The numerical semigroup of the integers which are bounded by a submonoid of $\mathbb{N}^2$ *

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Abstract. Let M be a submonoid of $(\mathbb{N}^2,+)$ such that

$$\mathrm{A}(M)=\{n\in\mathbb{N}\mid a<n<b \text{ for some } (a,b)\in M\}$$

is a non-empty set. Then $\mathrm{A}(M)\cup\{0\}$ is a numerical semigroup. We will show that a numerical semigroup S can be obtained in this way if and only if $\{a+b-1,a+b+1\}\subseteq S$ for all $a,b\in S\setminus\{0\}$. We will see that such numerical semigroups form a Frobenius variety and we will study it.

Keywords: submonoids, numerical semigroups, non-homogeneous patterns, Frobenius varieties, trees.

1 Introduction

Let $\mathbb{N}$ be the set of nonnegative integers and let M be a submonoid of $(\mathbb{N}^2,+)$ (that is, a subset of $\mathbb{N}^2$ that is closed for the addition and that contains the zero element $(0,0)$). We will say that a positive integer n is *bounded by* M if there exists $(a,b)\in M$ such that $a<n<b$. We will denote by $\mathrm{A}(M)=\{n\in\mathbb{N}\mid n \text{ is bounded by } M\}$.

A *numerical semigroup* is a submonoid S of $(\mathbb{N},+)$ such that $\gcd(S)=1$.

It is easy to check that, if M is a submonoid of $(\mathbb{N}^2,+)$ such that $\mathrm{A}(M)$ is not empty, then $\mathrm{A}(M)\cup\{0\}$ is a numerical semigroup. This fact allows us to give the concept of *numerical $\mathcal{A}$-semigroup*. Indeed, a numerical semigroup S is a *numerical $\mathcal{A}$-semigroup* if there exists M, submonoid of $(\mathbb{N}^2,+)$, such that $S=\mathrm{A}(M)\cup\{0\}$.

The purpose of this work is to study this family of numerical semigroups. First of all, we will show a characterization of them, via the concept of non-homogeneous pattern (see [1]), and how we can obtain all of them.

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We have that, if S is a numerical semigroup, then $\mathbb{N} \setminus S$ is finite. This fact allows us to define the *Frobenius number* of S, denoted by $\mathrm{F}(S)$, as the greatest integer that does not belong to S (see [4]).

A *Frobenius variety* (see [6]) is a non-empty family $\mathcal{V}$ of numerical semigroups that fulfills the following conditions,

1. if $S, T \in \mathcal{V}$, then $S \cap T \in \mathcal{V}$;
2. if $S \in \mathcal{V}$ and $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{V}$.

If we denote by $\mathcal{A} = \{S \mid S \text{ is a numerical } \mathcal{A}\text{-semigroup}\}$, then we will show that $\mathcal{A}$ is a Frobenius variety. As a consequence of this result, we will arrange the elements of $\mathcal{A}$ in a tree $\mathrm{G}(\mathcal{A})$ with root $\mathbb{N}$.

Finally, we will see that, if X is a non-empty set of positive integers, then there exists the smallest numerical $\mathcal{A}$-semigroup, $\mathcal{A}(X)$, that contains X. Moreover, we will design an algorithm to compute $\mathcal{A}(X)$ starting from X.

We end this introduction pointing out that this work is a short version of the manuscript [5] that is under review process.

2 Numerical $\mathcal{A}$-semigroups

Let us remember that, if M is a submonoid of $(\mathbb{N}^2, +)$, then

$$\mathrm{A}(M) = \{n \in \mathbb{N} \mid a < n < b \text{ for some } (a,b) \in \mathbb{N}^2\}.$$

Lemma 1. *Let M be a submonoid of $(\mathbb{N}^2, +)$. If $x, y \in \mathrm{A}(M)$, then we have that $\{x+y-1, x+y, x+y+1\} \subseteq \mathrm{A}(M)$.*

Remark 1. Let us observe that there exist numerical semigroups that are not numerical $\mathcal{A}$-semigroups. For example, the numerical semigroup $S = \{0, 5, 7, 9, 10, 12, 14, \rightarrow\}$ (where the symbol $\rightarrow$ means that every integer greater than 14 belongs to S) is not a numerical $\mathcal{A}$-semigroup because $5+7+1 \notin S$.

