} = \langle 1\rangle = \mathbb{N}\}$.
- $\Theta(\mathcal{F}) = \langle 4,5,6,7\rangle = S_{15} = S_{22}$.
- $\mathfrak{p}(\mathcal{F}) = \{S_{10}, S_{11}, S_{12}, S_{13}, S_{14}, S_{15} = S_{22}, S_{21}, S_{20}, S_{14}\cap S_{21}, S_{14}\cap S_{20},$
  $S_{13}\cap S_{21}, S_{13}\cap S_{20}, S_{12}\cap S_{21}, S_{11}\cap S_{21}, S_{10}\cap S_{21}, S_{12}\cap S_{20},$
  $S_{11}\cap S_{20}, S_{10}\cap S_{20}\}$.

$\mathcal{P}$-monoids

Definition

- *Let $\mathcal{P}$ be a pseudo-variety. We say that a submonoid M of $\mathbb{N}$ is a $\mathcal{P}$-*monoid *if it can be expressed as an intersection of elements of $\mathcal{P}$.*
- *If $A \subseteq \max(\mathcal{P})$, we define the $\mathcal{P}$-*monoid generated by A*, denoted by $\mathcal{P}(A)$, as the intersection of all the $\mathcal{P}$-monoids containing A (or, equivalently, the intersection of all elements of $\mathcal{P}$ containing A).*
- *If $M = \mathcal{P}(A)$, then we say that A is a $\mathcal{P}$-*system of generators *of M.*
- *If $M \neq \mathcal{P}(B)$ for every $B \subsetneq A$, then A is a* minimal $\mathcal{P}$-system of generators *of M.*

Theorem

- *Every $\mathcal{P}$-monoid has a unique minimal $\mathcal{P}$-system of generators.*

  Additionally, such a $\mathcal{P}$-system is finite.

Example

- $\mathcal{F} = \{S_1 = \langle 5,7,9\rangle, S_2 = \langle 4,6,7\rangle\}$.
- $\mathrm{C}(S_1) = \{S_{10} = \langle 5,7,9\rangle, S_{11} = \langle 5,7,9,13\rangle, S_{12} = \langle 5,7,9,11,13\rangle, S_{13} = \langle 5,7,8,9,11\rangle, S_{14} = \langle 5,6,7,8,9\rangle, S_{15} = \langle 4,5,6,7\rangle, S_{16} = \langle 3,4,5\rangle, S_{17} = \langle 2,3\rangle, S_{18} = \langle 1\rangle = \mathbb{N}\}$.
- $\mathrm{C}(S_2) = \{S_{20} = \langle 4,6,7\rangle, S_{21} = \langle 4,6,7,9\rangle, S_{22} = \langle 4,5,6,7\rangle, S_{23} = \langle 3,4,5\rangle, S_{24} = \langle 2,3\rangle, S_{25} = \langle 1\rangle = \mathbb{N}\}$.
- $\Theta(\mathcal{F}) = \langle 4,5,6,7\rangle = S_{15} = S_{22}$.
- $\mathcal{P} = \mathfrak{p}(\mathcal{F}) = \{S_{10} = \langle 5\rangle_{\mathcal{P}}, S_{11} = \langle 5,13\rangle_{\mathcal{P}}, S_{12} = \langle 5,11\rangle_{\mathcal{P}}, S_{13} = \langle 5,8\rangle_{\mathcal{P}}, S_{14} = \langle 5,6\rangle_{\mathcal{P}}, S_{15} = S_{22} = \langle 4,5\rangle_{\mathcal{P}}, S_{21} = \langle 4,9\rangle_{\mathcal{P}}, S_{20} = \langle 4\rangle_{\mathcal{P}}, S_{14}\cap S_{21} = \langle 6,9\rangle_{\mathcal{P}}, S_{14}\cap S_{20} = \langle 6\rangle_{\mathcal{P}}, S_{13}\cap S_{21} = \langle 8,9\rangle_{\mathcal{P}}, S_{13}\cap S_{20} = \langle 8\rangle_{\mathcal{P}}, S_{12}\cap S_{21} = \langle 9,11\rangle_{\mathcal{P}}, S_{11}\cap S_{21} = \langle 9,13\rangle_{\mathcal{P}}, S_{10}\cap S_{21} = \langle 9\rangle_{\mathcal{P}}, S_{12}\cap S_{20} = \langle 11\rangle_{\mathcal{P}}, S_{11}\cap S_{20} = \langle 13\rangle_{\mathcal{P}}, S_{10}\cap S_{20} = \langle \emptyset\rangle_{\mathcal{P}}\}$.

