A Saitô-Tomita-Lusin Theorem for JB*-triples and Applications

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Abstract

A Lusin’s theorem is proved in the non-ordered context of JB*-triples. This is applied to obtain versions of a general transitivity theorem and to deduce refinements of facial structure in closed unit balls of JB*-triples and duals.

1 Introduction and Preliminaries

If the open unit ball $D$ of a complex Banach space $E$ is a bounded symmetric domain the holomorphy of $D$ determines the geometry of $E$ and induces a ternary algebraic structure upon it. Banach spaces of this kind are known as JB*-triples [13]. The norm closed subspaces $E$ of C*-algebras for which $xx^*x$ lies in $E$ whenever $x$ does form a large class of examples of JB*-triples that, up to linear isometry, includes all Hilbert spaces, spin factors and many other familiar operator spaces. If $\mathbb{O}$ denotes the complex Cayley numbers then the space of all 1 by 2 matrices over $\mathbb{O}$, $M_{1,2}(\mathbb{O})$, appropriately normed, is an example of a JB*-triple not of this form. Despite a general lack of order and other constraints the ternary structure in JB*-triples, which generalises the binary structure in C*-algebras, has been shown to be a natural medium in diverse disciplines such as complex holomorphy, convexity and quantum mechanics [6, 8, 13].

Non-commutative versions of Egoroff’s and Lusin’s theorems for C*-algebras [17, 19] are extended in this paper to the non-ordered context of JB*-triples. A transitivity theorem for an arbitrary JB*-triple $E$, in the sense that the “$D$ - operator” associated with a finite rank tripotent of $E^{**}$

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has coincident image upon $E$ and $E^{**}$, is obtained as one application and consequences for facial structure discussed. In particular, it is deduced that the norm exposed faces of $E_1^*$ associated with finite rank tripotents in $E^{**}$ are weak* exposed whenever $E$ is separable and that, in general, the norm semi-exposed faces of $E_1$ are intersections of maximal norm closed faces.

We recall [13] that a JB*-triple is a complex Banach space together with a continuous triple product $\{.,.,.\} : E^3 \to E$ which is conjugate linear in the middle variable and symmetric bilinear in the outer variables such that, for the operator $D(a,b)$ given by $D(a,b)x = \{a,b,x\}$, we have

$$D(a,b)D(x,y) - D(x,y)D(a,b) = D(D(a,b)x,y) - D(x,D(b,a)y);$$

and that $D(a,a)$ is an hermitian operator with non-negative spectrum and $\|D(a,a)\| = \|a\|^2$.

Every C*-algebra is a JB*-triple via the triple product given by

$$2\{x,y,z\} = xy^*z + zy^*x,$$

and every JB*-algebra is a JB*-triple under the triple product

$$\{x,y,z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*.$$

A JBW*-triple is a JB*-triple with (unique) [3] predual. The second dual of a JB*-triple is a JBW*-triple [5]. Elements $a, b$ in a JB*-triple $E$ are orthogonal if $D(a,b) = 0$. With each tripotent $u$ (i.e. $u = \{u,u,u\}$) in $E$ is associated the Peirce decomposition

$$E = E_2(u) \oplus E_1(u) \oplus E_0(u),$$

where for $i = 0, 1, 2$ $E_i(u)$ is the $\frac{i}{2}$ eigenspace of $D(u,u)$. The Peirce rules are that $\{E_i(u), E_j(u), E_k(u)\}$ is contained in $E_{i-j+k}(u)$ if $i-j+k \in \{0, 1, 2\}$ and is zero otherwise. In addition,

$$\{E_2(u), E_0(u), E_1\} = \{E_0(u), E_2(u), E\} = 0.$$

The corresponding Peirce projections, $P_i(u) : E \to E_i(u)$, ($i = 0, 1, 2$) are contractive and satisfy

$$P_2(u) = D(2D - I), \quad P_1(u) = 4D(I - D), \quad \text{and} \quad P_0(u) = (I - D)(I - 2D),$$

where $D$ is the operator $D(u,u)$ (compare [9]).
Let $E$ be a JBW*-triple with tripotent $u$. The Peirce space $E_2(u)$ is a JBW*-algebra with Jordan product and involution given by $a \circ b = \{a, u, b\}$, $a^t = \{u, a, u\}$. Order amongst tripotents in $E$ arises as follows. Tripotents $u$ and $v$ satisfy $v \leq u$ if and only if $v$ is a projection in the JBW*-algebra $E_2(u)$. See [6], [14, §5] for several characterisations. A development [7] (see also [6], [9]) of triple functional calculus [13] is that if $x$ is norm-one element of $E$ there is a least tripotent $r(x)$ of $E$ such that $x$ belongs to the positive part of $E_2(r(x))$; the tripotent that arises as the greatest projection in $E_2(r(x))_+$ majorised by $x$ is denoted by $u(x)$. In particular, in $E_2(r(x))_+$ we have

\[ u(x) \leq x \leq r(x). \]

A non-zero tripotent $u$ in $E$ is said to be minimal if $E_2(u) = Cu$, and is said to have finite rank if it is an orthogonal sum of finitely many minimal tripotents. Given a convex set $K$ we denote by $\partial_e(K)$ its set of extreme points. If $E$ is a JB*-triple, for each $\rho \in \partial_e(E_1^*)$ there is a unique minimal tripotent, $u(\rho)$, of $E^{**}$ such that $\rho(u(\rho)) = 1$, and all minimal tripotents arise in this way [9].

