The numerical semigroup of the integers which are bounded by a submonoid of \( \mathbb{N}^2 \)

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3. Both of the authors are supported by FQM-143 (Junta de Andalucía), MTM2010-15959 (MCIINN, Spain), and FEDER funds.

The second author is also partially supported by Junta de Andalucía/FEDER Grant Number FQM-5849.

Preliminaries

Let \( H \) be the set of nonnegative integers. A submonoid of \((H, +)\) is a subset \( S \) of \( H \) that is closed for the addition and that contains the zero element. A numerical semigroup is a submonoid \( S \) of \((\mathbb{N}, +)\) such that \( 1 \in S \) is finite.

Definition 1. Let \( S \) be a numerical semigroup. The Frobenius number of \( S \) is \( \text{F}(S) = \text{max}(\mathbb{N} \setminus S) \) and the genus of \( S \) is \( g(S) = |\mathbb{N} \setminus S| \).

Definition 2. A Frobenius variety is a non-empty family \( \mathcal{V} \) of numerical semigroups that fulfills the following conditions.

1. If \( S, T \in \mathcal{V} \), then \( S \cap T \in \mathcal{V} \).
2. If \( S \in \mathcal{V} \) and \( s \in S \), then \( S \cup \{s\} \in \mathcal{V} \).

Example 3. Let \( S \) be the set formed by all numerical semigroups. Then \( S \) is a Frobenius variety.

Definition 4. A graph \( G(\mathcal{V}, E) \), where \( V \) is a non-empty set and \( E \subseteq \{v,w\} \in V \times V \) \( v \neq w \). The elements of \( V \) are called vertices of \( G \) and the elements of \( E \) are called edges of \( G \). A path of length \( n \) connecting the vertices \( x \) and \( y \) of \( G \) is a sequence of different edges of the form \((x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)\) such that \( x_0 = x \) and \( x_n = y \).

Definition 5. We say that a graph \( G \) is a tree if there exists a vertex \( v \) (the root of \( G \)) such that, for every other vertex \( x \) of \( G \), there exists a unique path connecting \( x \) and \( v \). If \( (x, y) \) is an edge of the tree, then we say that \( x \) is a child of \( y \).

Example 6. We can associate to \( S \) a tree in which the vertices are the elements of \( S \), \((T, S)\) is an edge if \( S = T \cup \{F(T)\} \) and \( H \) is the root. In addition, if \( S \) is a numerical semigroup, then the unique path connecting \( x \in S \) with \( H \) is given by the set \( C(S) = \{m \mid m \in S \cup \{0\} \} \).

Definition 7. A numerical semigroup \( S \) is a numerical \( A \)-semigroup if there exists \( M \), such that \( |x+y| = n \leq M \) if \( |x|, |y| \leq M \) for all \( x, y \in S \).

Theorem 9. A numerical semigroup \( S \) is a numerical \( A \)-semigroup if and only if \( \left\{ x+y \mid x, y \in S \right\} \subseteq \mathbb{Z} \), for all \( x, y \in S \).

Corollary 10. Let \( S \) be a numerical semigroup. The following conditions are equivalent.

1. \( S \) is a numerical \( A \)-semigroup.
2. \( S = \text{A}(S) \cup \{0\} \) for some finitely generated subsemigroup of \( \mathcal{M}(S) \).
3. There exist \( a_1, a_2, \ldots, a_p \in S \) such that \( S = \{n \in \mathbb{N} \mid a_1 n + \cdots + a_p n < n < b_1 n + \cdots + b_p n\} \) for some \( b_1, b_2, \ldots, b_p \in \mathbb{N} \).
4. \( S \) is a numerical semigroup and \( \{a_1, a_2, \ldots, a_p \} \subseteq \mathbb{N} \) is a system of generators of \( S \) such that \( |x| < |y| \) if \( x, y \in S \).
5. \( \text{ord}(S) = \text{ord}(S) \cup \{0\} \) and \( x \in S \) for all \( x \in \mathbb{Z} \) such that \( |x| < \text{ord}(S) \).

The Frobenius variety of the numerical \( A \)-Semigroups and the associated tree

Definition 12. \( \mathcal{A}(S) \) is a numerical \( A \)-semigroup if it is a Frobenius variety.

Theorem 13. \( \mathcal{A}(S) \) is a tree with root \( H \). Moreover, \( \mathcal{A}(S) \subseteq \text{F}(S) \), \( S \subseteq \text{F}(S) \), and \( \mathcal{A}(S) \subseteq \mathcal{F}(S) \).

Corollary 14. Let \( S \) be a numerical \( A \)-semigroup such that \( S \neq \mathcal{F}(S) \). Then \( S \) is a numerical \( A \)-semigroup if and only if \( |x| + 1 < \text{F}(S) \) and \( x \in \text{F}(S) \).

Corollary 15. Let \( S \) be a numerical \( A \)-semigroup such that \( S \neq \mathcal{F}(S) \), and let \( x \in S \) be a minimal generator of \( S \). Then \( \mathcal{A}(S) \cup \{x\} \) is a numerical \( A \)-semigroup (and only if \( |x| + 1 < \text{F}(S) \))

Example 16. Let \( S \) be the numerical semigroup \( S = \{4, 6, 7, 9\} = \{0, 4, 6, 9\} \). Then \( \mathcal{F}(S) = 5 \) and, by Proposition 11, \( S \) is a numerical \( A \)-semigroup. By applying Theorem 13 and Corollary 15, we get that \( S \) has a unique child in \( \mathcal{G}(S) \). Namely, \( S \cup \{6\} = \{4, 6, 7, 9\} \).

By applying Theorem 13 and Corollary 15, as in Example 16, we can build recursively the tree \( \mathcal{G}(S) \).

Corollary 17. Let \( \mathcal{A}(S) \) be the set of all positive integers. Then \( \mathcal{A}(S) \) is a numerical \( A \)-set if it can be expressed like the intersection of numerical \( A \)-semigroups.

Definition 18. Let \( X \subseteq \mathbb{N} \) be a subset of \( \mathbb{N} \). We say that \( X \) is a numerical \( A \)-monoid if it can be expressed like the intersection of numerical \( A \)-semigroups containing \( X \).

Theorem 19. \( \mathcal{A}(X) = \{\text{A}(X) \mid \text{X is a non-empty finite subset of} \mathbb{N} \} \).

Theorem 20. If \( a_1, \ldots, a_k \) are positive integers, then the smallest numerical \( A \)-semigroup that contains \( \{a_1, \ldots, a_k\} \) is \( \text{A}(\{a_1, \ldots, a_k\}) \).

Algorithm 21. INPUT: A finite set \( X \) of positive integers.

OUTPUT: The minimal system of generators of \( \mathcal{A}(X) \).

Algorithm 22. Example we are going to compute \( \mathcal{A}(X) \) by applying Algorithm 21.

References