PROPORTIONALLY MODULAR DIOPHANTINE INEQUALITIES WHICH ARE MODULAR

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Abstract. Let $\mathrm{S}(a,b,c)$ be the set of integer solutions of an expression of the form $ax \bmod b \leq cx$. We want to determine the conditions for which there exists another expression of the form $a^*x \bmod b^* \leq x$ such that $\mathrm{S}(a^*,b^*,1)=\mathrm{S}(a,b,c)$, and, in the affirmative case, compute algorithmically a^* and b^*.

Introduction

Let m,n be integers such that $n \neq 0$. We denote by $m \bmod n$ the remainder of the division of m by n. Following the notation of [7], we say that a *proportionally modular Diophantine inequality* is an expression of the form $ax \bmod b \leq cx$, where a,b,c are positive integers.

Let $\mathrm{S}(a,b,c)$ be the set of integer solutions of $ax \bmod b \leq cx$. It is easy to see that $\mathrm{S}(a,b,c)=\mathrm{S}(a \bmod b,b,c)$. Moreover, if $a \leq c$, then $\mathrm{S}(a,b,c)=\mathbb{N}$. Therefore, we can focus our attention to the case $0<c<a<b$.

A *modular Diophantine inequality* (see [8]) is an expression of the form $ax \bmod b \leq x$, where a,b are positive integers such that $1<a<b$. Obviously, a modular Diophantine inequality is a proportionally modular Diophantine inequality. A natural question is on the truth of the reciprocal assertion in the following sense: if a,b,c are given, do there exist a^*,b^* such that $\mathrm{S}(a,b,c)=\mathrm{S}(a^*,b^*,1)$? We will answer in a particular case (see [4]).

1. Numerical semigroups and Diophantine inequalities

A *numerical semigroup* is a subset S of $\mathbb{N}$ (set of nonnegative integers) such that it is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S$ is finite. It is well known (see [6]) that every numerical semigroup admits a unique set $X=\{n_1,n_2,\ldots,n_e\} \subseteq S$ such that $S=\langle X\rangle=\{\lambda_1 n_1+\lambda_2 n_2+\ldots+\lambda_e n_e \mid \lambda_1,\ldots,\lambda_n \in \mathbb{N}\}$. We say that X is the *minimal system of generators* of S. If $n_1<n_2<\ldots<n_e$, then n_1,n_2 are known as the *multiplicity* and the *ratio* of S, and $e=\mathrm{e}(S)$ is the *embedding dimension* of S.

In [7] it is proved that $\mathrm{S}(a,b,c)$ is a numerical semigroup. Such type of numerical semigroups are called *proportionally modular numerical semigroups* (PM-*semigroups*). If $c=1$, then we have the *modular numerical semigroups* (M-*semigroups*). Thus, every M-semigroup is a PM-semigroup, but the reciprocal statement is not true (see [7, Example 26]).

It is interesting to remark that proportionally modular numerical semigroups are tightly related to problems in classification theory of C^*-algebras via K_0-groups (see [10]). In particular, in [5] it is shown that a semigroup having a Toms' decomposition is a finite

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intersection of PM-semigroups. More in general, numerical semigroups have applications to Algebraic Geometry (see [1]).

As a consequence of [9, Theorem 31] (see its proof and [9, Corollary 18]) we have a characterization for PM-semigroups in terms of the minimal system of generators: a numerical semigroup S is a PM-semigroup if and only if there exists a convex arrangement $n_1, n_2, \ldots, n_e$ of its set of minimal generators that satisfies the following conditions,

(1) $\gcd\{n_i, n_{i+1}\} = 1$ for all $i \in \{1, \ldots, e-1\}$;
(2) $(n_{i-1} + n_{i+1}) \equiv 0 \bmod n_i$ for all $i \in \{2, \ldots, e-1\}$.

Let us observe that it is easy to determine whether or not a numerical semigroup is a PM-semigroup via the previous characterization. However, this question is more complicated for M-semigroups. In [8] there is an algorithm to give the answer to this problem, but we have not got a good characterization for M-semigroups.

As a first step to give such a characterization, and therefore an answer to the original question of this work, our aim will be to show the explicit descriptions of all the M-semigroups with embedding dimension equal to three. In order to do it, we consider two ideas: the relation between PM-semigroups and numerical semigroups associated with an interval (see [7]) on the one side, and the description of a PM-semigroup with embedding dimension equal to three when we fix the multiplicity and the ratio on the other side.

