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Modular numerical semigroups with embedding dimension equal to three

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Extended Abstract

Let $\mathbb{N}$ be the set of nonnegative integer numbers. A *numerical semigroup* is a subset S of $\mathbb{N}$ such that it is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S$ is finite. If $A \subseteq \mathbb{N}$, we denote by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by A, this is,

$$\langle A \rangle = \{\lambda_1 a_1 + \ldots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\}, a_1, \ldots, a_n \in A \text{ and } \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}.$$

It is well known (see [2]) that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd\{A\} = 1$, where gcd means *greatest common divisor*.

Let S be a numerical semigroup and let X be a subset of S. We say that X is a *system of generators* of S if $S = \langle X \rangle$. In addition, if no proper subset of X generates S, then we say that X is a *minimal system of generators* of S. Every numerical semigroup admits a unique minimal system of generators and, moreover, such system has finitely many elements (see [1, 2]). The cardinal of this system is known as the *embedding dimension* of S and it is denoted by $\mathrm{e}(S)$. On the other hand, if $X = \{n_1 < n_2 < \ldots < n_e\}$ is a minimal system of generators of S, then n_1, n_2 are known as the *multiplicity* and the *ratio* of S.

Let m, n be integers such that $n \neq 0$. We denote by $m \bmod n$ the remainder of the division of m by n. Following the notation of [3], we say that a *proportionally modular Diophantine inequality* is an expression of the form

$$ax \bmod b \leq cx \tag{1}$$

where a, b, c are positive integers. We call $a, b,$ and c the *factor*, the *modulus*, and the *proportion* of the inequality, respectively. Let $\mathrm{S}(a, b, c)$ be the set of integer solutions of (1). Then $\mathrm{S}(a, b, c)$ is a numerical semigroup (see [3]) that we call *proportionally modular numerical semigroup* (PM-*semigroup*).

As a consequence of [5, Theorem 31] (see its proof and [5, Corollary 18]) we have an easy characterization for PM-semigroups: a numerical semigroup S is a PM-semigroup if and only if there exists a convex arrangement $n_1, n_2, \ldots, n_e$ of its set of minimal generators that satisfies the following conditions

1. $\gcd\{n_i, n_{i+1}\} = 1$ for all $i \in \{1, \ldots, e-1\}$.
2. $(n_{i-1} + n_{i+1}) \equiv 0 \bmod n_i$ for all $i \in \{2, \ldots, e-1\}$.

A *modular Diophantine inequality* (see [4]) is an expression of the form

$$ax \bmod b \leq x, \tag{2}$$

this is, it is a proportionally modular Diophantine inequality with proportion equal to one. A numerical semigroup is a *modular numerical semigroup* (M-*semigroup*) if it is the set of integer solutions of a modular Diophantine inequality. Therefore, every M-semigroup is a PM-semigroup, but the reciprocal is false. In effect, from [3, Example 26], we have that the numerical semigroup $\langle 3, 8, 10 \rangle$ is a PM-semigroup, but is not an M-semigroup.

Let us observe that it is easy to determine whether or not a numerical semigroup is a PM-semigroup via the previous characterization. However, this question is more complicated for M-semigroups. In [4] there is an algorithm to give the answer to this problem, but we have not got a good characterization for M-semigroups.

As a first step to give an answer to this question, our aim is to show the explicit descriptions of all M-semigroups with embedding dimension equal to three. In order to do it, we consider two ideas. First, the relation between proportionally modular numerical semigroups and numerical semigroups associated with an interval (see [3]). Secondly, the description of a proportionally modular numerical semigroup with embedding dimension equal to three when we fix the multiplicity and the ratio.

References

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