Strong subdifferentiability of the norm on JB^*-triples

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We prove that the norm of a JB^*-triple X is strongly subdifferentiable at a norm-one element x if and only if 1 is an isolated point of the triple spectrum of x, if and only if the support of x in the bidual of X lies in X. Moreover we show that the JB^*-triples whose norms are strongly subdifferentiable at every point of their unit spheres are precisely the weakly compact JB^*-triples. Characterizations of those norm-one elements of a JB^*- or JBW^*-triple where the norm is Fréchet differentiable are also obtained.

1. Introduction

The norm of a Banach space X is said to be strongly subdifferentiable at an element $x \in X$ when the limit

$$\lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

(which always exists) is uniform for y in the closed unit ball of X. Since this condition is trivially satisfied for $x = 0$, and holds for ρx ($\rho > 0$) whenever it holds for x, we will only consider strong subdifferentiability of the norm at norm-one elements of the space. The notion of strong subdifferentiability of the norm of a Banach space was introduced by D. A. Gregory in [**17**], who proved that such a notion is equivalent to that of upper semicontinuity ($n{-}n$) of the duality mapping, previously considered by J. R. Giles, Gregory himself, and B. Sims in [**16**]. Both notions become natural succedanea of that of Fréchet differentiability of the norm when smoothness is not required. Thus, the norm of a Banach space X is Fréchet differentiable at a norm-one element x if and only if it is strongly subdifferentiable at x, and X is smooth at x. Strong subdifferentiability of the norm was rediscovered in [**1**] in relation to some questions on numerical ranges (see also [**25**]), and has been fully investigated in the paper of C. Franchetti and R. Payá [**14**].

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The most direct forerunner of the present paper is the work of M. Contreras, R. Payá, and W. Werner [**9**], where the strong subdifferentiability of the norm on C^*-algebras is studied in detail. The authors of [**9**] precisely determine those norm-one elements x, of a C^*-algebra X, such that the norm of X is strongly subdifferentiable at x, as well as those C^*-algebras X such that the norm of X is strongly subdifferentiable at every point of their unit spheres. In the present paper we deal with the strong subdifferentiability of the norm in the more general setting of JB^*-triples. It turns out that, sometimes, the new setting becomes more confortable than the original one, allowing us not only to generalize the results of [**9**], but even to improve some of them in its original context. Thus, for example, we prove in Theorem 2.7 that the norm of a JB^*-triple X is strongly subdifferentiable at a point if (and only if) the duality mapping of X is upper semicontinuous $(n-w)$ at such a point, a fact that has been not noticed in [**9**] in the particular case that X is a C^*-algebra. Actually, the result just quoted follows from the one, proved in Theorem 2.5, that the norm of a JBW^*-triple X is strongly subdifferentiable at a norm-one element x if (and only if) $D(X,x)\cap X_*$ is w^*-dense in $D(X,x)$. Here $D(X,\cdot)$ means the duality mapping of X, and X_* stands for the predual of X. The specialization of this last result to the particular case that X is a von Neumann algebra seems to be previously unknown.

Theorem 2.7 (which has been partially reviewed above) also contains precise determinations of those norm-one elements x, of a JB^*-triple X, such that the norm of X is strongly subdifferentiable at x. Such a determination can be made in terms of either the triple spectrum of x or the support of x in the bidual of X. Our concluding main result (see Theorem 2.12 and Remark 2.13) shows that the JB^*-triples whose norms are strongly subdifferentiable at every point of their unit spheres are precisely the weakly compact JB^*-triples. We note that weakly compact JB^*-triples are well-understood thanks to the work of L. J. Bunce and C. H. Chu [**7**].

Among the consequences of Theorem 2.7, we emphasize the following:

(1) The norm of a JBW^*-triple X is Fréchet differentiable at a norm-one element x if (and only if) X is smooth at x (Corollary 2.10).
(2) The norm of a JB^*-triple X is Fréchet differentiable at a norm-one element x if and only if the support of x in the bidual of X lies in X and is a minimal tripotent of X (Corollary 2.11).

We note that Result (1) above was previously known in the particular case that X is a von Neumann algebra [**26**], and that it proof relies on such a previous fact.