Given a JBW*-triple $M$, a norm-one element $\varphi$ of $M_*$ and a norm-one element $z$ in $M$ such that $\varphi(z) = 1$, it follows from [1, Proposition 1.2] that the assignment

\[(x, y) \mapsto \varphi \{x, y, z\}\]

defines a positive sesquilinear form on $M$, the values of which are independent of choice of $z$, and induces a prehilbert seminorm on $M$ given by

\[ \|x\|_\varphi := \left(\varphi \{x, x, z\}\right)^{\frac{1}{2}}. \]

As $\varphi$ ranges over the unit sphere of $M_*$ the topology induced by these seminorms is termed the strong*-topology of $M$. The strong*-topology was introduced in [2], and further developed in [16, 15]. In particular [16], the triple product is jointly strong*-continuous on bounded sets.

## 2 Saitô-Tomita-Lusin Theorem

The classical Lusin’s theorem states that if $\mu$ is a Radon measure on a locally compact Hausdorff space $T$ and if $f$ is a complex-valued measurable function on $T$ such that there exists a Borel set $A \subseteq T$ with $\mu(A) < \infty$ and $f(x) = 0$ for all $x \notin A$, then for each $\varepsilon > 0$ there exists a Borel set $E \subseteq T$ with $\mu(T \setminus E) < \varepsilon$ and a function $g \in C_0(T)$ such that $f$ and $g$ coincide on
A non-commutative analogue of Lusin’s theorem for general C*-algebras was given in [19] and considerably developed subsequently in [17]. The underlying strategy in our approach to non-ordered JB*-triple extensions is to release and exploit local order structures harboured by Peirce 1- spaces and Peirce-2 spaces. Our initial aim is to derive a novel inequality (see Proposition 2.4) involving D-operators and then to employ it as a controlling device thereafter.

The following result is proved in [9, Lemma 1.5] and remark prior to it.

**Lemma 2.1.** Let \( u \) be a tripotent in a JB*-triple \( E \) and let \( x \in E_1(u) \cup E_2(u) \). Then \( D(x, x)u \) is a positive element in the JB*-algebra \( E_2(u) \).

**Lemma 2.2.** Let \( e \) be a projection in a JB*-algebra \( E \) and let \( a \in E_1(e) \cup E_2(e) \) where \( a = a^* \). Then \( a^2 \circ e \geq 0 \) and \( \|a\|^2 = \|D(a, a)e\| \).

**Proof.** We have \( D(a, a)e = a^2 \circ e \). We may suppose without loss that \( E \) has an identity element, 1. If \( a \in E_2(e) \), then \( a^2 = a^2 \circ e \). Let \( a \in E_1(e) \).

By [20], the JB-subalgebra of \( E_{sa} \) generated by \( 1, e \) and \( a \) can be realised as a JC-subalgebra of the self-adjoint part of a C*-algebra \( B \) so that, in \( B \), we have \( a = 2D(e, e)a = ea + ae \) and therefore \( a^2 = ea^2 + aea \), giving \( ea^2 = a(1 - e)a = a^2e \). Consequently,

\[
\|ea^2\| = \|(1 - e)a^2(1 - e)\| = \|(1 - e)a^2\| = \|(1 - e)a\|^2
\]

and therefore

\[
\|a\|^2 = \max\{\|ea^2\|, \|(1 - e)a^2\|\} = \|ea^2\| = \|D(a, a)e\|.
\]

**Lemma 2.3.** Let \( E \) be a type I von Neumann factor or a finite dimensional simple JB*-algebra. Let \( u \) be a tripotent in \( E \). Then there exists a triple embedding \( \pi : E \to E \) such that \( \pi(u) \) is a projection.

**Proof.** In the first case we may suppose \( E = B(H) \) for some complex Hilbert space \( H \) [18, V.1.28] and that \( u \) is a partial isometry. From [12, Lemma 3.12] it follows that there exists a complete tripotent \( c \in B(H) \) such that \( c \geq u \).