Let X be a non-empty subset of a commutative monoid $(\mathfrak{M}, +)$. The monoid generated by X, denoted by $\langle X \rangle$, is the smallest (with respect to the set inclusion) submonoid of $(\mathfrak{M}, +)$ containing X. It is known (see [7]) that $\langle X \rangle = \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid n \in \mathbb{N} \setminus \{0\},\ x_1, \ldots, x_n \in X,\ \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$.

If $\mathfrak{M} = \langle X \rangle$, then we say that X is a *system of generators* of $\mathfrak{M}$ (or that $\mathfrak{M}$ is generated by X). It is well known (see [8]) that the submonoids of $(\mathbb{N}, +)$ (in particular, the numerical semigroups) are finitely generated.

Lemma 2. *Let S be the numerical semigroup generated by a set of positive integers $\{n_1, \ldots, n_p\}$. Let us assume that $\{x+y-1, x+y+1\} \subseteq S$, for every $x, y \in S \setminus \{0\}$. If $\lambda_1, \ldots, \lambda_p, x \in \mathbb{N}$ and $\lambda_1(n_1-1) + \cdots + \lambda_p(n_p-1) < x < \lambda_1(n_1+1) + \cdots + \lambda_p(n_p+1)$, then $x \in S$.*

In the next result (that is a consequence of Lemmas 1 and 2) we give a characterization of numerical $\mathcal{A}$-semigroups.

Theorem 1. *Let S be a numerical semigroup. The following conditions are equivalent.*

1. *S is a numerical $\mathcal{A}$-semigroup.*
2. *If $x, y \in S \setminus \{0\}$, then $\{x+y-1, x+y+1\} \subseteq S$.*

The next result guarantees us that, to study numerical $\mathcal{A}$-semigroups, we can focus in finitely generated submonoids of $(\mathbb{N}^2, +)$.

Corollary 1. *Let S be a numerical semigroup. The following conditions are equivalent.*

1. *S is a numerical $\mathcal{A}$-semigroup.*
2. *$S = \mathrm{A}(M) \cup \{0\}$ for some finitely generated submonoid M of $(\mathbb{N}^2, +)$.*
3. *There exist $a_1, b_1, \ldots, a_p, b_p \in \mathbb{N}$ such that $S = \{n \in \mathbb{N} \mid a_1x_1 + \cdots + a_px_p < n < b_1x_1 + \cdots + b_px_p$ for some $x_1, \ldots, x_p \in \mathbb{N}\} \cup \{0\}$.*

From Item 3 of Corollary 1, we can see a numerical $\mathcal{A}$-semigroup as a set that contains the integers n such that the system of inequalities

$$a_1x_1 + \cdots + a_px_p < n < b_1x_1 + \cdots + b_px_p$$

has solution in $\mathbb{N}^p$ (where $\{a_1, b_1, \ldots, a_p, b_p\} \subseteq \mathbb{N}$ is a fixed set).

Let $(\mathfrak{M}, +)$ be a commutative monoid and let X be a system of generators of $\mathfrak{M}$. If $\mathfrak{M} \neq \langle Y \rangle$ for all $Y \subset X$ (with $Y \neq X$), then we say that X is a *minimal system of generators* of $\mathfrak{M}$.

The following result is consequence of [8, Lemma 2.3, Corollary 2.8].

Lemma 3. *Let M be a submonoid of $(\mathbb{N}, +)$. Then the minimal system of generators of M is $X = (M \setminus \{0\}) \setminus ((M \setminus \{0\}) + (M \setminus \{0\}))$. In addition, X is finite and is contained in every system of generators of M.*

Let S be a numerical semigroup and let $\{n_1, \ldots, n_p\} \subseteq \mathbb{N} \setminus \{0\}$ be a system of generators of S. If $s \in S$, then it is defined the *order* of s in S by (see [2]) $\mathrm{ord}(s; S) = \max\{a_1 + \cdots + a_p \mid a_1n_1 + \cdots + a_pn_p = s, \text{ with } a_1, \ldots, a_p \in \mathbb{N}\}$. If no ambiguity is possible, then we write $\mathrm{ord}(s)$.