Let us finally comment on a result in [**9**] which has been not previously reviewed in this introduction. In fact, the paper [**9**] concludes with a purely algebraic characterization of those C^*-algebras whose norms are strongly subdifferentiable at every point of a dense subset of their unit spheres. Indeed, such C^*-algebras are precisely the ones with the property that every nonzero left ideal contains a nonzero idempotent. Keeping in mind [**19**, Theorem 6], this last property is equivalent to the one that every nonzero left ideal contains a nonzero self-adjoint idempotent. This suggests that JB^*-triples whose norms are strongly subdifferentiable at every point of a dense subset of their unit spheres could be characterized by the property that every nonzero inner ideal contains a nonzero tripotent. We have tried to prove such a characterization but, for the moment, we have not succeeded in this goal.

2. The results

Let X be a Banach space. We denote by S_X, B_X, and X^* the unit sphere, the closed unit ball, and the (topological) dual, respectively, of X. Let us fix u in B_X. We define the set $D(X,u)$ of all **states** of X relative to u by

$$D(X,u) := \{f \in B_{X^*} : f(u) = 1\}.$$

Note that $D(X,u)$ is nonempty if and only if u actually belongs to S_X, a fact that will be assumed along the remaining part of the present paragraph. Thus $D(X,u)$ is a non-empty $\sigma(X^*,X)$-closed face of B_{X^*}. For x in X, the mapping $\alpha \to \|u+\alpha x\|$ from $\mathbb{R}$ to $\mathbb{R}$ is convex, and hence, putting

$$\tau(u,x) := \inf\{\frac{\|u+\alpha x\|-1}{\alpha} : \alpha > 0\},$$

we have

$$\tau(u,x) = \lim_{\alpha\to 0^+} \frac{\|u+\alpha x\|-1}{\alpha}.$$

It is well-known that, for x in X, the equality

$$\tau(u,x) = \max\{\Re e(f(x)) : f \in D(X,u)\}$$

holds (see for instance [**11**, Theorem V.9.5]). Following [**17**], we say that the norm of X is **strongly subdifferentiable at** u if

$$\lim_{\alpha\to 0^+} \frac{\|u+\alpha x\|-1}{\alpha} = \tau(u,x) \text{ uniformly for } x \in B_X.$$

The reader is referred to [**17**], [**1**], and [**14**] for a comprehensive view of the usefulness of the strong subdifferentiability of the norm in the theory of Banach spaces. Now, recall that X is said to be **smooth at** u whenever $D(X,u)$ is reduced to a singleton, and **Fréchet-smooth at** u whenever there exists $\lim_{\alpha\to 0} \frac{\|u+\alpha x\|-1}{\alpha}$ uniformly for $x \in B_X$. It follows that *X is Fréchet-smooth at u if and only if the norm of X is strongly subdifferentiable at u and X is smooth at u.*

We recall that a JB^***-triple** is a complex Banach space X with a continuous triple product $\{\cdots\} : X \times X \times X \to X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

(1) For all x in X, the mapping $y \to \{xxy\}$ from X to X is a hermitian operator on X and has nonnegative spectrum.

(2) The **main identity**

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

holds for all a, b, x, y, z in X.

(3) $\parallel \{xxx\} \parallel = \parallel x \parallel^3$ for every x in X.

Concerning Condition (1) above, we also recall that a bounded linear operator T on a complex Banach space X is said to be **hermitian** if the equality $\parallel \exp(irT) \parallel = 1$ holds for every r in $\mathbb{R}$. Every C^*-algebra becomes a JB^*-triple under the triple product $\{xyz\} := \frac{1}{2}(xy^*z + zy^*x)$. The main interest of JB^*-triples relies on the fact that, up to biholomorphic equivalence, there are no bounded symmetric domains in complex Banach spaces others than the open unit balls of JB^*-triples (see [**22**] and [**23**]).

An element u of a JB^*-triple is said to be a **tripotent** if $\{uuu\} = u$. If X is a JB^*-triple, and if u is a tripotent of X, then the operator $P_2(u) : X \to X$, defined by $P_2(u)(x) := \{u\{uxu\}u\}$ for all $x \in X$, is a contractive projection on X. Our approach to the subdifferentiability of the norm in the setting of JB^*-triples starts with the following lemma taken from [**15**] (see [**15**, Proposition 1.(a)]).