Since \( c \) is complete in \( B(H) \) we have \( (1 - cc^*)B(H)(1 - c^*c) = 0 \) and hence \( cc^* = 1 \) or \( c^*c = 1 \). We may assume that \( cc^* = 1 \). Denoting \( q = c^*c \), it follows that \( q \circ H \to H \) is a surjective linear isometry. Thus, the mapping \( \pi : B(H) \to B(H, q(H)) \subseteq B(H) \) defined by \( \pi(x) = c^*x \) is a surjective linear
isometry from $B(H)$ onto $B(H,q(H))$ and hence a triple isomorphism. It is clear that $\pi(c) = q$ is a projection in $B(H)$. Moreover, since $\pi$ is a triple isomorphism and $c \geq u$ we have $\pi(u) \leq \pi(c) = q$, and hence $\pi(u)$ is a projection in $B(H)$.

The second case follows from [14, Corollary 5.12].

**Proposition 2.4.** Let $u$ be a tripotent in a JB*-triple $E$ and let $x \in E_1(u) \cup E_2(u)$. Then $\|x\|^2 \leq 4 \|D(x,x)u\|$.

**Proof.** By [10, Corollary 1] we may suppose that $E$ is a JB*-subtriple of an $\ell_\infty$-sum, $M \oplus N$, where $M$ is a type $I$ von Neumann factor and $N$ is an $\ell_\infty$-sum of finite dimensional simple JB*-algebras. Letting $F$ denote the JB*-algebra $M \oplus N$, it follows from Lemma 2.3 that there is a triple embedding, $\pi : F \rightarrow F$, such that $\pi(u)$ is a projection. Thus, since $E_i(u) \subseteq F_i(u)$ for $i = 1, 2$ and $\pi$ preserves the triple product we may assume without loss that $E$ is a JB*-algebra and that $u$ is a projection in $E$.

In which case, we have $x = a + ib$ where $a$ and $b$ are self-adjoint elements of $E$. Let $x \in E_1(u)$. Then $a, b$ and $x^*$ lie in $E_1(u)$. Via the equalities $2u \circ x = x$ and $2u \circ x^* = x^*$ and Lemma 2.2 we have

$$D(x,x)u = (x \circ x^*) \circ u = (a^2 + b^2) \circ u \geq a^2 \circ u, b^2 \circ u \geq 0$$

so that

$$4 \|D(x,x)u\| \geq 4 \max\{\|a\|^2, \|b\|^2\} \geq (\|a\| + \|b\|)^2 \geq \|x\|^2.$$  

If $x$ lies in $E_2(u)$ the assertion is verified by a similar (easier) argument. 

The following observations illustrate the geometric nature of the inequality in Proposition 2.4. Given a tripotent $u$ in a JB*-triple $E$ the weak*-closed face of $E_1^*$,

$$F_u = \{\varphi \in E_1^* : \varphi(u) = \|\varphi\| = 1\}$$

identifies with the state space of the JB*-algebra $E_2(u)$. Let $x \in E_1(u)$. Since $D(x,x)u \in (E_2(u))_+$ we therefore have that

$$\|D(x,x)u\| = \sup\{\varphi(D(x,x)u) : \varphi \in F_u\}.$$  

Thus, letting $\|x\|_u$ denote $\|D(x,x)u\|^\frac{1}{2}$ we have that

$$\|x\|_u = \sup\{\|x\|_{\varphi} : \varphi \in F_u\},$$  

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the seminorms $\|x\|_\varphi$ being as defined in the introduction. Further, $\|x\|_u = 0$ implies $x = 0$. Thus, $\|\cdot\|_u$ is a norm on $E_1(u)$ satisfying

$$\|\cdot\|_u \leq \|\cdot\| \leq 2\|\cdot\|_u,$$

the second inequality being given by Proposition 2.4. In particular, we record the following.

**Corollary 2.5.** If $u$ is a tripotent in a JB*-triple $E$ then $\|\cdot\|_u$ and $\|\cdot\|$ are equivalent norms on $E_1(u)$.

**Lemma 2.6.** Let $u$ be a tripotent in a JB*-triple $E$, let $x \in E$ and let $x_j = P_j(u)(x)$ for $j = 1, 2$. Then $P_2(u)D(x,x)u \geq 0$ in $E_2(u)$ and

$$\|D(x_j,x_j)u\| \leq \|P_2(u)D(x,x)u\| \quad \text{for } j = 1, 2.$$

**Proof.** Using the Peirce rules calculation gives

$$D(x_1,x_1)u + D(x_2,x_2)u = P_2(u)D(x,x)u.$$

Thus, by Lemma 2.1, in $E_2(u)$ we have

$$0 \leq D(x_j,x_j)u \leq P_2(u)D(x,x)u \quad \text{for } j = 1, 2,$$

from which the assertion follows.

