The numerical semigroup of the integers which are bounded by a submonoid of $\mathbb{N}^2$

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Preliminaries

Let $\mathbb{N}$ be the set of nonnegative integers. A *submonoid* of $(\mathbb{N},+)$ is a subset M of $\mathbb{N}$ that is closed for the addition and that contains the zero element. A *numerical semigroup* is a submonoid S of $(\mathbb{N},+)$ such that $\mathbb{N}\setminus S$ is finite.

Definition 1 *Let S be a numerical semigroup. The* Frobenius number *of S is* $\mathrm{F}(S)=\max(\mathbb{Z}\setminus S)$ *and the* genus *of S is* $\mathrm{g}(S)=\sharp(\mathbb{N}\setminus S)$.

Definition 2 *A* Frobenius variety *is a non-empty family $\mathcal{V}$ of numerical semigroups that fulfills the following conditions,*

1. if $S,T\in\mathcal{V}$, then $S\cap T\in\mathcal{V}$;

2. if $S\in\mathcal{V}$ and $S\neq\mathbb{N}$, then $S\cup\{\mathrm{F}(S)\}\in\mathcal{V}$.

Example 3 Let $\mathcal{S}$ be the set formed by all numerical semigroups. Then $\mathcal{S}$ is a Frobenius variety.

Definition 4 *A graph G is a pair (V,E), where V is a non-empty set and $E\subseteq\{(v,w)\in V\times V \mid v\neq w\}$. The elements of V are called* vertices *of G and the elements of E are called* edges *of G. A* path *(of length n) connecting the vertices x and y of G is a sequence of different edges of the form $(v_0,v_1),(v_1,v_2),\ldots,(v_{n-1},v_n)$ such that $v_0=x$ and $v_n=y$.*

Definition 5 *We say that a graph G is a* tree *if there exists a vertex v^* (the* root *of G) such that, for every other vertex x of G, there exists a unique path connecting x and v^*. If (x,y) is an edge of the tree, then we say that x is a* child *of y.*

Example 6 We can associate to $\mathcal{S}$ a tree in which the vertices are the elements of $\mathcal{S}$, (T,S) is an edge if $S=T\cup\{\mathrm{F}(T)\}$, and $\mathbb{N}$ is the root. In addition, if S is a numerical semigroup, then the unique path connecting S with $\mathbb{N}$ is given by the set $\mathrm{C}(S)=\{S_0\subsetneq S_1\subsetneq\cdots\subsetneq S_n\}$, where $S_0=S$, $S_{i+1}=S_i\cup\{\mathrm{F}(S_i)\}$, for all $i<n$, and $S_n=\mathbb{N}$. $\mathrm{C}(S)$ is called the *chain of numerical semigroups associated to S*.

Numerical $\mathcal{A}$-Semigroups

A *submonoid* of $(\mathbb{N}^2,+)$ is a subset M of $\mathbb{N}^2$ that is closed for the addition and such that $(0,0)\in M$.

Definition 7 *Let M be a submonoid of $(\mathbb{N}^2,+)$. We say that a positive integer n is* bounded *by M if there exists $(a,b)\in M$ such that $a<n<b$. We denote by $\mathrm{A}(M)=\{n\in\mathbb{N}\mid n$ is bounded by $M\}$.*

It is easy to check that, if M is a submonoid of $(\mathbb{N}^2,+)$ such that $\mathrm{A}(M)$ is not empty, then $\mathrm{A}(M)\cup\{0\}$ is a numerical semigroup.

Definition 8 *A numerical semigroup S is a* numerical $\mathcal{A}$-semigroup *if there exists M, submonoid of $(\mathbb{N}^2,+)$, such that $S=\mathrm{A}(M)\cup\{0\}$.*

Theorem 9 *A numerical semigroup S is a numerical $\mathcal{A}$-semigroup if and only if $\{x+y-1,x+y+1\}\subseteq S$, for all $x,y\in S\setminus\{0\}$.*

Let M be a submonoid (of $(\mathbb{N},+)$ or $(\mathbb{N}^2,+)$) and let X be a subset of M. We say that X is a *system of generators* of M if $M=\langle X\rangle=\{\lambda_1x_1+\cdots+\lambda_nx_n\mid n\in\mathbb{N}\setminus\{0\},\ x_1,\ldots,x_n\in X,\ \lambda_1,\ldots,\lambda_n\in\mathbb{N}\}$. The set X is a *minimal system of generators* of M (denoted by $\mathrm{msg}(M)$) if no proper subset of X is a system of generators of X. It is well known that the every submonoid of $(\mathbb{N},+)$ (in particular, numerical semigroup) has a unique minimal system of generators, which in addition is finite.

