Modular numerical semigroups with embedding dimension equal to three

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Diophantine inequalities.- General type

Definition

A **proportionally modular Diophantine inequality** *is an expression of the form*

$$ax \bmod b \leq cx$$

where a (the factor), *b* (the modulus), *and c* (the proportion) *are positive integer numbers.*

(Let m, n be integer numbers such that $n \neq 0$. Then $m \bmod n$ is the remainder of the division of m by n.)

Set of nonnegative integer solutions

$$\mathrm{S}(a,b,c) = \{x \in \mathbb{Z} \mid ax \bmod b \leq cx\}$$

A simplification

Lemma

Let a, b, c positive integer numbers.

1. $\mathrm{S}(a, b, c) = \mathrm{S}(a \bmod b, b, c)$.
2. $\mathrm{S}(a, b, c) = \mathbb{N}$ *if* $a \leq c$.

($\mathbb{N}$ is the set of nonnegative integer numbers.)

Not restrictive condition

$$c < a < b$$

Diophantine inequalities.- Particular type

Definition

A **modular Diophantine inequality** *is an expression of the form*

$$ax \bmod b \leq x,$$

where a, b are positive integer numbers (such that $a < b$).

Set of nonnegative integer solutions

$$\mathrm{S}(a,b) = \mathrm{S}(a,b,1) = \{x \in \mathbb{Z} \mid ax \bmod b \leq x\}$$

Diophantine inequalities.- The question

Problem

Let us have $\mathrm{S}(a,b,c)$. Does there exist positive integer numbers a^*, b^* such that $\mathrm{S}(a,b,c) = \mathrm{S}(a^*,b^*)$?

Example

1. $\mathrm{S}(21,189,3) = \mathrm{S}(7,63)$.
2. $\mathrm{S}(41,369,5) = \mathrm{S}(8,72)$.
3. $\mathrm{S}(51,459,6) \neq \mathrm{S}(a^*,b^*)$ for all $a^*, b^* \in \mathbb{N}$.

Tool: numerical semigroups

Definition

A **numerical semigroup** *is a subset S of $\mathbb{N}$ that is closed under addition, contains the zero element, and has finite complement in $\mathbb{N}$.*

Example

1. $S = \mathrm{S}(a,b,c) = \{x \in \mathbb{Z} \mid ax \bmod b \leq cx\}$ (PM-**semigroup**).
2. $S = \langle a_1, a_2, \dots, a_n \rangle$.
   - $A = \{a_1, a_2, \dots, a_n\} \subseteq \mathbb{N} \setminus \{0\}$ such that $\gcd\{a_1, a_2, \dots, a_n\} = 1$.
   - $\langle A \rangle = \{\lambda_1 a_1 + \ldots + \lambda_n a_n \mid \lambda_1, \dots, \lambda_n \in \mathbb{N}\} = \langle a_1, a_2, \dots, a_n \rangle$.
3. $S = \mathrm{S}([\alpha,\beta])$.
   - $\alpha,\beta \in \mathbb{Q}$ such that $0 < \alpha < \beta$; $J = [\alpha,\beta]$.
   - $\langle J \rangle = \{\lambda_1 a_1 + \ldots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\}, a_1, \dots, a_n \in J, \lambda_1, \dots, \lambda_n \in \mathbb{N}\}$.
   - $\langle J \rangle \cap \mathbb{N} = \mathrm{S}([\alpha,\beta])$.

Connection

Lemma

1. *Let a, b, c positive integer numbers such that $c < a < b$.*

$$\mathrm{S}(a,b,c) = \mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-c}\right]\right).$$

2. *Let a_1, a_2, b_1, b_2 be positive integer numbers such that $1 < \frac{b_1}{a_1} < \frac{b_2}{a_2}$.*

$$\mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_2}{a_2}\right]\right) = \mathrm{S}(a_1 b_2, b_1 b_2, a_1 b_2 - a_2 b_1).$$

Example

1. $\mathrm{S}(41,369,5) = \mathrm{S}\left(\left[\frac{369}{41}, \frac{369}{36}\right]\right) = \mathrm{S}\left(\left[\frac{9}{1}, \frac{41}{4}\right]\right).$
2. $\mathrm{S}(51,459,6) = \mathrm{S}\left(\left[\frac{459}{51}, \frac{459}{45}\right]\right) = \mathrm{S}\left(\left[\frac{9}{1}, \frac{51}{5}\right]\right).$