For subsequent purposes we remark that for tripotents $u,v$ in a JB*-triple with $v \leq u$ we have

$$P_i(u)P_j(v) = P_j(v)P_i(u) \quad \text{for } i,j = 1, 2;$$

$$P_1(v)P_0(u) = P_2(v)P_1(u) = 0 \quad \text{and } P_2(v)P_2(u) = P_2(v).$$

In particular,

$$P_i(v) (P_2(u) + P_1(u)) = P_i(v) \quad \text{for } i = 1, 2.$$

In addition,

$$P_2(u) + P_1(u) = 3D(u,u) - 2D(u,u)^2 \quad \text{and } 2P_2(u) + P_1(u) = 2D(u,u).$$

We employ the geometric inequality obtained in Proposition 2.4 as a key tool in arguments below that culminate with a non-ordered Lusin’s theorem for JB*-triples. The general process owes much to the scheme and ideas of Saitô [17]. We begin with a Saitô-Egoroff theorem for JB*-triples.
Theorem 2.7. Let $x$ belong to the strong*-closure of a bounded subset $X$ of $E^{**}$, where $E$ is a JB*-triple. Let $u$ be a tripotent in $E^{**}$, let $\varphi \in E^*$ and let $\varepsilon > 0$. Then there exist a sequence $(x_n)$ in $X$ and a tripotent $v$ in $E^{**}$ such that $v \leq u$, $|\varphi(u - v)| < \varepsilon$, and $\|S(x_n - x)\| \to 0$, for $S = P_2(v), P_1(v)$ and $D(v, v)$.

Proof. Choose a net $(x_\lambda)$ in $X$ with strong*-limit $x$. We may suppose that $X$ lies in the closed unit ball of $E^{**}$, that $\|\varphi\| = 1$ and (by translation) that $x = 0$.

For each $\lambda$ let $y_\lambda$ denote $P_2(u)D(x_\lambda, x_\lambda)u$. By the joint strong*-continuity of the triple product on bounded sets $(y_\lambda)$ is strong*-null. By Lemma 2.6 each $y_\lambda$ is a positive element of the JBW*-algebra $E^{**}_2(u)$, and $\|y_\lambda\| \leq 1$. In particular, for each $\lambda$, the JBW*-subtriple of $E^{**}$ generated by $y_\lambda$ and $u$ is an abelian W*-subalgebra of the JBW*-algebra $E^{**}_2(u)$. For each $\lambda$, let $u_\lambda$ denote $\chi(y_\lambda)$, where $\chi$ is the characteristic function of the interval $(-2^{-4}, 2^{-4})$. Then $u_\lambda$ is a projection in $E^{**}_2(u)$ satisfying

$$2^4 y_\lambda \geq u - u_\lambda \geq 0,$$

for all $\lambda$. Since $(y_\lambda)$ is strong*-null and hence weak*-null, $u - u_\lambda$ must be weak*-null in (the JBW*-algebra) $E^{**}_2(u)$ and thus weak*-null in $E^{**}$.

Choose $\lambda_1$ such that $|\varphi(u - u_{\lambda_1})| < 2^{-1}\varepsilon$. Denote $u_{\lambda_1}, x_{\lambda_1}$ and $y_{\lambda_1}$ by $u_1, x_1$ and $y_1$, respectively. Using Proposition 2.4 in the second inequality below and Lemma 2.6 in the third, we have

$$\|P_2(u_1)x_1 + P_1(u_1)x_1\|^2 \leq 2(\|P_2(u_1)x_1\|^2 + \|P_1(u_1)x_1\|^2)$$

$$\leq 2^3(\|D(P_2(u_1)x_1, P_2(u_1)x_1)u_1\| + \|D(P_1(u_1)x_1, P_1(u_1)x_1)u_1\|)$$

$$\leq 2^4\|P_2(u_1)D(x_1, x_1)u_1\| \leq 1.$$
The sequence \((u_n)\) decreases in the weak*-topology to a projection \(v\) of \(E_2^{**}(u)\) giving \(u - v = \sum_{i=1}^{+\infty}(u_{n-1} - u_n)\) and so \(|\varphi(u - v)| \leq \varepsilon\). Further, for each \(n\),
\[
\|P_i(v)(x_n)\| = \|P_i(v) (P_2(u_n)x_n + P_1(u_n)x_n)\| \leq n^{-1},
\]
for \(i = 1, 2\). The remaining assertions follow.

**Corollary 2.8.** Let \(u\) be a tripotent in \(E^{**}\), where \(E\) is a JB*-triple. Let \(x \in E^{**}\) and \(\varphi \in E^*\). Let \(\varepsilon > 0\) and \(\delta > 0\). Then there exists \(y \in E\) and a tripotent \(v\) in \(E^{**}\) such that \(v \leq u\), \(|\varphi(u - v)| \leq \varepsilon\), \(\|P_i(v)(x - y)\| \leq \delta\) for \(i = 1, 2\) and \(|\varphi| \leq \|P_2(u) + P_1(u)\|\).