Corollary 10 *Let S be a numerical semigroup. The following conditions are equivalent.*

1. S is a numerical $\mathcal{A}$-semigroup.

2. $S=\mathrm{A}(M)\cup\{0\}$ for some finitely generated submonoid M of $(\mathbb{N}^2,+)$.

3. There exist $a_1,b_1,\ldots,a_p,b_p\in\mathbb{N}$ such that $S=\{n\in\mathbb{N}\mid a_1x_1+\cdots+a_px_p<n<b_1x_1+\cdots+b_px_p$ for some $x_1,\ldots,x_p\in\mathbb{N}\}\cup\{0\}$.

Let S be a numerical semigroup and let $\{n_1,\ldots,n_p\}\subseteq\mathbb{N}\setminus\{0\}$ be a system of generators of S. If $s\in S$, then the *order of s in S* is $\mathrm{ord}(s;S)=\max\{a_1+\cdots+a_p\mid a_1n_1+\cdots+a_pn_p=s$, with $a_1,\ldots,a_p\in\mathbb{N}\}$. If no ambiguity is possible, then we write $\mathrm{ord}(s)$.

Proposition 11 *Let S be a numerical semigroup with minimal system of generators given by $\{n_1,\ldots,n_p\}$. The following conditions are equivalent.*

1. S is a numerical $\mathcal{A}$-semigroup.

2. If $i,j\in\{1,\ldots,p\}$, then $\{n_i+n_j-1,n_i+n_j+1\}\subseteq S$.

3. If $s\in S\setminus\{0,n_1,\ldots,n_p\}$, then $\{s-1,s+1\}\subseteq S$.

4. If $s\in S\setminus\{0\}$, then $s+z\in S$ for all $z\in\mathbb{Z}$ such that $|z|<\mathrm{ord}(s)$.

The Frobenius variety of the numerical $\mathcal{A}$-Semigroups and the associated tree

Proposition 12 $\mathcal{A}=\{S\mid S$ *is a numerical $\mathcal{A}$-semigroup$\}$ is a Frobenius variety.*

We define the graph $\mathrm{G}(\mathcal{A})$ in which $\mathcal{A}$ is the set of vertices of $\mathrm{G}(\mathcal{A})$ and $(S,S')\in\mathcal{A}\times\mathcal{A}$ is an edge if $S'=S\cup\{\mathrm{F}(S)\}$.

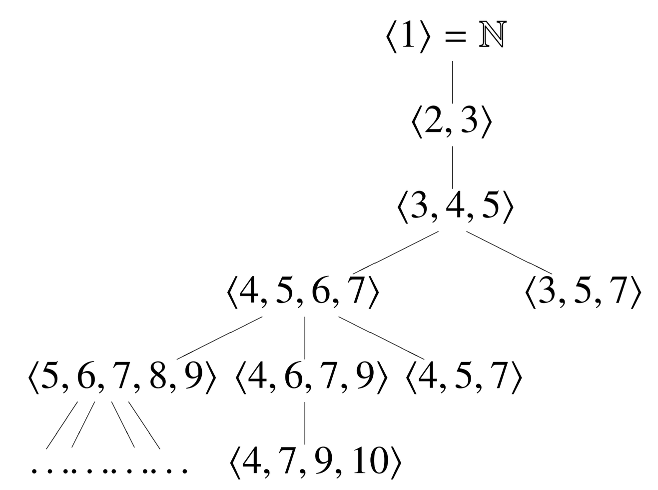
Theorem 13 $\mathrm{G}(\mathcal{A})$ *is a tree with root $\mathbb{N}$. Moreover, $\{S\setminus\{x\}\mid x\in\mathrm{msg}(S),\ x>\mathrm{F}(S),$ and $S\setminus\{x\}\in\mathcal{A}\}$ is the set of children of a vertex $S\in\mathcal{A}$.*

Proposition 14 *Let S be a numerical $\mathcal{A}$-semigroup such that $S\neq\mathbb{N}$, and let $x\in\mathrm{msg}(S)$. Then $S\setminus\{x\}$ is a numerical $\mathcal{A}$-semigroup if and only if $\{x-1,x+1\}\subseteq\{0\}\cup(\mathbb{N}\setminus S)\cup\mathrm{msg}(S)$.*

