Proportionally modular Diophantine inequalities which are modular

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Diophantine inequalities.- General type

Definition

A **proportionally modular Diophantine inequality** *is an expression of the form*

$$ax \bmod b \leq cx$$

where a (the factor), b (the modulus), *and* c (the proportion) *are positive integers.*

(Let m, n be integers such that $n \neq 0$. Then $m \bmod n$ is the remainder of the division of m by n.)

Set of nonnegative integer solutions

$$\mathrm{S}(a,b,c) = \{x \in \mathbb{Z} \mid ax \bmod b \leq cx\}$$

A simplification

Lemma

Let a, b, c positive integers.

1. $\mathrm{S}(a,b,c) = \mathrm{S}(a \bmod b, b, c)$.
2. $\mathrm{S}(a,b,c) = \mathbb{N}$ *if* $a \leq c$.

($\mathbb{N}$ is the set of nonnegative integers.)

Not restrictive condition

$$c < a < b$$

Diophantine inequalities.- Particular type

Definition

A **modular Diophantine inequality** *is an expression of the form*

$$ax \bmod b \leq x,$$

where a, b are positive integers (such that $a < b$).

Set of nonnegative integer solutions

$$\mathrm{S}(a,b) = \mathrm{S}(a,b,1) = \{x \in \mathbb{Z} \mid ax \bmod b \leq x\}$$

Diophantine inequalities.- The question

Problem

Let us have $\mathrm{S}(a,b,c)$. Does there exist positive integers a^*, b^* such that $\mathrm{S}(a,b,c) = \mathrm{S}(a^*,b^*)$?

Example

1. $\mathrm{S}(21,189,3) = \mathrm{S}(7,63)$.
2. $\mathrm{S}(41,369,5) = \mathrm{S}(8,72)$.
3. $\mathrm{S}(51,459,6) \neq \mathrm{S}(a^*,b^*)$ for all $a^*, b^* \in \mathbb{N}$.

Tool: numerical semigroups

Definition

A **numerical semigroup** *is a subset S of $\mathbb{N}$ that is closed under addition, contains the zero element, and has finite complement in $\mathbb{N}$.*

Example

1. $S = \mathrm{S}(a,b,c) = \{x \in \mathbb{Z} \mid ax \bmod b \leq cx\}$ (PM-**semigroup**).
2. $S = \mathrm{S}([\alpha,\beta])$.
   - $\alpha,\beta \in \mathbb{Q}$ such that $0 < \alpha < \beta$; $J = [\alpha,\beta]$.
   - $\langle J \rangle = \{\lambda_1 a_1 + \ldots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\}, a_1,\ldots,a_n \in J, \lambda_1,\ldots,\lambda_n \in \mathbb{N}\}$.
   - $\langle J \rangle \cap \mathbb{N} = \mathrm{S}([\alpha,\beta])$.
3. $S = \langle a_1, a_2, \ldots, a_n \rangle$.
   - $A = \{a_1,a_2,\ldots,a_n\} \subseteq \mathbb{N} \setminus \{0\}$ such that $\gcd\{a_1,a_2,\ldots,a_n\} = 1$.
   - $\langle A \rangle = \{\lambda_1 a_1 + \ldots + \lambda_n a_n \mid \lambda_1,\ldots,\lambda_n \in \mathbb{N}\} = \langle a_1,a_2,\ldots,a_n \rangle$.

Connection

Lemma

1. *Let a,b,c positive integers such that $c<a<b$. Then*

$$\mathrm{S}(a,b,c)=\mathrm{S}\left(\left[\frac{b}{a},\frac{b}{a-c}\right]\right).$$

2. *Let a_1,a_2,b_1,b_2 be positive integers such that $1<\frac{b_1}{a_1}<\frac{b_2}{a_2}$. Then*

$$\mathrm{S}\left(\left[\frac{b_1}{a_1},\frac{b_2}{a_2}\right]\right)=\mathrm{S}(a_1b_2,b_1b_2,a_1b_2-a_2b_1).$$

