Modular translations and retractions of numerical semigroups

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- $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Definition

- *A* numerical semigroup *is a subset S of $\mathbb{N}$ that is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S$ is finite.*

- $\mathrm{H}(S) = \mathbb{N} \setminus S$ (*gaps*)
- $\mathrm{F}(S) = \max(\mathbb{Z} \setminus S)$ (*Frobenius number*)
  (if $S \neq \mathbb{N}$, then $\mathrm{F}(S) = \max(\mathrm{H}(S))$)
- $\mathrm{g}(S) = \sharp(\mathrm{H}(S))$ (*genus*)
- $\mathrm{m}(S) = \min(S \setminus \{0\})$ (*multiplicity*)

- If $A \subseteq \mathbb{N}$ is a nonempty set,

$$\langle A \rangle = \{\lambda_1 a_1 + \ldots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\},\ a_1, \ldots, a_n \in A,\ \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}.$$

Lemma

- *$\langle A \rangle$ is a numerical semigroup if and only if* $\gcd\{A\} = 1$.

- If $S = \langle A \rangle$, then A is a *system of generators* of S.
- In addition, if no proper subset of A generates S, then A is a *minimal system of generators* of S.

Lemma

- *Every numerical semigroup admits a unique minimal system of generators, which in addition is finite.*

- The cardinality of the minimal system of generators of S is called the *embedding dimension* of S and will be denoted by $\mathrm{e}(S)$.

Example

$$S = \{0,5,7,9,10,12,14,\rightarrow\} =$$

$$\{0,5,7,9,10,12\} \cup \{z \in \mathbb{N} \mid z \geq 14\}$$

- $\mathrm{H}(S) = \{1,2,3,4,6,8,11,13\}$
- $\mathrm{F}(S) = 13$
- $\mathrm{g}(S) = 8$
- $\langle 5,7,9 \rangle$ is the minimal system of generators of S
- $\mathrm{e}(S) = 3$

- Let S be a numerical semigroup, $m \in S \setminus \{0\}$.

Definition

- *The* Apéry set *of m in S is the set* $\mathrm{Ap}(S,m) = \{s \in S \mid s - m \notin S\}$.

Lemma

- $\mathrm{Ap}(S,m) = \{w(0) = 0, w(1), \ldots, w(m-1)\}$, *where $w(i)$ is the least element of S congruent with i modulo m, for all $i \in \{0,1,\ldots,m-1\}$*
- *Moreover,*
  - *$z \in S$ if and only if $z \geq w(z \bmod m)$.*
  - $\mathrm{F}(S) = \max\{\mathrm{Ap}(S,m)\} - m$.
  - $\mathrm{g}(S) = \frac{1}{m}\left(w(0) + w(1) + \cdots + w(m-1)\right) - \frac{m-1}{2}$.

Example

$$S = \{0,5,7,9,10,12,14,\rightarrow\} = \langle 5,7,9 \rangle$$

- $\mathrm{H}(S) = \{1,2,3,4,6,8,11,13\}$
- $\mathrm{F}(S) = 13$
- $\mathrm{g}(S) = 8$
- $\mathrm{Ap}(S,10) = \{0,21,12,23,14,5,16,7,18,9\}$
- $\max\{\mathrm{Ap}(S)\} - 10 = 23 - 10 = 13$
- $\frac{0+21+12+23+14+5+16+7+18+9}{10} - \frac{10-1}{2} = 8$

- If x, y are integers and $y \neq 0$, then we denote by $x \bmod y$ the remainder of the division of x by y.

- Let S be a numerical semigroup, $m \in S \setminus \{0\}$, and $a \in \mathbb{N}$.

Definition

- *Modular translation:* $\mathrm{T}(S, a, m) = \{s + as \bmod m \mid s \in S\}$.
- *Modular retraction:* $\mathrm{R}(S, a, m) = \{s - as \bmod m \mid s \in S\}$.

- We can take $a < m$ without loss of generality.

Some examples with modular translations

$$S = \{0, 5, 7, 9, 10, 12, 14, \to\} = \langle 5, 7, 9 \rangle$$

- $\mathrm{T}(S, 1, 10) = \langle 10, 14, 18, 22, 26 \rangle$ (no numerical semigroup)
- $\mathrm{T}(S, 2, 10) = \langle 5, 11, 17, 18, 24 \rangle$ (numerical semigroup)
- $\mathrm{T}(S, 3, 10) = \langle 8, 10, 22 \rangle$ (no numerical semigroup)
- $\mathrm{T}(S, 4, 10) = \langle 5 \rangle$ (no numerical semigroup)
- $\mathrm{T}(S, 5, 10) = \langle 10, 12, 14, 16, 18 \rangle$ (no numerical semigroup)
- $\mathrm{T}(S, 6, 10) = \langle 5, 9, 13 \rangle$ (numerical semigroup)
- $\mathrm{T}(S, 7, 10) = \langle 10, 12, 16, 18 \rangle$ (no numerical semigroup)
- $\mathrm{T}(S, 8, 10) = \langle 5, 11, 13, 16 \rangle$ (numerical semigroup)
- $\mathrm{T}(S, 9, 10) = \langle 10 \rangle$ (no numerical semigroup)

Observe that $\mathrm{T}(S, a, 10)$ is always a submonoid of $(\mathbb{N}, +)$.