Lemma 2.1. *Let X be a JB^*-triple, let u be a nonzero tripotentent in X, and let f be in S_{X^*} such that $\|P_2(u)^*(f)\| = 1$. Then we have $P_2(u)^*(f) = f$.*

To get an added value from Lemma 2.1 above, we invoke some notions and results from the theory of (Banach) ultraproducts [**20**]. Let $\mathcal{U}$ be an ultrafilter on a nonempty set I, and let $\{X_i\}_{i\in I}$ be a family of Banach spaces. We can consider the Banach space $\oplus_{i\in I}^{\ell_\infty} X_i$, together with its closed subspace

$$N_{\mathcal{U}} := \{\{x_i\}_{i\in I} \in \oplus_{i\in I}^{\ell_\infty} X_i : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The quotient space $(\oplus_{i\in I}^{\ell_\infty} X_i)/N_{\mathcal{U}}$ is called the **ultraproduct** of the family $\{X_i\}_{i\in I}$ relative to the ultrafilter $\mathcal{U}$, and is denoted by $(X_i)_{\mathcal{U}}$. Let (x_i) stand for the element of $(X_i)_{\mathcal{U}}$ containing a given family $\{x_i\} \in \oplus_{i\in I}^{\ell_\infty} X_i$. It is easy to check that $\|(x_i)\| = \lim_{\mathcal{U}} \|x_i\|$. Moreover, the ultraproduct $(X_i^*)_{\mathcal{U}}$ can be seen as a subspace of $((X_i)_{\mathcal{U}})^*$ by identifying each element $(f_i) \in (X_i^*)_{\mathcal{U}}$ with the (well-defined) functional on $(X_i)_{\mathcal{U}}$ given by $(x_i) \to \lim_{\mathcal{U}}(f_i(x_i))$. On the other hand, as noticed in [**10**], if X_i is a JB^*-triple for every $i \in I$, then $(X_i)_{\mathcal{U}}$ is a JB^*-triple in a natural way.

Lemma 2.2. *Given $\varepsilon > 0$, there exists $\delta > 0$ such that, for every JB^*-triple X, every nonzero tripotent u in X, and every f in B_{X^*} with $\|P_2(u)^*(f)\| > 1 - \delta$, we have $\|P_2(u)^*(f) - f\| < \varepsilon$.*

Proof. Assume that the result is not true. Then there exists $\varepsilon_0 > 0$ such that, for every $n \in \mathbb{N}$, we can find a JB^*-triple X_n, a nonzero tripotent $u_n \in X_n$, and an element $f_n \in B_{X_n^*}$ satisfying $\|P_2(u_n)^*(f_n)\| > 1 - \frac{1}{n}$ and $\|P_2(u_n)^*(f_n) - f_n\| \geq \varepsilon_0$. Take a nontrivial ultrafilter $\mathcal{U}$ in $\mathbb{N}$, and consider the JB^*-triple $X := (X_n)_{\mathcal{U}}$. It follows that $u := (u_n)$ is a nonzero tripotent in X, that $f := (f_n)$ is an element of S_{X^*}, and that, since $P_2(u)^*(f) = (P_2(u_n)^*(f_n))$, we have $\|P_2(u)^*(f)\| = 1$ and $\|P_2(u)^*(f) - f\| \geq \varepsilon_0$. This contradicts Lemma 2.1. ∎

PROPOSITION 2.3. *Given $\varepsilon > 0$, there exists $\delta > 0$ such that, for every JB^*-triple X, every nonzero tripotent u in X, and every x in B_X, we have*

$$\frac{\|u + \alpha x\| - 1}{\alpha} - \tau(u, x) < \varepsilon$$

whenever $0 < \alpha < \delta$.