Example

1. $\mathrm{S}(41,369,5)=\mathrm{S}\left(\left[\frac{369}{41},\frac{369}{36}\right]\right)=\mathrm{S}\left(\left[\frac{9}{1},\frac{41}{4}\right]\right)$
2. $\mathrm{S}(51,459,6)=\mathrm{S}\left(\left[\frac{459}{51},\frac{459}{45}\right]\right)=\mathrm{S}\left(\left[\frac{9}{1},\frac{51}{5}\right]\right)$

Bézout sequences

Example

1. $\frac{9}{1} < \frac{10}{1} < \frac{21}{2}$

$$\mathrm{S}(21,189,3) = \mathrm{S}\left(\left[\frac{189}{21}, \frac{189}{18}\right]\right) = \mathrm{S}\left(\left[\frac{9}{1}, \frac{21}{2}\right]\right) = \langle 9,10,21 \rangle$$

2. $\frac{9}{1} < \frac{10}{1} < \frac{41}{4}$

$$\mathrm{S}(41,369,5) = \mathrm{S}\left(\left[\frac{369}{41}, \frac{369}{36}\right]\right) = \mathrm{S}\left(\left[\frac{9}{1}, \frac{41}{4}\right]\right) = \langle 9,10,41 \rangle$$

3. $\frac{9}{1} < \frac{10}{1} < \frac{51}{5}$

$$\mathrm{S}(51,459,6) = \mathrm{S}\left(\left[\frac{459}{51}, \frac{459}{45}\right]\right) = \mathrm{S}\left(\left[\frac{9}{1}, \frac{51}{5}\right]\right) = \langle 9,10,51 \rangle$$

Bézout sequences (more ...)

Example

1. $\frac{9}{5} < \frac{2}{1} < \frac{9}{4}$

   $$\mathrm{S}(5,9,1) = \mathrm{S}\left(\left[\frac{9}{5},\frac{9}{4}\right]\right) = \langle 2,9 \rangle$$

2. $\frac{23}{13} < \frac{16}{9} < \frac{9}{5} < \frac{2}{1} < \frac{9}{4} < \frac{16}{7}$

   $$\mathrm{S}(208,368,47) = \mathrm{S}\left(\left[\frac{368}{208},\frac{368}{161}\right]\right) = \mathrm{S}\left(\left[\frac{23}{13},\frac{16}{7}\right]\right) = \langle 2,9 \rangle$$

3. $\frac{9}{1} < \frac{10}{1} < \frac{11}{1} < \frac{12}{1}$

   $$\mathrm{S}(4,36,3) = \mathrm{S}\left(\left[\frac{36}{4},\frac{36}{3}\right]\right) = \mathrm{S}\left(\left[\frac{9}{1},\frac{12}{1}\right]\right) = \langle 9,10,11,12 \rangle$$

Algorithm

Algorithm

1. $\mathrm{S}(a,b,c) \Rightarrow \mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-c}\right]\right)$ *$(1<c<a<b)$*
2. $\mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-c}\right]\right) \Rightarrow \mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_p}{a_p}\right]\right)$ $(\gcd\{a_1,b_1\}=\gcd\{a_p,b_p\}=1)$
3. $\mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_p}{a_p}\right]\right) \Rightarrow \frac{b_1}{a_1} < \frac{b_2}{a_2} < \cdots < \frac{b_p}{a_p}$ *(Bézout sequence)*
4. $\frac{b_1}{a_1} < \frac{b_2}{a_2} < \cdots < \frac{b_p}{a_p} \Rightarrow \langle b_1, b_2, \ldots, b_p\rangle$ *(System of generators)*
5. $\langle b_1, b_2, \ldots, b_p\rangle \Rightarrow \langle n_1, n_2, \ldots, n_e\rangle$ *(Minimal system of generators)*
6. $\langle n_1, n_2, \ldots, n_e\rangle \Rightarrow$ *Is* $\mathrm{S}(a,b,c)$ *modular?*