- Let S be a numerical semigroup, $m \in S \setminus \{0\}$, and $a \in \mathbb{N}$.

Proposition

- $\mathrm{T}(S,a,m)$ *is a submonoid of* $(\mathbb{N},+)$.
- $\mathrm{T}(S,a,m)$ *is a numerical semigroup if and only if* $\gcd(a+1,m)=1$.

Problems with modular retractions

$$S = \{0,5,7,9,10,12,14,\rightarrow\} = \langle 5,7,9\rangle$$

- $\mathrm{R}(S,1,10) = \langle 10\rangle$ (no numerical semigroup)
- $\mathrm{R}(S,2,10) : 1 \in, 2 \notin$ (no submonoid of $(\mathbb{N},+)$)
- $\mathrm{R}(S,3,10) : 2 \in, 4 \notin$ (no submonoid of $(\mathbb{N},+)$)
- $\mathrm{R}(S,4,10) : -1 \in$ (no submonoid of $(\mathbb{N},+)$)
- $\mathrm{R}(S,5,10) : 4 \in, 8 \notin$ (no submonoid of $(\mathbb{N},+)$)
- $\mathrm{R}(S,6,10) = \langle 5\rangle$ (no numerical semigroup)
- $\mathrm{R}(S,7,10) : -2 \in$ (no submonoid of $(\mathbb{N},+)$)
- $\mathrm{R}(S,8,10) : 1 \in, 2 \notin$ (no submonoid of $(\mathbb{N},+)$)
- $\mathrm{R}(S,9,10) = \langle 4,10\rangle$ (no numerical semigroup)

A better example with modular retractions

$$S = \{0,5,7,9,10,12,14,\rightarrow\} = \langle 5,7,9\rangle$$

- $\mathrm{R}(S,1,9) = \langle 9\rangle$ (no numerical semigroup)
- $\mathrm{R}(S,2,9) = \langle 2,9\rangle$ (numerical semigroup)
- $\mathrm{R}(S,3,9) : -1 \in$ (no submonoid of $(\mathbb{N},+)$)
- $\mathrm{R}(S,4,9) = \langle 3\rangle$ (no numerical semigroup)
- $\mathrm{R}(S,5,9) : -2 \in$ (no submonoid of $(\mathbb{N},+)$)
- $\mathrm{R}(S,6,9) : 1,2 \in, 3 \notin$ (no submonoid of $(\mathbb{N},+)$)
- $\mathrm{R}(S,7,9) : -3 \in$ (no submonoid of $(\mathbb{N},+)$)
- $\mathrm{R}(S,8,9) : 1,2 \in, 3 \notin$ (no submonoid of $(\mathbb{N},+)$)

- Let S be a numerical semigroup, $m \in S \setminus \{0\}$, and $a \in \mathbb{N}$.

Proposition

$\mathrm{R}(S,a,m)$ *is a numerical semigroup if and only if*

- $\gcd(a-1,m) = 1$;
- *if* $s_1, s_2 \in S$ *and* $as_1 \bmod m + as_2 \bmod m \geq m$, *then* $s_1 + s_2 - m \in S$.

- Let $\mathrm{Ap}(S,m) = \{w(0) = 0, w(1), \ldots, w(m-1)\}$.

Proposition

$\mathrm{R}(S,a,m)$ *is a numerical semigroup if and only if*

- $\gcd(a-1,m) = 1$;
- *if* $w(i) + w(j) \in \mathrm{Ap}(S,m)$, *then* $ai \bmod m + aj \bmod m < m$.

- Let S be a numerical semigroup, $m \in S \setminus \{0\}$, $a \in \mathbb{N}$ such that $\gcd(a+1, m) = 1$, and $d = \gcd(a, m)$.
- Let $\mathrm{Ap}(S, m) = \{w(0) = 0, w(1), \ldots, w(m-1)\}$.

Theorem

- $\mathrm{Ap}(\mathrm{T}(S, a, m), m) = \{w(i) + ai \bmod m \mid i \in \{0, 1, \ldots, m-1\}\}$.

