PROPORTIONALLY MODULAR DIOPHANTINE INEQUALITIES WHICH ARE MODULAR

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Abstract. Let $S(a, b, c)$ be the set of integer solutions of an expression of the form $ax \mod b \leq cx$. We want to determine the conditions for which there exists another expression of the form $a^*x \mod b^* \leq x$ such that $S(a^*, b^*, 1) = S(a, b, c)$, and, in the affirmative case, compute algorithmically $a^*$ and $b^*$.

Introduction

Let $m, n$ be integers such that $n \neq 0$. We denote by $m \mod n$ the remainder of the division of $m$ by $n$. Following the notation of [7], we say that a proportionally modular Diophantine inequality is an expression of the form $ax \mod b \leq cx$, where $a, b, c$ are positive integers.

Let $S(a, b, c)$ be the set of integer solutions of $ax \mod b \leq cx$. It is easy to see that $S(a, b, c) = S(a \mod b, b, c)$. Moreover, if $a \leq c$, then $S(a, b, c) = \mathbb{N}$. Therefore, we can focus our attention to the case $0 < c < a < b$.

A modular Diophantine inequality (see [8]) is an expression of the form $ax \mod b \leq x$, where $a, b$ are positive integers such that $1 < a < b$. Obviously, a modular Diophantine inequality is a proportionally modular Diophantine inequality. A natural question is on the truth of the reciprocal assertion in the following sense: if $a, b, c$ are given, do there exist $a^*, b^*$ such that $S(a, b, c) = S(a^*, b^*, 1)$? We will answer in a particular case (see [4]).

1. Numerical semigroups and Diophantine inequalities

A numerical semigroup is a subset $S$ of $\mathbb{N}$ (set of nonnegative integers) such that it is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S$ is finite. It is well known (see [6]) that every numerical semigroup admits a unique set $X = \{n_1, n_2, \ldots, n_e\} \subseteq S$ such that $S = \langle X \rangle = \{\lambda_1n_1 + \lambda_2n_2 + \ldots + \lambda_en_e \mid \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$. We say that $X$ is the minimal system of generators of $S$. If $n_1 < n_2 < \ldots < n_e$, then $n_1, n_2$ are known as the multiplicity and the ratio of $S$, and $e = e(S)$ is the embedding dimension of $S$.

In [7] it is proved that $S(a, b, c)$ is a numerical semigroup. Such type of numerical semigroups are called proportionally modular numerical semigroups (PM-semigroups). If $c = 1$, then we have the modular numerical semigroups (M-semigroups). Thus, every M-semigroup is a PM-semigroup, but the reciprocal statement is not true (see [7, Example 26]).

It is interesting to remark that proportionally modular numerical semigroups are tightly related to problems in classification theory of $C^\ast$-algebras via $K_0$-groups (see [10]). In particular, in [5] it is shown that a semigroup having a Toms’ decomposition is a finite

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intersection of PM-semigroups. More in general, numerical semigroups have applications to Algebraic Geometry (see [1]).

As a consequence of [9, Theorem 31] (see its proof and [9, Corollary 18]) we have a characterization for PM-semigroups in terms of the minimal system of generators: a numerical semigroup \( S \) is a PM-semigroup if and only if there exists a convex arrangement \( n_1, n_2, \ldots, n_e \) of its set of minimal generators that satisfies the following conditions,

\[
\begin{align*}
(1) \quad & \gcd\{n_i, n_{i+1}\} = 1 \quad \text{for all} \quad i \in \{1, \ldots, e-1\}; \\
(2) \quad & (n_{i-1} + n_{i+1}) \equiv 0 \mod n_i \quad \text{for all} \quad i \in \{2, \ldots, e-1\}.
\end{align*}
\]

Let us observe that it is easy to determine whether or not a numerical semigroup is a PM-semigroup.

As a first step to give such a characterization, and therefore an answer to the original question of this work, our aim will be to show the explicit descriptions of all the \( M \)-semigroups. In [8] there is an algorithm to give the answer to this problem, but we have not got a good characterization for \( M \)-semigroups.

As a first step to give such a characterization, and therefore an answer to the original question of this work, our aim will be to show the explicit descriptions of all the \( M \)-semigroups with embedding dimension equal to three. In order to do it, we consider two ideas: the relation between PM-semigroups and numerical semigroups associated with an interval (see [7]) on the one side, and the description of a PM-semigroup with embedding dimension equal to three when we fix the multiplicity and the ratio on the other side.