   6.1 *No* $\Rightarrow$ *Other questions?*

   6.2 *Yes* $\Rightarrow S(a^*,b^*)$*?*

Some answers

Remark

1. $\mathrm{S}(a,b,c)=\langle n_1,n_2\rangle \Rightarrow \mathrm{S}(a,b,c)=\mathrm{S}(un_2,n_1n_2)$ $(un_2-vn_1=1)$
2. $\mathrm{S}(a,b,c)=\langle n_1,n_2,n_3\rangle \Rightarrow$ *Whole answer*
3. $\mathrm{S}(a,b,c)=\langle n_1,n_2,\ldots,n_e\rangle$, $e\geq 4 \Rightarrow$ *Partial conjecture*

Example

$\mathrm{S}(208,368,47)=\langle 2,9\rangle$

1. $1\times 9-4\times 2=1 \Rightarrow \mathrm{S}(208,368,47)=\mathrm{S}(9,18)$
2. $5\times 2-1\times 9=1 \Rightarrow \mathrm{S}(208,368,47)=\mathrm{S}(10,18)$

And remember that $\mathrm{S}(5,9,1)=\mathrm{S}(5,9)=\langle 2,9\rangle$.

Characterization for PM-semigroups

Lemma

A numerical semigroup S is a PM-semigroup if and only if there exists a convex arrangement $n_1, n_2, \ldots, n_e$ of its set of minimal generators that satisfies the following conditions

1. $\gcd\{n_i, n_{i+1}\} = 1$ *for all* $i \in \{1, \ldots, e-1\}$,
2. $(n_{i-1} + n_{i+1}) \equiv 0 \bmod n_i$ *for all* $i \in \{2, \ldots, e-1\}$.

Definition

A sequence of integers $x_1, x_2, \ldots, x_q$ is arranged in a convex form if one of the following conditions is satisfied,

1. $x_1 \leq x_2 \leq \ldots \leq x_q$;
2. $x_1 \geq x_2 \geq \ldots \geq x_q$;
3. *there exists* $h \in \{2, \ldots, q-1\}$ *such that* $x_1 \geq \ldots \geq x_h \leq \ldots \leq x_q$.

By generators via characterization

Proposition

1. *Let λ, d, d' be integers greater than one such that* $\gcd\{d, d'\} = \gcd\{\lambda, d+d'\} = 1$.
   Then $S = \langle \lambda d, d+d', \lambda d' \rangle$ is an M-semigroup with $\mathrm{e}(S) = 3$.

2. *Let m_1, m_2 be positive integers such that $\gcd\{m_1, m_2\} = 1$. Let q be a divisor of $\gcd\{m_2 - 1, m_1 + m_2\}$ such that* $2 \leq q < \min\{m_1, m_2\}$.
   Then $S = \langle m_1, \frac{m_1+m_2}{q}, m_2 \rangle$ is an M-semigroup with $\mathrm{e}(S) = 3$.

Example

1. $\langle 9, 10, 21 \rangle = \langle 3 \cdot 3, 3+7, 3 \cdot 7 \rangle$
2. $\langle 9, 10, 41 \rangle = \langle 9, \frac{9+41}{5}, 41 \rangle$ $\quad (5 = \gcd\{41-1, 9+41\})$
3. $\langle 9, 10, 51 \rangle$ Not possible!

By generators via characterization

Example

1. $\langle 9,10,21\rangle = \langle 3\cdot 3, 3+7, 3\cdot 7\rangle$

$$9\cdot(3\cdot 3) - 8\cdot(3+7) = 1$$

$$\langle 9,10,21\rangle = S((9\cdot 3-8)\cdot 3, 3\cdot 3\cdot 7) = S(57,63)$$

2. $\langle 9,10,41\rangle = \langle 9, \frac{9+41}{5}, 41\rangle$

$$2\cdot 41 - 9\cdot 9 = 1 \Rightarrow 10\cdot 41 - 45\cdot 9 = 5$$

$$\langle 9,10,41\rangle = S\left(\frac{41-1}{5}\cdot 10, \frac{41-1}{5}\cdot 9\right) = S(80,72) = S(8,72)$$

By generators via closed intervals

Proposition

Let $S = \mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-1}\right]\right)$ be a numerical semigroup such that $\mathrm{e}(S)=3$. Let us have $\gcd\{a,b\}=d$ and $\gcd\{a-1,b\}=d'$.