Remark

- *It would be interesting to arrange the elements of* $\mathfrak{p}(\mathcal{F})$ *in a nice way.*

The tree associated to a pseudo-variety

Definition

- *Let $\mathcal{P}$ be a pseudo-variety with $\Delta = \max(\mathcal{P})$. We denote by $\mathrm{G}(\mathcal{P})$ the tree associated to $\mathcal{P}$. In this tree*
  - *the vertices are the elements of $\mathcal{P}$,*
  - *(T, S) is an edge if $S = T \cup \{\mathrm{F}(T)\}$,*
  - *and Δ is the root.*

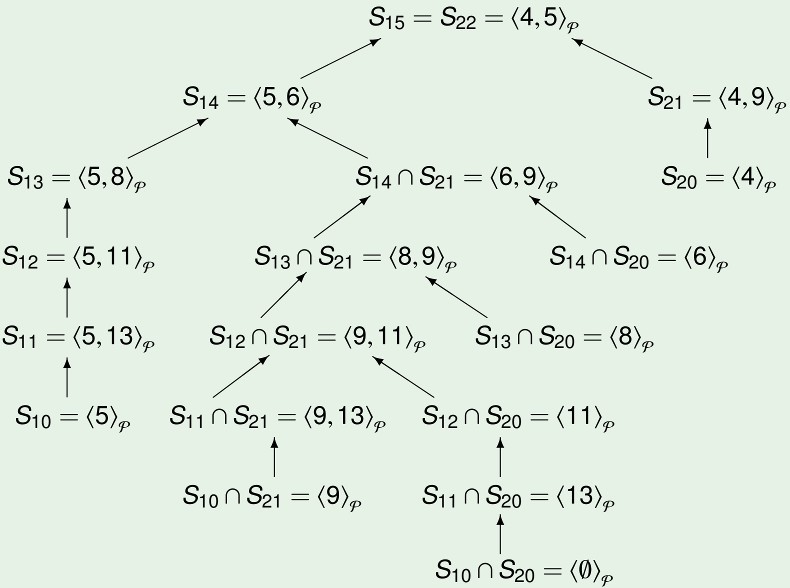
Proposition

- *If $S \in \mathcal{P}$, then the unique path connecting S with Δ is given by $\mathrm{C}_{\mathcal{P}}(S) = \{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n\}$, where $S_0 = S$, $S_{i+1} = S_i \cup \{\mathrm{F}(S_i)\}$, for all $i < n$, and $S_n = \Delta$.*

Theorem

- *The children of $S \in \mathcal{P}$ are $S \setminus \{x_1\}, \ldots, S \setminus \{x_r\}$, where $x_1, \ldots, x_r$ are elements of the minimal $\mathcal{P}$-system of generators of S that are greater than $\mathrm{F}(S)$.*