## 2. Preliminaries

Let \( \alpha, \beta \) be two positive rational numbers with \( \alpha < \beta \) and let \( T \) be the submonoid of \( (\mathbb{Q}_0^+, +) \) generated by the interval \([\alpha, \beta]\). In [7] it is shown that \( T \cap \mathbb{N} \) is a PM-semigroup and that every PM-semigroup is of this form. We will denote \( T \cap \mathbb{N} = S([\alpha, \beta]) \).

**Lemma 2.1.** [7, Corollary 9]

1. Let \( a, b, c \) be positive integers such that \( c < a < b \). Then \( S(a, b, c) = S\left(\frac{b}{a_1}, \frac{b}{a_2}\right) \).

2. Let \( a_1, a_2, b_1, b_2 \) be positive integers such that \( 1 < \frac{b_1}{a_1} < \frac{b_2}{a_2} \). Then \( S\left(\frac{b_1}{a_1}, \frac{b_2}{a_2}\right) = S(a_1b_2, b_1b_2, a_1b_2 - a_2b_1) \).

If \( a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_p \) are positive integers such that \( a_ib_{i+1} - a_{i+1}b_i = 1 \) for all \( i \in \{1, 2, \ldots, p-1\} \), then the sequence of rational numbers \( \frac{b_1}{a_1} < \frac{b_2}{a_2} < \ldots < \frac{b_p}{a_p} \) is a Bézout sequence. The fractions \( \frac{b_1}{a_1} \) and \( \frac{b_p}{a_p} \) are the ends of the sequence. We say that a Bézout sequence is proper if \( a_ib_{i+h} - a_{i+h}b_i \geq 2 \) for all \( h \geq 2 \) such that \( i, i+h \in \{1, 2, \ldots, p\} \). On the other hand, we will say that two fractions \( \frac{b_1}{a_1} < \frac{b_2}{a_2} \) are adjacent if

\[
\frac{b_2}{a_2+1} < \frac{b_1}{a_1}, \quad \text{and either} \quad a_1 = 1 \quad \text{or} \quad \frac{b_2}{a_2} < \frac{b_1}{a_1-1}.
\]

We note that an useful tool for computing Bézout sequences is the Stern-Brocot tree (see [2, 3]).

**Lemma 2.2.** [9, Theorem 20, Theorem 23]

1. Let \( \frac{b_1}{a_1} < \frac{b_2}{a_2} < \ldots < \frac{b_p}{a_p} \) be a proper Bézout sequence with adjacent ends. Then \( \{b_1, b_2, \ldots, b_p\} \) is the minimal system of generators of \( S\left(\frac{b_1}{a_1}, \frac{b_p}{a_p}\right) \),
(2) Let $S$ be a PM-semigroup with $e(S) = p \geq 2$. Then there exist an arrangement of the set of minimal generators of $S$, $n_1, \ldots, n_p$, and positive integers $a_1, \ldots, a_p$ such that $\frac{n_1}{a_1} < \frac{n_2}{a_2} < \ldots < \frac{n_p}{a_p}$ is a proper Bézout sequence with adjacent ends.

3. ALL M-SEMIGROUPS WITH EMBEDDING DIMENSION EQUAL TO THREE

Firstly, we give two families of M-semigroups with embedding dimension equal to three.

**Proposition 3.1.**

1. Let $\lambda, d, d'$ be integers greater than one such that $\gcd(d, d') = 1$. Then $S = \langle \lambda d, d + d', \lambda d' \rangle$ is an M-semigroup with $e(S) = 3$.
2. Let $m_1, m_2$ be positive integers such that $\gcd\{m_1, m_2\} = 1$. Let $q$ be a divisor of $\gcd\{m_2 - 1, m_1 + m_2\}$ such that $2 \leq q < \min\{m_1, m_2\}$. Then $S = \langle m_1, \frac{m_1 + m_2}{q}, m_2 \rangle$ is an M-semigroup with $e(S) = 3$.

Secondly, we show that every M-semigroup with embedding dimension equal to three belongs to one of the families described in the previous proposition.

**Proposition 3.2.** Let $S = S\left(\left[ \frac{b}{a}, \frac{b}{a-1} \right] \right)$ be a numerical semigroup such that $e(S) = 3$. Let us have $\gcd\{a, b\} = d$ and $\gcd\{a-1, b\} = d'$.