1. *If $d\neq 1$ and $d'\neq 1$, then there exists an integer λ greater than one such that $\gcd\{d,d'\}=\gcd\{\lambda,d+d'\}=1$ and $S=\langle \lambda d, d+d', \lambda d'\rangle$.*
2. *If $d=1$ and/or $d'=1$, then there exist three positive integers m_1,m_2,q such that $\gcd\{m_1,m_2\}=1$, q is a divisor of $\gcd\{m_2-1,m_1+m_2\}$, $2\leq q<\min\{m_1,m_2\}$, and $S=\langle m_1, \frac{m_1+m_2}{q}, m_2\rangle$.*

Remark

Observe that $S = \mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-1}\right]\right)$ is always a modular numerical semigroup.

Families by generators

Theorem

S is an M-semigroup with $\mathrm{e}(S)=3$ if and only if

(T1) *$S=\langle \lambda d, d+d', \lambda d'\rangle$, where λ, d, d' are integers greater than one such that $\gcd\{d,d'\}=\gcd\{\lambda, d+d'\}=1$,*

(T2) *or $S=\langle m_1, \frac{m_1+m_2}{q}, m_2\rangle$, where m_1, m_2, q are positive integers such that $\gcd\{m_1,m_2\}=1$, q is a divisor of $\gcd\{m_2-1, m_1+m_2\}$, and $2\leq q<\min\{m_1,m_2\}$.*

Remark

If S is an M-semigroup of type (T1) then it is not (T2). Consequently, if S is (T2) then it is not (T1).

PM-semigroups with n_1, n_2 fixed

Lemma

Let n_1, n_2, n_3 be integers such that $3 \leq n_1 < n_2 < n_3$, $\gcd\{n_1, n_2\} = 1$, and $n_3 \notin \langle n_1, n_2 \rangle$. Then $\langle n_1, n_2, n_3 \rangle$ is a PM-semigroup if and only if n_3 belongs to one of the following sets.

1. $C_1 = \{kn_2 - n_1 \mid k \in A(n_1)\}$.
2. $C_2 = \{tn_1 - n_2 \mid t \in A(n_1, n_2)\}$.

Moreover, $C_1 \cap C_2 = \{n_1 n_2 - n_1 - n_2\}$.

Definition

Let n_1, n_2 be integers such that $3 \leq n_1 < n_2$ and $\gcd\{n_1, n_2\} = 1$.

- $A(n_1) = \{2, \ldots, n_1 - 1\}$.
- $A(n_1, n_2) = \left\{\left\lceil \frac{2n_2}{n_1} \right\rceil, \ldots, n_2 - 1\right\}$.
- $D(n) = \{k \in \mathbb{N}$ *such that* $k \mid n\}$.

(If $q \in \mathbb{Q}$, then $\lceil q \rceil = \min\{z \in \mathbb{Z} \mid q \leq z\}$)

M-semigroups with n_1, n_2 fixed

Theorem

Let n_1, n_2, n_3 be integers such that $3 \leq n_1 < n_2 < n_3$, $\gcd\{n_1, n_2\} = 1$, and $n_3 \notin \langle n_1, n_2 \rangle$. Then $S = \langle n_1, n_2, n_3 \rangle$ is an M-semigroup if and only if n_3 belongs to

1. $B_1 = \{kn_2 - n_1 \mid k \in A(n_1) \cap [D(n_1 - 1) \cup D(n_1) \cup D(n_1 + 1)]\}$,
2. *or* $B_2 = \{tn_1 - n_2 \mid t \in A(n_1, n_2) \cap [D(n_2 - 1) \cup D(n_2) \cup D(n_2 + 1)]\}$.