**Proof.** Since, by \([2, \text{Corollary 3.3}]\), the closed unit ball \(E_1\) of \(E\) is strong*-dense in the closed unit ball of \(E^{**}\), the assertions follow from replacing \(x\) and \(X\) in Theorem 2.7 with \((P_2(u) + P_1(u))(x)\) and \(|(P_2(u) + P_1(u))(x)\| E_1\), respectively.

A Lusin’s theorem for JB*-triples is proved next.

**Theorem 2.9.** Let \(E\) be a JB*-triple, let \(\varphi \in E^*\) and let \(x \in E^{**}\). Let \(u\) be a tripotent in \(E^{**}\) and let \(\varepsilon > 0\) and \(\delta > 0\). Then there exists \(y \in E\) and a tripotent \(v \in E^{**}\) such that \(v \leq u\), \(|\varphi(u - v)| \leq \varepsilon\), \(S(x - y) = 0\) for \(S = P_2(v), P_1(v)\) and \(D(v, v)\), and \(|\varphi| \leq \|P_2(u) + P_1(u)\|\).

**Proof.** We may assume without loss that \(|(P_2(u) + P_1(u))(x)\| = 1\). By Corollary 2.8, there is an element \(y_1\) in \(E\) and a tripotent \(u_1\) \(\leq u\) in \(E^{**}\) satisfying
\[
|\varphi(u - v)| < 2^{-1} \varepsilon, \quad \|P_i(u_1)(x - y_1)\| < 2^{-2} \delta \quad \text{for} \quad i = 1, 2
\]
and \(\|y_1\| \leq \|P_2(u) + P_1(u)\| = 1\).

Replacing \(u\) with \(u_1\) and \(x\) with \((P_2(u_1) + P_1(u_1))(x - y_1)\) in Corollary 2.8 now gives an element \(y_2\) in \(E\) and a tripotent \(u_2\) in \(E^{**}\) such that \(u_2 \leq u_1\) satisfying
\[
|\varphi(u_1 - u_2)| < 2^{-2} \varepsilon, \quad \|P_i(u_2)(x - y_1 - y_2)\| = \|P_i(u_2)(P_2(u_1) + P_1(u_1))(x - y_1 - y_2)\| < 2^{-3} \delta \quad \text{for} \quad i = 1, 2
\]
and \(\|y_2\| \leq \|P_2(u_1) + P_1(u_1)\|(x - y_1)\| < 2^{-1} \delta\).
Proceeding in this way gives rise to a sequence \((y_n)\) in \(E\) and a decreasing sequence \((u_n)\) of tripotents in \(E_2^{**}(\mathcal{U})\), which, for \(u_0 = u\), and all \(n \geq 1\) satisfies
\[
|\varphi(u_{n-1} - u_n)| < 2^{-n} \varepsilon, \quad \|P_i(u_n)(x - \sum_{k=1}^{n} y_k)\| < 2^{-(n+1)} \delta \quad \text{for } i = 1, 2
\]
and \(\|y_{n+1}\| \leq \|(P_2(u_n) + P_1(u_n))(x - \sum_{k=1}^{n} y_k)\| < 2^{-n} \delta.\)

Letting \(v\) denote the weak*-limit of \((u_n)\) in \(E_2^{**}(\mathcal{U})\), and \(y = \sum_{n=1}^{+\infty} y_n\), we have that \(v\) is a tripotent in \(E^{**}\) with \(v \leq u\) and \(y \in E\) such that \(|\varphi(u - v)| < \varepsilon\) and \(\|y\| \leq 1 + \delta.\)

Finally, for \(i = 1, 2\)
\[
\|P_i(v)(x - \sum_{k=1}^{n} y_k)\| = \|P_i(v)(P_2(u_n) + P_1(u_n))(x - \sum_{k=1}^{n} y_k)\| < 2^{-n} \delta
\]
for all \(n \geq 1\), so that \(P_i(v)(x - y) = 0.\) In turn, this implies
\[
D(v, v)(x - y) = 0.
\]

\[\square\]

3 Applications

In [6] Edwards and Rüttimann investigated facial structure of unit balls of a JBW*-triple and predual giving a complete description, and made significant inroads into corresponding general JB*-triple theory in the subsequent treatise [7]. In this section we exploit Theorem 2.9 to obtain versions of Kadison transitivity for JB*-triples (c.f. [18, II.4.18]) and use it to contribute observations on facial structure.