Characterization for PM-semigroups

Lemma

A numerical semigroup S is a PM*-semigroup if and only if there exists a convex arrangement $n_1, n_2, \ldots, n_e$ of its set of minimal generators that satisfies the following conditions*

1. $\gcd\{n_i, n_{i+1}\} = 1$ *for all* $i \in \{1, \ldots, e-1\}$,
2. $(n_{i-1} + n_{i+1}) \equiv 0 \mod n_i$ *for all* $i \in \{2, \ldots, e-1\}$.

Definition

A sequence of integer numbers $x_1, x_2, \ldots, x_q$ is arranged in a convex form if one of the following conditions is satisfied,

1. $x_1 \leq x_2 \leq \ldots \leq x_q$;
2. $x_1 \geq x_2 \geq \ldots \geq x_q$;
3. *there exists* $h \in \{2, \ldots, q-1\}$ *such that* $x_1 \geq \ldots \geq x_h \leq \ldots \leq x_q$.

Bézout sequences

Example

1. $S\left(\left[\frac{9}{1},\frac{21}{2}\right]\right)$.
$$\frac{9}{1}<\frac{10}{1}<\frac{21}{2}$$
$$S(21,189,3)=S\left(\left[\frac{189}{21},\frac{189}{18}\right]\right)=S\left(\left[\frac{9}{1},\frac{21}{2}\right]\right)=\langle 9,10,21\rangle.$$
2. $S\left(\left[\frac{9}{1},\frac{41}{4}\right]\right)$.
$$\frac{9}{1}<\frac{10}{1}<\frac{41}{4}$$
$$S(41,369,5)=S\left(\left[\frac{369}{41},\frac{369}{36}\right]\right)=S\left(\left[\frac{9}{1},\frac{41}{4}\right]\right)=\langle 9,10,41\rangle.$$
3. $S\left(\left[\frac{9}{1},\frac{51}{5}\right]\right)$.
$$\frac{9}{1}<\frac{10}{1}<\frac{51}{5}$$
$$S(51,459,6)=S\left(\left[\frac{459}{51},\frac{459}{45}\right]\right)=S\left(\left[\frac{9}{1},\frac{51}{5}\right]\right)=\langle 9,10,51\rangle.$$

(Minor troubles with) Bézout sequences

Example

1. $S\left(\left[\frac{9}{5},\frac{9}{4}\right]\right)$.

$$\frac{9}{5} < \frac{2}{1} < \frac{9}{4}$$

$$S(9,5,1) = S\left(\left[\frac{9}{5},\frac{9}{4}\right]\right) = \langle 2,9\rangle.$$

2. $S\left(\left[\frac{9}{1},\frac{12}{1}\right]\right)$.

$$\frac{9}{1} < \frac{10}{1} < \frac{11}{1} < \frac{12}{1}$$

$$S(4,36,3) = S\left(\left[\frac{36}{4},\frac{36}{3}\right]\right) = S\left(\left[\frac{9}{1},\frac{12}{1}\right]\right) = \langle 9,10,11,12\rangle.$$

By generators (via characterization)

Proposition

1. *Let* m_1, m_2 *be integer numbers such that* $m_1, m_2 \geq 3$ *and* $\gcd\{m_1, m_2\} = 1$.
   Then $S = \langle m_1, m_2, m_1 m_2 - m_1 - m_2 \rangle$ *is an* M*-semigroup with* $\mathrm{e}(S) = 3$.
2. *Let* λ, d, d' *be integer numbers such that* $\lambda, d, d' \geq 2$ *and* $\gcd\{d, d'\} = \gcd\{\lambda, d + d'\} = 1$.
   Then $S = \langle \lambda d, d + d', \lambda d' \rangle$ *is an* M*-semigroup with* $\mathrm{e}(S) = 3$.
3. *Let* m_1, m_2 *be positive integer numbers such that* $\gcd\{m_1, m_2\} = 1$. *Let* q *be a divisor of* $\gcd\{m_2 - 1, m_1 + m_2\}$ *such that* $2 \leq q < \min\{m_1, m_2\}$.
   Then $S = \langle m_1, \frac{m_1+m_2}{q}, m_2 \rangle$ *is an* M*-semigroup with* $\mathrm{e}(S) = 3$.