Moreover,

1. *S is $(T1)$ if and only if $k \in D(n_1)$ or $t \in D(n_2)$.*
2. *S is $(T2)$ if and only if $k \in D(n_1 - 1) \cup D(n_1 + 1)$ or $t \in D(n_2 - 1) \cup D(n_2 + 1)$.*

M-semigroups with $n_1 = 9$, $n_2 = 10$

Example

1. PM-semigroups
   - $\langle 9, 10, 10k - 9 \rangle$ with $k \in \{2, 3, 4, 5, 6, 7, 8\}$
   - $\langle 9, 10, 9t - 10 \rangle$ with $t \in \{3, 4, 5, 6, 7, 8, 9\}$
2. M-semigroups
   - $\langle 9, 10, 10k - 9 \rangle$ with $k \in \{2, 3, 4, 5, 8\}$
   - $\langle 9, 10, 9t - 10 \rangle$ with $t \in \{3, 5, 9\}$

   (Observe that $\langle 9, 10, 10 \times 8 - 9 \rangle = \langle 9, 10, 9 \times 9 - 10 \rangle = \langle 9, 10, 71 \rangle$.)

Example

1. $\langle 9, 10, 21 \rangle = \langle 9, 10, 3 \times 10 - 9 \rangle$ is (T1)
2. $\langle 9, 10, 41 \rangle = \langle 9, 10, 5 \times 10 - 9 \rangle$ is (T2)
3. $\langle 9, 10, 51 \rangle = \langle 9, 10, 6 \times 10 - 9 \rangle$ is not modular

M-semigroups with $n_1 = 9$, $n_2 = 10$

Example

1. $(T1) : \mathrm{S}(21,189,3) = \langle 9,10,21\rangle = \langle 3\cdot 3, 3+7, 3\cdot 7\rangle$

$$9\cdot(3\cdot 3) - 8\cdot(3+7) = 1$$

$$\mathrm{S}(21,189,3) = \mathrm{S}((9\cdot 3-8)\cdot 3, 3\cdot 3\cdot 7) = \mathrm{S}(57,63)$$

2. $(T2) : \mathrm{S}(41,369,5) = \langle 9,10,41\rangle = \langle 9, \frac{9+41}{5}, 41\rangle$

$$2\cdot 41 - 9\cdot 9 = 1 \Rightarrow 10\cdot 41 - 45\cdot 9 = 5$$

$$\mathrm{S}(41,369,5) = \mathrm{S}\left(\frac{41-1}{5}\cdot 10, \frac{41-1}{5}\cdot 9\right) = \mathrm{S}(80,72) = \mathrm{S}(8,72)$$

Conjecture

Let $S=\langle n_1,n_2,\ldots,n_e\rangle$ be a PM-semigroup. Let us suppose that $n_1,n_2,\ldots,n_e$ are arranged according the characterization of PM-semigroups. Let us consider the notation

- $n_3=k_2n_2-n_1=\alpha_3n_2-\beta_3n_1$;
- $n_4=k_3n_3-n_2=k_3(\alpha_3n_2-\beta_3n_1)-n_2=\alpha_4n_2-\beta_4n_1$;
- $n_5=k_4n_4-n_3=k_4(\alpha_4n_2-\beta_4n_1)-(\alpha_3n_2-\beta_3n_1)=\alpha_5n_2-\beta_5n_1$;
- $\ldots$
- $n_e=\alpha_en_2-\beta_en_1$.

Then, $S=\langle n_1,n_2,\ldots,n_e\rangle$ is an M-semigroup if and only if

$$\alpha_e\in D(n_1-1)\cup D(n_1)\cup D(\beta_en_1+1).$$

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Proposition (J.M. Urbano-Blanco, Ph.D. Thesis)

Let m,c,k positive integers such that $\gcd\{m,c\}=1$. Then, $S=\langle m,m+c,\ldots,m+kc\rangle$ is a PM-semigroup. Moreover, S is modular if and only if $m \bmod k\in\{0,1\}$.

1. $m+kc=k(m+c)-(k-1)m$;
2. $k\in D(m-1)\Leftrightarrow m \bmod k=1$;
3. $k\in D(m)\Leftrightarrow m \bmod k=0$;
4. $k\in D((k-1)m)\Leftrightarrow m \bmod k=0$.
5. Moreover, $k\in D(m+1)\Leftrightarrow m \bmod k=k-1$

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