Proposition

- $\mathrm{g}(\mathrm{T}(S, a, m)) = \mathrm{g}(S) + \frac{m-d}{2}$.
- $\mathrm{F}(\mathrm{T}(S, a, m)) = \max\{w(i) + ai \bmod m \mid i \in \{0, 1, \ldots, m-1\}\} - m$. *Moreover,*
  - $\mathrm{F}(\mathrm{T}(S, a, m)) \geq \mathrm{F}(S) + a\mathrm{F}(S) \bmod m$.
  - $\mathrm{F}(\mathrm{T}(S, a, m)) \leq \mathrm{F}(S) + m - d$.
  - $\mathrm{F}(\mathrm{T}(S, a, m)) = \mathrm{F}(S) + m - d$ *if and only if* $a\mathrm{F}(S) \bmod m = m - d$.

Example

$$S = \{0,5,7,9,10,12,14,\rightarrow\} = \langle 5,7,9\rangle,\ \mathrm{F}(S) = 13,\ \mathrm{g}(S) = 8$$

- $\mathrm{T}(S,2,10) = \langle 5,11,17,18,24\rangle, \quad \mathrm{g}(\mathrm{T}(S,2,10)) = 12^{(*)}$

  $$13 + 2\times 13 \bmod 10 = 19 = \mathrm{F}(\mathrm{T}(S,2,10)) < 21 = 13 + 10 - 2$$

- $\mathrm{T}(S,6,10) = \langle 5,9,13\rangle, \quad \mathrm{g}(\mathrm{T}(S,6,10)) = 12^{(*)}$

  $$13 + 6\times 13 \bmod 10 = 21 = \mathrm{F}(\mathrm{T}(S,6,10)) = 13 + 10 - 2$$

- $\mathrm{T}(S,8,10) = \langle 5,11,13,16\rangle, \quad \mathrm{g}(\mathrm{T}(S,8,10)) = 12^{(*)}$

  $$13 + 8\times 13 \bmod 10 = 17 < 19 = \mathrm{F}(\mathrm{T}(S,8,10)) < 21 = 13 + 10 - 2$$

$^{(*)}\, 8 + \frac{10-2}{2} = 12$

- Let S be a numerical semigroup, $m \in S \setminus \{0\}$, $a \in \mathbb{N}$ such that $\gcd(a-1, m) = 1$, and $d = \gcd(a, m)$.
- Let $\mathrm{Ap}(S, m) = \{w(0) = 0, w(1), \ldots, w(m-1)\}$.
- Let $\mathrm{R}(S, a, m)$ be a numerical semigroup.

Theorem

- $\mathrm{Ap}(\mathrm{R}(S, a, m), m) = \{w(i) - ai \bmod m \mid i \in \{0, 1, \ldots, m-1\}\}$.

Proposition

- $\mathrm{g}(\mathrm{R}(S, a, m)) = \mathrm{g}(S) - \frac{m-d}{2}$.
- $\mathrm{F}(\mathrm{R}(S, a, m)) = \max\{w(i) - ai \bmod m \mid i \in \{0, 1, \ldots, m-1\}\} - m$. *Moreover,*
  - $\mathrm{F}(\mathrm{R}(S, a, m)) \geq \mathrm{F}(S) - a\mathrm{F}(S) \bmod m$.
  - $\mathrm{F}(\mathrm{R}(S, a, m)) \leq \mathrm{F}(S)$.
  - $\mathrm{F}(\mathrm{R}(S, a, m)) = \mathrm{F}(S)$ *if and only if* $a\mathrm{F}(S) \bmod m = 0$.

Example

$$S = \{0, 8, 12, 15, \rightarrow\} = \langle 8, 15, 17, 18, 19, 12, 21, 22\rangle, \ \mathrm{F}(S) = 14, \ \mathrm{g}(S) = 12$$

- $\mathrm{R}(S,2,8) = \langle 8, 9, 12, 13, 14, 15, 19\rangle, \quad \mathrm{g}(\mathrm{R}(S,2,8)) = 9 = 12 - \frac{8-2}{2}$

  $$14 - 2 \times 14 \bmod 8 = 10 < 11 = \mathrm{F}(\mathrm{R}(S,2,8)) < 14$$

- $\mathrm{R}(S,4,8) = \langle 8, 11, 12, 13, 15, 17, 18\rangle, \quad \mathrm{g}(\mathrm{R}(S,4,8)) = 10 = 12 - \frac{8-4}{2}$

  $$14 - 4 \times 14 \bmod 8 = 14 = \mathrm{F}(\mathrm{R}(S,4,8)) = 14$$

- $\mathrm{R}(S,6,8) = \langle 8, 11, 12, 13, 14, 15, 17, 18\rangle, \quad \mathrm{g}(\mathrm{R}(S,4,5)) = 9 = 12 - \frac{8-2}{2}$

  $$14 - 6 \times 14 \bmod 8 = 10 = \mathrm{F}(\mathrm{T}(S,8,10)) < 14$$

- Let p, q be nonnegative integers with $q \neq 0$.