By generators via closed intervals

Proposition

Let $S = \mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-1}\right]\right)$ *be a numerical semigroup such that* $\mathrm{e}(S) = 3$.

1. *If* $\gcd\{a,b\} = \gcd\{a-1,b\} = 1$, *then there exist* $m_1, m_2 \geq 3$ *such that* $\gcd\{m_1,m_2\} = 1$ *and* $S = \langle m_1, m_2, m_1 m_2 - m_1 - m_2 \rangle$.
2. *If* $\gcd\{a,b\} = d \neq 1$ *and* $\gcd\{a-1,b\} = d' \neq 1$, *then there exist* $\lambda \geq 2$ *such that* $\gcd\{d,d'\} = \gcd\{\lambda, d+d'\} = 1$ *and* $S = \langle \lambda d, d+d', \lambda d' \rangle$.
3. *If* $\gcd\{a,b\} \neq 1$ *and* $\gcd\{a-1,b\} = 1$, *then there exist* $m_1, m_2, q \geq 2$ *such that* $\gcd\{m_1,m_2\} = 1$, $q \mid \gcd\{m_2 - 1, m_1 + m_2\}$, $2 \leq q < \min\{m_1, m_2\}$, *and* $S = \langle m_1, \frac{m_1+m_2}{q}, m_2 \rangle$.

Remark

Because $\mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-1}\right]\right) = \mathrm{S}\left(\left[\frac{b}{b+1-a}, \frac{b}{b-a}\right]\right)$, *the case* $\gcd\{a,b\} = 1$ *and* $\gcd\{a-1,b\} \neq 1$ *is analogous to the third one in the proposition.*

Families by generators

Theorem

S is an M-semigroup with $\mathrm{e}(S) = 3$ if and only if it is one of the following types.

(T1) $S = \langle m_1, m_2, m_1 m_2 - m_1 - m_2 \rangle$ *with* $m_1, m_2 \geq 3$ *such that* $\gcd\{m_1, m_2\} = 1$.

(T2) $S = \langle \lambda d, d + d', \lambda d' \rangle$ *with* $\lambda, d, d' \geq 2$ *such that* $\gcd\{d, d'\} = \gcd\{\lambda, d + d'\} = 1$.

(T3) $S = \langle m_1, \frac{m_1 + m_2}{q}, m_2 \rangle$ *with* $m_1, m_2, q \geq 2$ *such that* $\gcd\{m_1, m_2\} = 1$, $q | \gcd\{m_2 - 1, m_1 + m_2\}$, *and* $2 \leq q < \min\{m_1, m_2\}$.

Remark

1. $m_1 = m_1' m_2' - m_1' - m_2'$, $m_2 = m_2'$, *and* $q = m_2' - 1$ *in (T3)* $\Rightarrow$ *(T1).*
2. *There is no relation between (T1) and (T2).*
3. *There is no relation between (T2) and (T3).*

PM-semigroups with n_1, n_2 fixed

Lemma

Let n_1, n_2, n_3 be integer numbers such that $3 \leq n_1 < n_2 < n_3$, $\gcd\{n_1, n_2\} = 1$, and $n_3 \notin \langle n_1, n_2 \rangle$. Then $\langle n_1, n_2, n_3 \rangle$ is a PM-semigroup if and only if n_3 belongs to one of the following sets.

1. $C_1 = \{kn_2 - n_1 \mid k \in A(n_1)\}$.
2. $C_2 = \{tn_1 - n_2 \mid t \in A(n_1, n_2)\}$.

Moreover, $C_1 \cap C_2 = \{n_1 n_2 - n_1 - n_2\}$.

Definition

Let n_1, n_2 be integer numbers such that $3 \leq n_1 < n_2$ and $\gcd\{n_1, n_2\} = 1$.