Remark 2. From Lemma 3, the definition of $\mathrm{ord}(s; S)$ is independent of the considered system of generators of S, that is, only depends on s and S. Thus, we can take the minimal system of generators of S in order to define $\mathrm{ord}(s; S)$.

Lemma 4. *Let S be the numerical semigroup with minimal system of generators $\{n_1, \ldots, n_p\}$ and let $s \in S$.*

1. *If $i \in \{1, \ldots, p\}$ and $s - n_i \in S$, then $\mathrm{ord}(s - n_i) \leq \mathrm{ord}(s) - 1$.*
2. *If $s = a_1n_1 + \cdots + a_pn_p$, with $\mathrm{ord}(s) = a_1 + \cdots + a_p$ and $a_i \neq 0$, then $\mathrm{ord}(s - n_i) = \mathrm{ord}(s) - 1$.*

As a consequence of Theorem 1 and Lemma 4, we have another characterization of numerical $\mathcal{A}$-semigroups.

Proposition 1. *Let S be a numerical semigroup with minimal system of generators given by $\{n_1,\dots,n_p\}$. The following conditions are equivalent.*

1. *S is a numerical $\mathcal{A}$-semigroup.*
2. *If $i,j \in \{1,\dots,p\}$, then $\{n_i+n_j-1, n_i+n_j+1\} \subseteq S$.*
3. *If $s \in S \setminus \{0,n_1,\dots,n_p\}$, then $\{s-1,s+1\} \subseteq S$.*
4. *If $s \in S \setminus \{0\}$, then $s+z \in S$ for all $z \in \mathbb{Z}$ such that $|z| < \mathrm{ord}(s)$.*

Remark 3. Item 2 of the above proposition allows us to decide, faster than with Theorem 1, if a numerical semigroup is a numerical $\mathcal{A}$-semigroup.

Remark 4. Let us remember that the gaps of a numerical semigroup S are the elements of the set $\mathbb{N} \setminus S$. As a consequence of Item 3 of Proposition 1, we have that a numerical $\mathcal{A}$-semigroups S can be characterized as a numerical semigroup satisfying that, if $s \in S$ is a maximum or a minimum of an interval of non-gaps, then s is a minimal generator of S or $s=0$.

Example 1. Let S be the numerical semigroup minimally generated by $\{3,5,7\}$. Since $\{3+3\}+\{-1,1\}$, $\{3+5\}+\{-1,1\}$, $\{3+7\}+\{-1,1\}$, $\{5+5\}+\{-1,1\}$, $\{5+7\}+\{-1,1\}$, and $\{7+7\}+\{-1,1\}$ are subsets of S, then we can assert that S is a numerical $\mathcal{A}$-semigroup (remember Remark 3).

On the other hand, we have that $S=\{0,3,5,6,7,\rightarrow\}$ and, thereby, its intervals of non-gaps are $\{0\}$, $\{3\}$ and $\{5,\rightarrow\}$. Inasmuch as the maximum and the minimum of such a sets are zero or a minimal generator, from Remark 4, we have another way to state that S is a numerical $\mathcal{A}$-semigroup.

Example 2. Let T be the numerical semigroup minimally generated by $\{5,7,9\}$. Then $T=\{0,5,7,9,10,12,14,\rightarrow\}$ and its intervals of non-gaps are $\{0\}$, $\{5\}$, $\{7\}$, $\{9,10\}$, $\{12\}$, and $\{14,\rightarrow\}$. Since $\max\{9,10\}=10$ is different from zero and it is not a minimal generator of T, then T is not a numerical $\mathcal{A}$-semigroup.