2. Preliminaries

Let α, β be two positive rational numbers with $\alpha < \beta$ and let T be the submonoid of $(\mathbb{Q}_0^+, +)$ generated by the interval $[\alpha, \beta]$. In [7] it is shown that $\mathrm{T} \cap \mathbb{N}$ is a PM-semigroup and that every PM-semigroup is of this form. We will be denote $\mathrm{T} \cap \mathbb{N} = \mathrm{S}\left([\alpha, \beta]\right)$.

Lemma 2.1. [7, Corollary 9]

(1) *Let a, b, c be positive integers such that $c < a < b$. Then* $\mathrm{S}(a, b, c) = \mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-c}\right]\right)$.
(2) *Let a_1, a_2, b_1, b_2 be positive integers such that $1 < \frac{b_1}{a_1} < \frac{b_2}{a_2}$. Then* $\mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_2}{a_2}\right]\right) = \mathrm{S}(a_1 b_2, b_1 b_2, a_1 b_2 - a_2 b_1)$.

If $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_p$ are positive integers such that $a_i b_{i+1} - a_{i+1} b_i = 1$ for all $i \in \{1, 2, \ldots, p-1\}$, then the sequence of rational numbers $\frac{b_1}{a_1} < \frac{b_2}{a_2} < \ldots < \frac{b_p}{a_p}$ is a *Bézout sequence.* The fractions $\frac{b_1}{a_1}$ and $\frac{b_p}{a_p}$ are the *ends* of the sequence. We say that a Bézout sequence is *proper* if $a_i b_{i+h} - a_{i+h} b_i \geq 2$ for all $h \geq 2$ such that $i, i+h \in \{1, 2, \ldots, p\}$. On the other hand, we will say that two fractions $\frac{b_1}{a_1} < \frac{b_2}{a_2}$ are *adjacent* if

$$\frac{b_2}{a_2+1} < \frac{b_1}{a_1}, \text{ and either } a_1 = 1 \text{ or } \frac{b_2}{a_2} < \frac{b_1}{a_1 - 1}.$$

We note that an useful tool for computing Bézout sequences is the Stern-Brocot tree (see [2, 3]).

Lemma 2.2. [9, Theorem 20, Theorem 23]

(1) *Let $\frac{b_1}{a_1} < \frac{b_2}{a_2} < \ldots < \frac{b_p}{a_p}$ be a proper Bézout sequence with adjacent ends. Then $\{b_1, b_2, \ldots, b_p\}$ is the minimal system of generators of* $\mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_p}{a_p}\right]\right)$.

(2) *Let S be a* PM*-semigroup with* $\mathrm{e}(S) = p \geq 2$. *Then there exist an arrangement of the set of minimal generators of S, $n_1, \ldots, n_p$, and positive integers $a_1, \ldots, a_p$ such that $\frac{n_1}{a_1} < \frac{n_2}{a_2} < \ldots < \frac{n_p}{a_p}$ is a proper Bézout sequence with adjacent ends.*

3. All M-semigroups with embedding dimension equal to three

Firstly, we give two families of M-semigroups with embedding dimension equal to three.

Proposition 3.1. (1) *Let λ, d, d' be integers greater than one such that* $\gcd\{d, d'\} = \gcd\{\lambda, d + d'\} = 1$. *Then $S = \langle \lambda d, d + d', \lambda d' \rangle$ is an* M*-semigroup with* $\mathrm{e}(S) = 3$.

(2) *Let m_1, m_2 be positive integers such that* $\gcd\{m_1, m_2\} = 1$. *Let q be a divisor of* $\gcd\{m_2 - 1, m_1 + m_2\}$ *such that* $2 \leq q < \min\{m_1, m_2\}$. *Then* $S = \langle m_1, \frac{m_1+m_2}{q}, m_2 \rangle$ *is an* M*-semigroup with* $\mathrm{e}(S) = 3$.

Secondly, we show that every M-semigroup with embedding dimension equal to three belongs to one of the families described in the previous proposition.

Proposition 3.2. *Let* $S = \mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-1}\right]\right)$ *be a numerical semigroup such that* $\mathrm{e}(S) = 3$. *Let us have* $\gcd\{a, b\} = d$ *and* $\gcd\{a - 1, b\} = d'$.

(1) *If $d \neq 1$ and $d' \neq 1$, then there exists an integer λ greater than one such that* $\gcd\{d, d'\} = \gcd\{\lambda, d + d'\} = 1$ *and* $S = \langle \lambda d, d + d', \lambda d' \rangle$.