Let \(x \in E\) where \(E\) is a JB*-triple. Let \(E_x\) and \(E(x)\), respectively denote the JB*-subtriple and norm closed inner ideal of \(E\) generated by \(x\). We have \(E(x)^{**} = E_2^{**}(\mathcal{R}(x))\) and, when the latter is realised as a JBW*-algebra, \(E(x)\) is a JB*-algebra with \(x \in E(x)_+\) and \(E_x\) is the abelian C*-algebra (i.e. associative JB*-algebra) of \(E(x)\) generated by \(x\), with corresponding spectrum \(\sigma(x)\). To avoid possible confusion below, given a continuous real-valued function \(f\) on \(\sigma(x) \cup \{0\}\) vanishing at 0, \(f(x)\) shall have its usual meaning when \(E_x\) is regarded as an abelian C*-algebra and \(f_t(x)\) shall denote...
the same element of $E_x$ when the latter is regarded as a JB*-subtriple of $E$. Thus, for any real odd polynomial, $P(\lambda) = \sum_{k=0}^{n} \alpha_k \lambda^{2k+1}$, we have $P_t(x) = \sum_{k=0}^{n} \alpha_k D(x, x)^k(x)$.

We remark that if $\|x\| = 1$, then the tripotents $u(x)$ and $r(x)$ in $E^{**}$ are projections in the abelian von Neumann algebra $(E_x)^{**}$, $r(x)$ being the identity element.

**Lemma 3.1.** Let $x \in E$ and $u \in E^{**}$ where $E$ is a JB*-triple and $u$ is a tripotent such that $D(u, u)x = u$. Then there is an element $a \in E_x$ such that $\|a\| = 1$ and $D(u, u)a = u$. Moreover, $D(u, u)u(a) = D(u, u)r(a) = u$ and $u \leq u(a) \leq a \leq r(a)$ (in $E_x^{**}(r(a))$).

**Proof.** We may suppose that $u$ is non-zero and therefore that $\|x\| \geq 1$. Since $P_0(u)(x - u) = 0$, $u$ and $x - u$ are orthogonal which implies

$$D(u, x) = D(u, u) = D(x, u).$$

By Peirce arithmetic $D(x, x)$ and $D(u, u)$ commute and, by induction, we have

$$D(u, u)D(x, x)^nx = u,$$

for all $n \geq 0$. Thus,

$$D(u, u)P_t(x) = P(1)u,$$

for all real odd polynomials $P$. If $f$ is a continuous real valued function on $[0, \|x\|]$ vanishing at 0 it follows that

$$D(u, u)f_t(x) = f(1)u.$$

Putting $a = f_t(x)$, where $f(\lambda) = \min\{\lambda, 1\}$, we have that $D(u, u)a = u$ and $\|a\| = 1$.

Further, since $(f_n)_{t}(a) \to r(a)$ and $(g_n)_{t}(a) \to u(a)$ in the weak*-topology, where $f_n(\lambda) = \lambda^{2n+1}$ and $g_n(\lambda) = \lambda^{2n+1}$ ($0 \leq \lambda \leq 1$) we have

$$D(u, u)r(a) = u = D(u, u)u(a).$$

The final assertion follows from this and the above remarks. \qed

**Lemma 3.2.** Let $u$ be a tripotent in $E^{**}$ where $E$ is a JB*-triple. The sets

$\{x \in E_1 : D(u, u)x = u\}$, $(u + E_0^{**}(u)) \cap E_1$ and $\{x \in E : u \leq x \leq r(x)\}$ coincide.
Proof. The coincidence of the first two sets is evident from the fact that, for $x \in E^{**}$, $D(u,u)x = u$ if and only if $x - u \in \ker D(u,u) = E_0^{**}(u)$. The first set is contained in the third by Lemma 3.1. Conversely, given $x \in E$ with $u \leq x \leq r(x)$ we have, since $r(x) - x \geq 0$ in $E_2^{**}(r(x))$ and $u$ is a projection there satisfying $\{u, r(x) - x, u\} = 0$, that $u$ and $r(x) - x$ must be orthogonal so that

$$0 = D(u,u)(r(x) - x) = u - D(u,u)x.$$

\[\square\]

We shall now prove a transitivity theorem for JB*-triples.

**Theorem 3.3.** Let $E$ be a JB*-triple and let $u_1, \ldots, u_n$ be orthogonal minimal tripotents in $E^{**}$ with sum $u$. Then

(a) $D(u,u)E = D(u,u)E^{**}$ and $P_j(u)E = P_j(u)E^{**}$ for $j = 1, 2$.

(b) There exists $a$ in $E$ such that $\|a\| = 1$ and $D(u,u)a = u$.

(c) There exist orthogonal elements, $a_1, \ldots, a_n$ in $E$ such that $D(u_i, u_i)a_i = u_i$ and $\|a_i\| = 1$ for $i = 1, \ldots, n$.