1. If $d \neq 1$ and $d' \neq 1$, then there exists an integer $\lambda$ greater than one such that $\gcd\{d, d'\} = \gcd\{\lambda, d + d'\} = 1$ and $S = \langle \lambda d, d + d', \lambda d' \rangle$.
2. If $d = 1$ and/or $d' = 1$, then there exist three positive integers $m_1, m_2, q$ such that $\gcd\{m_1, m_2\} = 1$, $q$ is a divisor of $\gcd\{m_2 - 1, m_1 + m_2\}$, $2 \leq q < \min\{m_1, m_2\}$, and $S = \langle m_1, \frac{m_1 + m_2}{q}, m_2 \rangle$.

Finally, we summarize both results in the next theorem.

**Theorem 3.3.** $S$ is an M-semigroup with $e(S) = 3$ if and only if

(T1) $S = \langle \lambda d, d + d', \lambda d' \rangle$, where $\lambda, d, d'$ are integers greater than one such that $\gcd\{d, d'\} = \gcd\{\lambda, d + d'\} = 1$,

(T2) $S = \langle m_1, \frac{m_1 + m_2}{q}, m_2 \rangle$, where $m_1, m_2, q$ are positive integers such that $\gcd\{m_1, m_2\} = 1$, $q$ is a divisor of $\gcd\{m_2 - 1, m_1 + m_2\}$, and $2 \leq q < \min\{m_1, m_2\}$.

4. MULTIPPLICITY AND RATIO FIXED

Let $\{n_1, n_2, n_3\}$ be the minimal system of generators of a numerical semigroup $S$. Let us suppose that $n_1 < n_2 < n_3$. The aim of this section is to describe all M-semigroups with embedding dimension equal to three when we fix the multiplicity and the ratio of $S$, this is, when $n_1$ and $n_2$ are fixed. We use the following sets,

* $A(n_1) = \{2, \ldots, n_1 - 1\}$;
* $A(n_1, n_2) = \left\{ \left[ \frac{2n_2}{n_1} \right], \ldots, n_2 - 1 \right\}$, where $\left[ \frac{2n_2}{n_1} \right] = \min\{z \in \mathbb{Z} \mid q \leq z\}$;
* $D(n) = \{k \in \mathbb{N} \mid k \mid n\}$.

**Theorem 4.1.** Let $n_1, n_2, n_3$ be integers such that $3 \leq n_1 < n_2 < n_3$, $\gcd\{n_1, n_2\} = 1$, and $n_3 \notin \langle n_1, n_2 \rangle$. Then $S = \langle n_1, n_2, n_3 \rangle$ is an M-semigroup if and only if $n_3$ belongs to

1. $B_1 = \{kn_2 - n_1 \mid k \in A(n_1) \cap [D(n_1 - 1) \cup D(n_1) \cup D(n_1 + 1)]\}$
2. $B_2 = \{tn_1 - n_2 \mid t \in A(n_1, n_2) \cap [D(n_2 - 1) \cup D(n_2) \cup D(n_2 + 1)]\}$. 


Moreover,

1. $S$ is (T1) if and only if $k \in D(n_1)$ or $t \in D(n_2)$.

2. $S$ is (T2) if and only if $k \in D(n_1 - 1) \cup D(n_1 + 1)$ or $t \in D(n_2 - 1) \cup D(n_2 + 1)$.

5. Conclusion

Let $S = S(a,b,c)$ be the set of solutions of $ax \text{mod} b \leq cx$, where $a,b,c$ are positive integers such that $c < a < b$. By Lemma 2.1, $S = S\left(\left[\frac{b}{a}, \frac{b}{a - c}\right]\right)$. If $e(S) = 3$, applying Lemma 2.2, $S = \langle n_1, n_2, n_3 \rangle$. From Theorem 4.1, we can determine if $S$ is an M-semigroup. Finally, since the proofs of the results in Section 3 are constructive, we find $a^*, b^*$ such that $S = S(a^*, b^*, 1)$.

At this point, the proposed problem is partially solved. In order to give the complete answer, we need to remove the condition $e(S) = 3$ in Sections 3 and 4. In fact, we believe that the key is to show a general characterization for M-semigroups.

Let us have $S$ with minimal system of generators $n_1, n_2, \ldots, n_e$ which are arranged according the characterization of PM-semigroups. Then $n_{i+1} = k_in_i - n_{i-1}$, $2 \leq i \leq e - 1$. If we replace $n_3$ in $n_4$, $(n_3, n_4)$ in $n_5$, and so on, we have $n_e = \alpha_en_2 - \beta_en_1$. Thus, we finish with the following statement.

**Conjecture.** $S$ is an M-semigroup if and only if $\alpha_e \in D(n_1 - 1) \cup D(n_1) \cup D(\beta_en_1 + 1)$.

**References**


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