Lemma

- $\mathrm{M}(p,q) = \{x \in \mathbb{N} \mid px \bmod q \leq x\}$ *is a numerical semigroup.*

- Let S be a numerical semigroup, $m \in S \setminus \{0\}$, and $a \in \mathbb{N}$ such that $\gcd(a-1, m) = 1$.

Proposition (Families of retractable numerical semigroups)

- $\mathrm{R}(S,a,m) = \mathbb{N}$ *if and only if* $S = \mathrm{M}(a,m)$.
- *If* S *is a numerical semigroup with maximal embedding dimension and multiplicity* m*, then* $\mathrm{R}(S,a,m)$ *is a numerical semigroup.*

- Let S be a numerical semigroup, $m \in S \setminus \{0\}$, $a \in \mathbb{N}$ such that $\gcd(a-1, m) = 1$, and $\gcd(a, m) = d$.
- Let $\langle n_1, n_2, \ldots, n_p \rangle$ be the minimal system of generators of S.

Proposition

- $\langle n_1 + an_1 \bmod m, n_2 + an_2 \bmod m, \ldots, n_p + an_p \bmod m \rangle$ *is a subset of the minimal system of generators of* $\mathrm{T}(S, a, m)$.
- $\mathrm{e}(S) \leq \mathrm{e}(\mathrm{T}(S, a, m))$.

- Let $\mathrm{R}(S, a, m)$ be a numerical semigroup.

Proposition

- $\langle n_1 - an_1 \bmod m, n_2 - an_2 \bmod m, \ldots, n_p - an_p \bmod m \rangle$ *is a system of generators of* $\mathrm{R}(S, a, m)$.
- $\mathrm{e}(S) \geq \mathrm{e}(\mathrm{R}(S, a, m))$.

Example

$$S = \langle 5, 7, 9 \rangle$$

- $\mathrm{T}(S, 2, 10) = \langle 5, 11, 17, 18, 24 \rangle$
- $\mathrm{T}(S, 6, 10) = \langle 5, 9, 13 \rangle$
- $\mathrm{T}(S, 8, 10) = \langle 5, 11, 13, 16 \rangle$

$$S = \langle 5, 6, 7, 8, 9 \rangle$$

- $\mathrm{R}(S, 2, 5) = \mathrm{R}(S, 3, 5) = \langle 3, 4, 5 \rangle$
- $\mathrm{R}(S, 4, 5) = \langle 2, 5 \rangle$

$$S = \langle 5, 11, 12, 13, 9 \rangle$$

- $\mathrm{R}(S, 2, 5) = \langle 5, 6, 8, 9 \rangle$
- $\mathrm{R}(S, 3, 5) = \mathrm{R}(S, 4, 5) = \langle 5, 7, 8, 9, 11 \rangle$

- Let S be a numerical semigroup, $m \in S \setminus \{0\}$.

Lemma

- *Let* $a, u \in \mathbb{N}$ *such that* $\gcd(a+1, m) = 1$ *and* $(1+a)u \equiv 1 \pmod{m}$. *Then* $S = \mathrm{R}(\mathrm{T}(S, a, m), au, m)$.

Lemma

- *Let* $a, u \in \mathbb{N}$ *such that* $\gcd(a-1, m) = 1$ *and* $(1-a)u \equiv 1 \pmod{m}$. *Then* $S = \{x + aux \bmod m \mid x \in \mathrm{R}(S, a, m)\}$.
- *Moreover, if* $\mathrm{R}(S, a, m)$ *is a numerical semigroup, then* $\mathrm{T}(\mathrm{R}(S, a, m), au, m) = S$.

Theorem

- *Let* S, B *be two numerical semigroups,* $m \in (S \cap B) \setminus \{0\}$*, and* $a, b \in \mathbb{N}$ *such that* $(1+b)(1-a) \equiv 1 \pmod{m}$*. Then* $\mathrm{R}(S,a,m) = B$ *if and only if* $S = \mathrm{T}(B,b,m)$*.*

Example

$$S = \langle 5, 11, 12, 13, 9 \rangle$$

- $\mathrm{R}(S,2,5) = \langle 5,6,8,9 \rangle = B$; $S = \mathrm{T}(B,3,5)$
  $(1+3)(1-2) \equiv 1 \pmod{5}$
- $\mathrm{R}(S,3,5) = \langle 5,7,8,9,11 \rangle = B$; $S = \mathrm{T}(B,1,5)$
  $(1+1)(1-3) \equiv 1 \pmod{5}$
- $\mathrm{R}(S,4,5) = \langle 5,7,8,9,11 \rangle = B$; $S = \mathrm{T}(B,2,5)$
  $(1+2)(1-4) \equiv 1 \pmod{5}$

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