Corollary 15 *Let S be a numerical $\mathcal{A}$-semigroup such that $S\neq\mathbb{N}$, and let x be a minimal generator of S greater than $\mathrm{F}(S)$. Then $S\setminus\{x\}$ is a numerical $\mathcal{A}$-semigroup if and only if $\{x-1,x+1\}\subseteq\mathrm{msg}(S)\cup\{\mathrm{F}(S)\}$.*

Example 16 Let S be the numerical semigroup $S=\langle 4,6,7,9\rangle=\{0,4,6,\rightarrow\}$. Therefore, $\mathrm{F}(S)=5$ and, by Proposition 11, S is a numerical $\mathcal{A}$-semigroup. By applying Theorem 13 and Corollary 15, we get that S has a unique child in $\mathrm{G}(\mathcal{A})$. Namely, $S\setminus\{6\}=\langle 4,7,9,10\rangle$.

By applying Theorem 13 and Corollary 15, as in Example 16, we can build recursively the tree $\mathrm{G}(\mathcal{A})$.



The smallest numerical $\mathcal{A}$-semigroup containing a given set of positive integers

Definition 17 *Let M be a submonoid of $(\mathbb{N},+)$. We say that M is an* $\mathcal{A}$-monoid *if it can be expressed like the intersection of numerical $\mathcal{A}$-semigroups.*

Let X be a subset of $\mathbb{N}$. Being that the intersection of $\mathcal{A}$-monoids is an $\mathcal{A}$-monoid, we can define the *$\mathcal{A}$-monoid generated by X* (denoted by $\mathcal{A}(X)$) as the intersection of all $\mathcal{A}$-monoids containing X. Let us observe that $\mathcal{A}(X)$ is the smallest $\mathcal{A}$-monoid containing X.

Proposition 18 *If $X\subseteq\mathbb{N}$, then $\mathcal{A}(X)$ is the intersection of all numerical $\mathcal{A}$-semigroups that contain X. Moreover, if $X\neq\emptyset$ and $X\subseteq\mathbb{N}\setminus\{0\}$, then $\mathcal{A}(X)$ is a numerical $\mathcal{A}$-semigroup.*

Theorem 19 $\mathcal{A}=\{\mathcal{A}(X)\mid X$ *is a non-empty finite subset of* $\mathbb{N}\setminus\{0\}\}$.

Theorem 20 *If $x_1,\ldots,x_t$ are positive integers, then the smallest numerical $\mathcal{A}$-semigroup that contains $\{x_1,\ldots,x_t\}$ is $S(x_1,\ldots,x_t)=\{a_1x_1+\cdots+a_tx_t+z\mid a_1,\ldots,a_t\in\mathbb{N},\ z\in\mathbb{Z},$ and $|z|<a_1+\cdots+a_t\}\cup\{0\}$.*

Algorithm 21 INPUT: A finite set X of positive integers.

OUTPUT: The minimal system of generators of $\mathcal{A}(X)$.

(1) $Y=\mathrm{msg}(\langle X\rangle)$.

(2) $Z=Y\cup\left(\bigcup_{a,b\in Y}\{a+b-1,a+b+1\}\right)$.

(3) If $\mathrm{msg}(\langle Z\rangle)=Y$, then return Y.

(4) Set $Y=\mathrm{msg}(\langle Z\rangle)$ and go to (2).

Example 22 We are going to compute $\mathcal{A}(\{5,7\})$ by applying Algorithm 21.

- $Y=\{5,7\}$.
- $Z=\{5,7,9,11,13,15\}$.
- $\mathrm{msg}(\langle Z\rangle)=\{5,7,9,11,13\}$.
- $Y=\{5,7,9,11,13\}$.
- $Z=\{5,7,9,11,13,15,17,19,21,23,25,27\}$.
- $\mathrm{msg}(\langle Z\rangle)=\{5,7,9,11,13\}$.
- $\mathcal{A}(\{5,7\})=\langle 5,7,9,11,13\rangle$.

The most complex process in Algorithm 21 is the computation of $\mathrm{msg}(\langle Z\rangle)$, that is, the computation of the minimal system of generators of a numerical semigroup S starting from a system of generators of it. For this purpose, we can use the GAP package called `numericalsgps`.

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