Example


$S_{15} = S_{22} = \langle 4, 5 \rangle_{\mathcal{P}}$
$S_{14} = \langle 5, 6 \rangle_{\mathcal{P}}$
$S_{21} = \langle 4, 9 \rangle_{\mathcal{P}}$
$S_{13} = \langle 5, 8 \rangle_{\mathcal{P}}$
$S_{14} \cap S_{21} = \langle 6, 9 \rangle_{\mathcal{P}}$
$S_{20} = \langle 4 \rangle_{\mathcal{P}}$
$S_{12} = \langle 5, 11 \rangle_{\mathcal{P}}$
$S_{13} \cap S_{21} = \langle 8, 9 \rangle_{\mathcal{P}}$
$S_{14} \cap S_{20} = \langle 6 \rangle_{\mathcal{P}}$
$S_{11} = \langle 5, 13 \rangle_{\mathcal{P}}$
$S_{12} \cap S_{21} = \langle 9, 11 \rangle_{\mathcal{P}}$
$S_{13} \cap S_{20} = \langle 8 \rangle_{\mathcal{P}}$
$S_{10} = \langle 5 \rangle_{\mathcal{P}}$
$S_{11} \cap S_{21} = \langle 9, 13 \rangle_{\mathcal{P}}$
$S_{12} \cap S_{20} = \langle 11 \rangle_{\mathcal{P}}$
$S_{10} \cap S_{21} = \langle 9 \rangle_{\mathcal{P}}$
$S_{11} \cap S_{20} = \langle 13 \rangle_{\mathcal{P}}$
$S_{10} \cap S_{20} = \langle \emptyset \rangle_{\mathcal{P}}$

A related question

Problem

- *Can we determine automatically the elements of the minimal $\mathcal{P}$-system of generators of S that are greater than $\mathrm{F}(S)$?*

Answers (Some answers)

- *Finitely generated pseudo-variety: pseudo-variety $\mathfrak{p}(\mathcal{F})$ generated by a finite family of numerical semigroups $\mathcal{F}$ (equivalently, pseudo-variety which is finite).*
- *Family of numerical semigroups defined by non-homogeneous patterns.*

Finitely generated pseudo-variety

Proposition

- *Let $\mathcal{F}$ be a non-empty finite family of numerical semigroups, $\mathcal{P} = \mathfrak{p}(\mathcal{F})$, and $A \subseteq \max(\mathcal{P})$.*
  For each $S \in \mathcal{F}$ such that $A \nsubseteq S$, let $x_S = \min\{a \in A \mid a \notin S\}$.
  Then $B = \{x_S \mid S \in \mathcal{F}$ and $A \nsubseteq S\}$ is the minimal $\mathcal{P}$-system of generators of $\mathcal{P}(A)$.

Example

- $\mathcal{F} = \{S_1 = \langle 5,7,9\rangle, S_2 = \langle 4,6,7\rangle\}$.
- $S = S_{11} \cap S_{21} = \langle 7,9,10,12,13,15\rangle$, $A = \{7,9,10,12,13,15\}$, $\mathrm{F}(S) = 11$.
  - $x_{s_1} = \min\{a \in A \mid a \notin S_1\} = 13$.
  - $x_{s_2} = \min\{a \in A \mid a \notin S_2\} = 9$.
- $S = S_{12} \cap S_{21} = \langle 9,13\rangle_{\mathcal{P}}$.
- 13 is the unique minimal $\mathcal{P}$-generator of S greater than $\mathrm{F}(S)$.

Variety defined by non-homogeneous patterns

Definition (R.-P. and Rosales, preprint)

- *We say that a numerical semigroup S is a numerical $\mathcal{A}$-semigroup if $\{x+y-1, x+y+1\} \subseteq S$, for all $x, y \in S \setminus \{0\}$.*

  (Bras-Amorós, Stokes, 2012. Bras-Amorós, García-Sánchez, Vico-Oton, 2013. Stokes, Bras-Amorós, 2014.)

Proposition

- *Let S be a numerical $\mathcal{A}$-semigroup such that $S \neq \mathbb{N}$, and let $x \in \mathrm{msg}(S)$. Then $S \setminus \{x\}$ is a numerical $\mathcal{A}$-semigroup if and only if $\{x-1, x+1\} \subseteq \{0\} \cup (\mathbb{N} \setminus S) \cup \mathrm{msg}(S)$.*

  ($\mathrm{msg}(S)$ is the minimal system fo generators of S.)

Example

- $S = \langle 5, 7, 8, 9, 11 \rangle$, $\mathrm{F}(S) = 6$.
- $S \setminus \{5\}$, $S \setminus \{7\}$, $S \setminus \{8\}$ are numerical $\mathcal{A}$-semigroups.
- In the tree associated to the variety of numerical $\mathcal{A}$-semigroups, $S \setminus \{7\}$ and $S \setminus \{8\}$ are the children of S.

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