- $A(n_1) = \{2, \ldots, n_1 - 1\}$.
- $A(n_1, n_2) = \left\{\left\lceil \frac{2n_2}{n_1} \right\rceil, \ldots, n_2 - 1\right\}$.
- $D(n) = \{k \in \mathbb{N}$ *such that* $k \mid n\}$.

(If $q \in \mathbb{Q}$, then $\lceil q \rceil = \min\{z \in \mathbb{Z} \mid q \leq z\}$)

M-semigroups with n_1, n_2 fixed

Lemma

1. $S = \langle n_1, n_2, n_1 n_2 - n_1 - n_2 \rangle$ *is* $(T3)$.
2. *Let us have* $S = \langle n_1, n_2, kn_2 - n_1 \rangle$ *with* $k \in A(n_1) \setminus \{n_1 - 1\}$.
   - i) S *is* $(T2)$ *if and only if* $k \mid n_1$.
   - ii) S *is* $(T3)$ *if and only if* $k \mid (n_1 - 1)$ *or* $k \mid (n_1 + 1)$.
3. *Let us have* $S = \langle n_1, n_2, tn_1 - n_2 \rangle$ *whit* $t \in A(n_1, n_2) \setminus \{n_2 - 1\}$.
   - i) S *is* $(T2)$ *if and only if* $t \mid n_2$.
   - ii) S *is* $(T3)$ *if and only if* $t \mid (n_2 - 1)$ *or* $t \mid (n_2 + 1)$.

M-semigroups with n_1, n_2 fixed

Theorem

Let n_1, n_2, n_3 be integer numbers such that $3 \leq n_1 < n_2 < n_3$, $\gcd\{n_1, n_2\} = 1$, and $n_3 \notin \langle n_1, n_2 \rangle$. Then $\langle n_1, n_2, n_3 \rangle$ is an M-semigroup if and only if n_3 belongs to one of the following sets.

1. $B_1 = \{kn_2 - n_1 \mid k \in A(n_1) \cap [D(n_1 - 1) \cup D(n_1) \cup D(n_1 + 1)]\}$.
2. $B_2 = \{tn_1 - n_2 \mid t \in A(n_1, n_2) \cap [D(n_2 - 1) \cup D(n_2) \cup D(n_2 + 1)]\}$.

Moreover, $B_1 \cap B_2 = \{n_1 n_2 - n_1 - n_2\}$.

M-semigroups with $n_1 = 9$, $n_2 = 10$

Example

1. PM-semigroups
   - $\langle 9, 10, 10k - 9 \rangle$ with $k \in \{2, 3, 4, 5, 6, 7, 8\}$
   - $\langle 9, 10, 9t - 10 \rangle$ with $t \in \{3, 4, 5, 6, 7, 8, 9\}$
2. M-semigroups
   - $\langle 9, 10, 10k - 9 \rangle$ with $k \in \{2, 3, 4, 5\}$
   - $\langle 9, 10, 9t - 10 \rangle$ with $t \in \{3, 5\}$
   - $\langle 9, 10, 71 \rangle$

M-semigroups with $n_1 = 9$, $n_2 = 10$

Example

1. $(T2): \mathcal{S}(21,189,3) = \langle 9,10,21\rangle = \langle 3\cdot 3, 3+7, 3\cdot 7\rangle$

$$9\cdot(3\cdot 3) - 8\cdot(3+7) = 1$$

$$\mathcal{S}(21,189,3) = \mathcal{S}((9\cdot 3-8)\cdot 3, 3\cdot 3\cdot 7) = \mathcal{S}(57,63)$$

2. $(T3): \mathcal{S}(41,369,5) = \langle 9,10,41\rangle = \langle 9, \frac{9+41}{5}, 41\rangle$

$$2\cdot 41 - 9\cdot 9 = 1 \Rightarrow 10\cdot 41 - 45\cdot 9 = 5$$

$$\mathcal{S}(41,369,5) = \mathcal{S}\left(\frac{41-1}{5}\cdot 10, \frac{41-1}{5}\cdot 9\right) = \mathcal{S}(80,72) = \mathcal{S}(8,72)$$

3. $\langle 9,10,51\rangle = \langle 9,10,6\cdot 10-9\rangle$ and $6 \notin D(8)\cup D(9)\cup D(10)$

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