(2) *If $d = 1$ and/or $d' = 1$, then there exist three positive integers m_1, m_2, q such that* $\gcd\{m_1, m_2\} = 1$, *q is a divisor of* $\gcd\{m_2 - 1, m_1 + m_2\}$, $2 \leq q < \min\{m_1, m_2\}$, *and* $S = \langle m_1, \frac{m_1+m_2}{q}, m_2 \rangle$.

Finally, we summarize both results in the next theorem.

Theorem 3.3. *S is an* M*-semigroup with* $\mathrm{e}(S) = 3$ *if and only if*

(T1) $S = \langle \lambda d, d+d', \lambda d' \rangle$, *where λ, d, d' are integers greater than one such that* $\gcd\{d, d'\} = \gcd\{\lambda, d + d'\} = 1$,

(T2) *or* $S = \langle m_1, \frac{m_1+m_2}{q}, m_2 \rangle$, *where m_1, m_2, q are positive integers such that* $\gcd\{m_1, m_2\} = 1$, *q is a divisor of* $\gcd\{m_2 - 1, m_1 + m_2\}$, *and* $2 \leq q < \min\{m_1, m_2\}$.

4. Multiplicity and ratio fixed

Let $\{n_1, n_2, n_3\}$ be the minimal system of generators of a numerical semigroup S. Let us suppose that $n_1 < n_2 < n_3$. The aim of this section is to describe all M-semigroups with embedding dimension equal to three when we fix the multiplicity and the ratio of S, this is, when n_1 and n_2 are fixed. We use the following sets,

- $A(n_1) = \{2, \ldots, n_1 - 1\}$;
- $A(n_1, n_2) = \left\{\left\lceil \frac{2n_2}{n_1} \right\rceil, \ldots, n_2 - 1\right\}$, where $\left\lceil \frac{2n_2}{n_1} \right\rceil = \min\{z \in \mathbb{Z} \mid q \leq z\}$;
- $D(n) = \{k \in \mathbb{N} \text{ such that } k \mid n\}$.

Theorem 4.1. *Let n_1, n_2, n_3 be integers such that* $3 \leq n_1 < n_2 < n_3$, $\gcd\{n_1, n_2\} = 1$, *and* $n_3 \notin \langle n_1, n_2 \rangle$. *Then $S = \langle n_1, n_2, n_3 \rangle$ is an* M*-semigroup if and only if n_3 belongs to*

(1) $B_1 = \{kn_2 - n_1 \mid k \in A(n_1) \cap [D(n_1 - 1) \cup D(n_1) \cup D(n_1 + 1)]\}$

(2) *or* $B_2 = \{tn_1 - n_2 \mid t \in A(n_1, n_2) \cap [D(n_2 - 1) \cup D(n_2) \cup D(n_2 + 1)]\}$.

Moreover,

(1) *S is $(T1)$ if and only if $k \in D(n_1)$ or $t \in D(n_2)$.*
(2) *S is $(T2)$ if and only if $k \in D(n_1 - 1) \cup D(n_1 + 1)$ or $t \in D(n_2 - 1) \cup D(n_2 + 1)$.*

5. Conclusion

Let $S = \mathrm{S}(a, b, c)$ be the set of solutions of $ax \bmod b \leq cx$, where a, b, c are positive integers such that $c < a < b$. By Lemma 2.1, $S = \mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-c}\right]\right)$. If $\mathrm{e}(S) = 3$, applying Lemma 2.2, $S = \langle n_1, n_2, n_3 \rangle$. From Theorem 4.1, we can determine if S is an M-semigroup. Finally, since the proofs of the results in Section 3 are constructive, we find a^*, b^* such that $S = \mathrm{S}(a^*, b^*, 1)$.

At this point, the proposed problem is partially solved. In order to give the complete answer, we need to remove the condition $\mathrm{e}(S) = 3$ in Sections 3 and 4. In fact, we believe that the key is to show a general characterization for M-semigroups.

Let us have S with minimal system of generators $n_1, n_2, \ldots, n_e$ which are arranged according the characterization of PM-semigroups. Then $n_{i+1} = k_i n_i - n_{i-1}$, $2 \leq i \leq e - 1$. If we replace n_3 in n_4, (n_3, n_4) in n_5, and so on, we have $n_e = \alpha_e n_2 - \beta_e n_1$. Thus, we finish with the following statement.

Conjecture. *S is an* M*-semigroup if and only if $\alpha_e \in D(n_1 - 1) \cup D(n_1) \cup D(\beta_e n_1 + 1)$.*

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