**Proof.** (a) The JBW*-algebra $E_2^{**}(u)$ is an $\ell_\infty$-sum of JBW*-algebras $M_1, \ldots, M_k$ where each $M_i$ is a type $I_{n_i}$ factor with $n_i < \infty$. For each $i$, let $\psi_i$ be the (unique) faithful tracial state on $M_i$, and let $\varphi$ denote $\psi P_2(u)$ where $\psi$ is the faithful tracial state on $E_2^{**}(u)$ given by $k^{-1} \sum_{i=1}^k \psi_i$. By construction, as $v$ ranges over all tripotents in $E^{**}$ such that $v \leq u$ and $v \neq u$, the values of $\varphi(v)$ form a finite set of rational numbers with supremum $\alpha < \varphi(u) = 1$.

Let $x \in E^{**}$. By Theorem 2.9 there exists an element $a \in E$ and a tripotent $v \in E^{**}$ with $v \leq u$ such that $D(v, v)(x - a) = 0$ and $1 - \varphi(v) < 1 - \alpha$. Since $\varphi(v) > \alpha$, we must have $v = u$ and hence $D(u,u)E = D(u,u)E^{**}$. The remaining equalities are immediate from the identities $2P_j(u)D(u,u) = JP_j(u)$ for $j = 1, 2$.

(b) By the arguments of the previous paragraph, and Lemma 3.1, there is a norm-one element $a \in E$ such that $u - D(u,u)a = D(u,u)(u - a) = 0$.

(c) Let $a$ be as in (b) and let $\rho_i \in \partial_\varphi(E_i^*)$ such that $\rho_i(u_i) = 1$ for each $i$ [9, Proposition 4]. By restriction, the $\rho_i$ are pure states of the JB*-algebra $E(a)$ with support projections $u_i$ in $E(a)^{**}$. Hence, by [11, Proposition 2.3], there exist orthogonal norm-one elements $b_1, \ldots, b_n \in E(a)_+$ with $\rho_i(b_i) = 1$ for $i = 1, \ldots, n$. Since each $u_i$ is now a minimal tripotent of $E(b_i)^{**}$ we can
apply (b) to find a norm-one element $a_i \in E(b_i)$ such that $D(u_i, u_i) a_i = u_i$. Since the inner ideals $E(b_1), \ldots, E(b_n)$ are mutually orthogonal, so are the elements $a_1, \ldots, a_n$.

Let $E$ be a JB*-triple. In the terminology introduced in [7] the tripotents of $E^{**}$ of the form $u(a)$ where $a \in E$ with $\|a\| = 1$, are referred to as compact $G_\delta$’s relative to $E$ and each tripotent of $E^{**}$ that is the weak* limit of a decreasing net of compact $G_\delta$’s relative to $E$ is called compact relative to $E$. Let $x \in E^{**}$ with $\|x\| = 1$. The norm-exposed face of $E_1^*$

$$F_x = \{ \varphi \in E_1^* : \varphi(x) = 1 \}$$

satisfies $F_x = F_{u(x)}$ [7, Lemma 3.3]. The face $F_x$ is weak*-exposed if $x \in E$. By [7, Corollary 4.4] (c.f. [6, Lemma 3.2, Theorem 4.6]) the assignments

$$u \mapsto F_u \quad \text{and} \quad u \mapsto (u + E_0^{**}(u)) \cap E_1$$

are respectively an order isomorphism, and an anti-order isomorphism, from the non-zero compact tripotents relative to $E$ onto the proper weak*-semi-exposed faces of $E_1^*$, and onto the proper norm-semi-exposed faces of $E_1$.

**Theorem 3.4.** Let $E$ be a JB*-triple and let $u$ be a finite rank tripotent of $E^{**}$. Then

(a) $u$ is compact relative to $E$;

(b) $F_u$ is a weak*-semi-exposed face of $E_1^*$.

Moreover, if $E$ is separable, then

(c) $u$ is a compact $G_\delta$ relative to $E$;

(d) $F_u$ is a weak*-exposed face of $E_1^*$;

(e) $\{ \rho \}$ is weak*-exposed for all $\rho \in \partial_c(E_1^*)$.

**Proof.** (a) Let $u$ be the sum of orthogonal minimal tripotents $u_1, \ldots, u_n$ in $E^{**}$. Via Theorem 3.3 (c), choose orthogonal norm-one elements $a_1, \ldots, a_n$ in $E$ such that

$$u_i \leq a_i \leq r(a_i) \quad \text{for } i = 1, \ldots, n.$$ 

Each $u_i$ is a minimal projection of $E(a_i)^{**} = E_2^{**}(r(a_i))$. Thus $r(a_1) - a_1, \ldots, r(a_n) - a_n$ are, respectively, weak* limits of increasing nets
Let \( x \) be the sum of \( a_1, \ldots, a_n \). Via mutual orthogonality of the \( a_i \) we have
\[
 u \leq x \leq r(x) = \sum_{i=1}^{n} r(a_i),
\]
and \( r(x) - \sum_{i=1}^{n} (x_i)_{\lambda} \) is a decreasing net in \( A(x)^{**} \) with weak* limit \( u \). Thus,
\[
\left\{ x, r(x) - \sum_{i=1}^{n} (x_i)_{\lambda}, x \right\}
\]
is a decreasing net in \( E(x) \) with weak*-limit \( \{ x, u, x \} = u \). It follows that \( u \) is compact relative to \( E \).