3 The Frobenius variety of the numerical $\mathcal{A}$-semigroups

Lemma 5. *Let S, T be numerical semigroups.*

1. *$S \cap T$ is a numerical semigroup.*
2. *If $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\}$ is a numerical semigroup.*

From Lemma 5 and Theorem 1, we have the following result.

Proposition 2. *The set $\mathcal{A}=\{S \mid S$ is a numerical $\mathcal{A}$-semigroup$\}$ is a Frobenius variety.*

A *graph* G is a pair (V, E), where V is a non-empty set and E is a subset of $\{(v, w) \in V \times V \mid v \neq w\}$. The elements of V are called *vertices* of G and the elements of E are called *edges* of G. A *path (of length n)* connecting the vertices x and y of G is a sequence of different edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ such that $v_0 = x$ and $v_n = y$.

We say that a graph G is a *tree* if there exists a vertex v^* (the *root* of G) such that, for every other vertex x of G, there exists a unique path connecting x and v^*. If (x, y) is an edge of the tree, then we say that x is a *child* of y.

We define the graph $\mathrm{G}(\mathcal{A})$ in the following way,

- $\mathcal{A}$ is the set of vertices of $\mathrm{G}(\mathcal{A})$;
- $(S, S') \in \mathcal{A} \times \mathcal{A}$ is an edge of $\mathrm{G}(\mathcal{A})$ if $S' = S \cup \{\mathrm{F}(S)\}$.

If S is a numerical semigroup, then we will denote by $\mathrm{msg}(S)$ the minimal system of generators of S. It is easy to show (see [8, Exercise 2.1]) that, if S is a numerical semigroup and $x \in S$, then $S \setminus \{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(S)$.

The next result is consequence of [6, Proposition 24, Theorem 27].

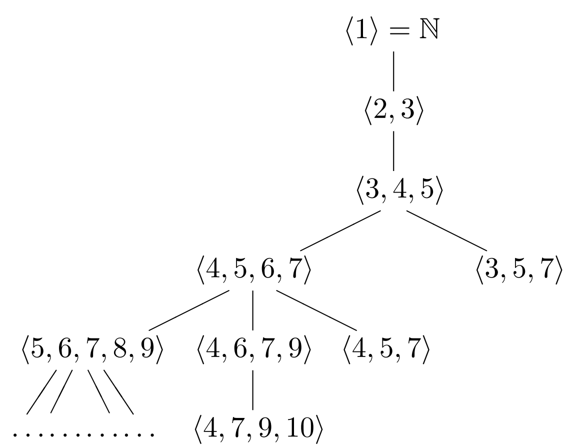
Theorem 2. $\mathrm{G}(\mathcal{A})$ *is a tree with root* $\mathbb{N}$. *Moreover, the set of children of a vertex* $S \in \mathcal{A}$ *is* $\{S \setminus \{x\} \mid x \in \mathrm{msg}(S),\ x > \mathrm{F}(S),$ *and* $S \setminus \{x\} \in \mathcal{A}\}$.

Now, we will characterize the elements $x \in \mathrm{msg}(S)$ such that $S \setminus \{x\} \in \mathcal{A}$.

Proposition 3. *Let S be a numerical $\mathcal{A}$-semigroup such that $S \neq \mathbb{N}$, and let $x \in \mathrm{msg}(S)$. Then $S \setminus \{x\}$ is a numerical $\mathcal{A}$-semigroup if and only if* $\{x-1, x+1\} \subseteq \{0\} \cup (\mathbb{N} \setminus S) \cup \mathrm{msg}(S)$.

Corollary 2. *Let S be a numerical $\mathcal{A}$-semigroup such that $S \neq \mathbb{N}$, and let x be a minimal generator of S greater than $\mathrm{F}(S)$. Then $S \setminus \{x\}$ is a numerical $\mathcal{A}$-semigroup if and only if* $\{x-1, x+1\} \subseteq \mathrm{msg}(S) \cup \{\mathrm{F}(S)\}$.

By applying Theorem 2 and Corollary 2 (see Example 3) we can build recursively the tree $\mathrm{G}(\mathcal{A})$ such as it is shown in the following figure.