(b) This is immediate from (a) and [7, Corollary 4.4].

Suppose now that \( E \) is separable.

(c) Let the minimal tripotents \( u_1, \ldots, u_n \) be as in (a) and let \( \rho_1, \ldots, \rho_n \) in \( \partial_e(E_1^*) \) such that \( \{ \rho_i \} = F_{u_i} \) for each \( i \) [6, Proposition 4]. As in part (a) we can choose \( x \) in \( E \) such that \( u \leq x \leq r(x) \). Passing to the separable JB*-algebra \( E(x) \), it follows from [11, Propositions 2.3 and 3.1] that there exist orthogonal norm-one elements \( a_1, \ldots, a_n \) in \( E(x)_+ \) such that for each \( i \),
\[
\{ \rho_i \} = F_{a_i} = F_{u(a_i)};
\]
the second equality coming from [6, Lemma 3.3], so that \( u_i = u(a_i) \) by [6, Theorem 4.4]. Thus, letting \( a = \sum_{i=1}^{n} a_i \), in the JB*-algebra \( E(x) \) we have that
\[
a^{2n+1} = \sum_{i=1}^{n} a_i^{2n+1}
\]
is a decreasing sequence with weak* limit \( \sum_{i=1}^{n} u_i = u \) in \( E(x)^{**} \). Therefore, \( u = u(a) \).

(d) With \( a \) in \( E \) as in the proof of (c), we have \( F_u = F_a \), as required.

(e) This is contained in the proof of (c). \( \square \)

We conclude with an observation on the facial structure of the closed unit ball \( E_1 \) of a JB*-triple \( E \). If \( G \) is a norm-semi-exposed face of \( E_1 \) then (see [7])
\[
G' = \{ \rho \in E_1^* : \rho(x) = 1 \text{ for all } x \in G \}
\]
is a weak*-semi-exposed face of \( E_1^* \) and

\[
G = \{ x \in E_1 : \rho(x) = 1 \text{ for all } \rho \in G' \}.
\]

Let \( u(a) \) be a compact \( G_\delta \) in \( E^{**} \) relative to \( E \) (where \( a \) lies in \( E \) with \( \| a \| = 1 \)) and let \( \rho \in \partial_c(F_a) \). Then \( u(a) \) majorises the support tripotent \( v \) of \( \rho \) in \( E^{**} \), and we note that \( v \) is a minimal tripotent since \( \rho \in \partial_c(E_1^*) \). It follows by definition that each element in the set, \( S \), of all non-zero compact tripotents in \( E^{**} \) relative to \( E \), majorises a minimal tripotent in \( E^{**} \). Since, by Theorem 3.4 \( a \), all minimal tripotents of \( E^{**} \) are compact relative to \( E \) we deduce that the minimal elements of the set \( S \) (see [7, Theorem 4.5]) are, precisely, the minimal tripotents of \( E^{**} \). By Theorem 3.3 \( b \) each \( \rho \in \partial_c(E_1^*) \) attains its norm on \( E_1 \) so that

\[
E_\rho = \{ x \in E_1 : \rho(x) = 1 \}
\]

is a non-empty (norm-exposed) face of \( E_1 \).

**Corollary 3.5.** Let \( E \) be a \( JB^* \)-triple.

\( a \) The \( E_\rho \) are the maximal proper norm-closed faces of \( E_1 \) as \( \rho \) ranges over \( \partial_c(E_1^*) \).

\( b \) Each norm semi-exposed face of \( E_1 \) is an intersection of maximal norm closed faces of \( E_1 \).

**Proof.** \( a \) Each maximal proper norm closed face of \( E_1 \) is norm exposed by [7, Lemma 2.1]. Given \( \rho \in \partial_c(E_1^*) \) with minimal support tripotent \( u \) in \( E^{**} \) we clearly have that \( E_\rho \) contains \( (u + E_0^{**}(u)) \cap E_1 \). By the above remarks together with [7, Corollary 4.4 \( ii \)] the assertion now follows.

\( b \) Let \( G \) be a norm semi-exposed face of \( E_1 \). By the Krein-Milman theorem

\[
G = \{ x \in E_1 : \rho(x) = 1 \text{ for all } \rho \in \partial_c(G') \}.
\]

Further, \( \partial_c(G') = \partial_c(E_1^*) \cap G' \), since \( G' \) is a face. Hence,

\[
G = \bigcap \{ E_\rho : \rho \in \partial_c(E_1^*) \cap G' \}.
\]
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