Example 3. Let S be the numerical semigroup with minimal system of generators $\{4,6,7,9\}$. Then $S=\{0,4,6,\rightarrow\}$ and, therefore, $\mathrm{F}(S)=5$. From Proposition 1, we deduce that S is a numerical $\mathcal{A}$-semigroup. By applying Theorem 2 and Corollary 2, we get that S has a unique child in $\mathrm{G}(\mathcal{A})$. Namely, $S\setminus\{6\}=\langle 4,7,9,10\rangle$.

Remark 5. Let us observe that, if S' is a child of S in $\mathrm{G}(\mathcal{A})$, then $\mathrm{F}(S')>\mathrm{F}(S)$ and $\mathrm{g}(S')=\mathrm{g}(S)+1$. Therefore, when we go on along the branches of the tree $\mathrm{G}(\mathcal{A})$, we get numerical semigroups with greater Frobenius number and genus. Thus, we can use this construction in order to obtain all the numerical $\mathcal{A}$-semigroups with a given Frobenius number or genus.

4 The smallest numerical $\mathcal{A}$-semigroup that contains a given set of positive integers

Since $\mathcal{A}$ is a Frobenius variety, we have that the finite intersection of numerical $\mathcal{A}$-semigroups is a numerical $\mathcal{A}$-semigroup. However, the infinite intersection of numerical $\mathcal{A}$-semigroups is not always a numerical $\mathcal{A}$-semigroup. On the other hand, it is clear that the (finite or infinite) intersection of numerical semigroups is always a submonoid of $(\mathbb{N},+)$.

If M is a submonoid of $(\mathbb{N},+)$, then we will say that M is an $\mathcal{A}$-monoid if it can be expressed like the intersection of numerical $\mathcal{A}$-semigroups. Being that the intersection of $\mathcal{A}$-monoids is an $\mathcal{A}$-monoid, we can give the following definition.

Definition 1. *Let X be a subset of* $\mathbb{N}$. *The* $\mathcal{A}$-monoid generated by X *(denoted by $\mathcal{A}(X)$) is the intersection of all $\mathcal{A}$-monoids containing X.*

Let us observe that $\mathcal{A}(X)$ is the smallest $\mathcal{A}$-monoid containing X.

Proposition 4. *If $X\subseteq\mathbb{N}$, then $\mathcal{A}(X)$ is the intersection of all numerical $\mathcal{A}$-semigroups that contain X. Moreover, if $X\neq\emptyset$ and $X\subseteq\mathbb{N}\setminus\{0\}$, then $\mathcal{A}(X)$ is a numerical $\mathcal{A}$-semigroup.*

As a consequence of Proposition 4, we have that M is an $\mathcal{A}$-monoid if and only if M is a numerical $\mathcal{A}$-semigroup or $M=\{0\}$. Moreover, from Lemma 3 and Proposition 4, we can prove the next result.

Theorem 3. $\mathcal{A}=\{\mathcal{A}(X)\mid X$ *is a non-empty finite subset of* $\mathbb{N}\setminus\{0\}\}$.

If M is an $\mathcal{A}$-monoid and X is a subset of $\mathbb{N}$ such that $M=\mathcal{A}(X)$, then we will say that X is an *$\mathcal{A}$-system of generators* of M. In addition, if $M\neq\mathcal{A}(Y)$ for all $Y\subset X$ (with $Y\neq X$), then we will say that X is a *minimal $\mathcal{A}$-system of generators* of M. From [6, Corollary 19], we have the following result.

Proposition 5. *Every $\mathcal{A}$-monoid has a unique minimal $\mathcal{A}$-system of generators, which in addition is finite.*

The next result follows from [6, Proposition 24].

Proposition 6. *Let M be an $\mathcal{A}$-monoid and let $x \in M$. Then $M \setminus \{x\}$ is an $\mathcal{A}$-monoid if and only if x is a minimal $\mathcal{A}$-system generator of M.*

Corollary 3. *Let X be a non-empty subset of $\mathbb{N} \setminus \{0\}$. Then the set given by $\{x \in X \mid \mathcal{A}(X) \setminus \{x\}$ is a numerical $\mathcal{A}$-semigroup$\}$ is the minimal $\mathcal{A}$-system of generators of $\mathcal{A}(X)$.*

Example 4. Beginning in Example 3, we know that $S = \langle 4, 6, 7, 9\rangle$ is a numerical $\mathcal{A}$-semigroup. By applying Proposition 3, we easily deduce that

$$\{x \in \{4, 6, 7, 9\} \mid S \setminus \{x\} \text{ is a numerical } \mathcal{A}\text{-semigroup}\} = \{4, 6\}.$$

Therefore, $S = \mathcal{A}(\{4, 6\})$ and $\{4, 6\}$ is its minimal $\mathcal{A}$-system of generators.

Let $x_1, \dots, x_t$ be positive integers. We will denote by $S(x_1, \dots, x_t)$ the set $\{a_1x_1 + \cdots + a_tx_t + z \mid a_1, \dots, a_t \in \mathbb{N},\ z \in \mathbb{Z}, \text{ and } |z| < a_1 + \cdots + a_t\} \cup \{0\}$.

Theorem 4. *If $x_1, \dots, x_t$ are positive integers, then $S(x_1, \dots, x_t)$ is the smallest numerical $\mathcal{A}$-semigroup that contains the set $\{x_1, \dots, x_t\}$.*

Corollary 4. *If m is a positive integer, then*

$$\mathcal{A}(\{m\}) = \{km + z \mid k \in \mathbb{N} \setminus \{0\} \text{ and } z \in \{-(k-1), \dots, k-1\}\} \cup \{0\}.$$

Example 5. The smallest numerical $\mathcal{A}$-semigroup containing $m = 10$ is the set $\mathcal{A}(\{10\}) = \{k \cdot 10 + z \mid k \in \mathbb{N} \setminus \{0\} \text{ and } z \in \{-(k-1), \dots, k-1\}\} \cup \{0\} = \{0, 10, 19, 20, 21, 28, 29, 30, 31, 32, 37, 38, 39, 40, 41, 42, 43, 46, \rightarrow\}$. By using minimal generators, $\mathcal{A}(\{10\}) = \langle 10, 19, 21, 28, 32, 37, 43, 46, 54, 55\rangle$.

Let us observe that, in Theorem 4, are described the elements of the smallest numerical $\mathcal{A}$-semigroup containing a given set of positive integers. However, in order to compute such a numerical $\mathcal{A}$-semigroup, we propose the following algorithm that is justified by Proposition 1.

Algorithm 1. *INPUT: A finite set X of positive integers.*

OUTPUT: The minimal system of generators of $\mathcal{A}(X)$.

(1) $Y = \operatorname{msg}(\langle X\rangle)$.

(2) $Z = Y \cup \left(\bigcup_{a,b \in Y} \{a + b - 1, a + b + 1\}\right)$.

(3) If $\operatorname{msg}(\langle Z\rangle) = Y$, *then return* Y.

(4) Set $Y = \operatorname{msg}(\langle Z\rangle)$ *and go to (2).*

Example 6. We are going to compute $\mathcal{A}(\{5,7\})$ applying Algorithm 1.

- $Y = \{5,7\}$.
- $Z = \{5,7,9,11,13,15\}$.
- $\mathrm{msg}(\langle Z \rangle) = \{5,7,9,11,13\}$.
- $Y = \{5,7,9,11,13\}$.
- $Z = \{5,7,9,11,13,15,17,19,21,23,25,27\}$.
- $\mathrm{msg}(\langle Z \rangle) = \{5,7,9,11,13\}$.
- $\mathcal{A}(\{5,7\}) = \langle 5,7,9,11,13 \rangle$.

The most complex process in Algorithm 1 is the computation of $\mathrm{msg}(\langle Z \rangle)$, that is, the computation of the minimal system of generators of a numerical semigroup S starting from a system of generators of it. For this purpose, we can use the `GAP` package called `numericalsgps` (see [3]).

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