PROOF. Let $\varepsilon > 0$, let X be a JB^*-triple, and let u be a nonzero tripotent of X. It is well known that $P_2(u)(X)$, endowed with the product $x \diamond y := \{xuy\}$, becomes a norm-unital complete normed (possibly non associative) algebra whose unit is precisely u. Therefore, by the Bishop-Phelps-Bollobás theorem [**4**, Theorem 15.1] and the proof of [**24**, Proposition 4.5], there exists $\delta_1 > 0$ (depending only on ε) such that

(a) $d(h, D(P_2(u)(X), u)) < \frac{\varepsilon}{2}$ whenever h is in $B_{(P_2(u)(X))^*}$ with $|h(u) - 1| < \delta_1$.

On the other hand, by Lemma 2.2, there exists $\delta_2 > 0$ (depending only on ε) such that

(b) $\|P_2(u)^*(f) - f\| < \frac{\varepsilon}{2}$ whenever f is in B_{X^*} with $\|P_2(u)^*(f)\| > 1 - \delta_2$.

Put $\delta_3 := \min\{\delta_1, \delta_2\}$, and let f be in B_{X^*} with $|f(u) - 1| < \delta_3$. Since $|f(u) - 1| < \delta_1$, Property (a) and the Hahn-Banach theorem provide us with some g in $D(X, u)$ satisfying

$$\|P_2(u)^*(f - g)\| = \|f_{|P_2(u)(X)} - g_{|P_2(u)(X)}\| < \frac{\varepsilon}{2}.$$

Since $|f(u) - 1| < \delta_2$, we have

$$\|P_2(u)^*(f)\| \geq |(P_2(u)^*(f))(u)| = |f(u)| > 1 - \delta_2,$$

and hence, by Property (b),

$$\|P_2(u)^*(f) - f\| < \frac{\varepsilon}{2}.$$

It follows that $\|f - P_2(u)^*(g)\| < \varepsilon$, and, since $P_2(u)^*(g)$ lies $D(X, u)$, we realize that $d(f, D(X, u)) < \varepsilon$. Since f is arbitrary in B_{X^*} with $|f(u) - 1| < \delta_3$, we have in particular

(c) $d(\varphi, D(X, u)) < \varepsilon$ whenever φ is in $D(X, v)$ for some v in S_X with $\|v - u\| < \delta_3$.

Finally, putting $\delta := \min\{\frac{\delta_3}{4}, \frac{1}{2}\}$, Property (c) and the proof of $(i) \Rightarrow (ii)$ in [**1**, Theorem 5.1] show that

$$\frac{\|u + \alpha x\| - 1}{\alpha} - \tau(u, x) < \varepsilon$$

whenever x lies in B_X and $0 < \alpha < \delta$. ∎

Given a Banach space X and a subset U of S_X, we say that the norm of X is **uniformly strongly subdifferentiable on** U if

$$\lim_{\alpha\to 0^+} \frac{\|u+\alpha x\|-1}{\alpha} = \tau(u,x) \text{ uniformly for } (u,x)\in U\times B_X.$$

The following corollary follows straightforwardly from Proposition 2.3

Corollary 2.4. *Let X be a JB^*-triple. Then the norm of X is uniformly strongly subdifferentiable on the set of all nonzero tripotents of X.*

JBW^***-triples** are defined as those JB^*-triples having a (complete) predual. Let X be a JBW^*-triple. Then the predual of X (denoted by X_*) is unique, and the triple product of X becomes $\sigma(X,X_*)$-continuous in each of its variables [**2**, Theorem 2.1]. On the other hand, for x in S_X, $D(X,x)\cap X_*$ is a (possibly empty) proper closed face of B_{X_*}, and therefore, by [**12**, Theorem 4.4], there is a unique tripotent u (possibly equal to zero) such that $D(X,x)\cap X_* = D(X,u)\cap X_*$. Such a tripotent u is called **the support of** x **in** X, and will be denoted by $u(X,x)$.

Theorem 2.5. *Let X be a JBW^*-triple, and let x be in S_X. Then the norm of X is strongly subdifferentiable at x if and only if $D(X,x)\cap X_*$ is $\sigma(X^*,X)$-dense in $D(X,x)$.*

Proof. That the strong subdifferentiability of the norm of X at x implies the $\sigma(X^*,X)$-density of $D(X,x)\cap X_*$ in $D(X,x)$ is known to be true even if X is any Banach space with a predual X_*, and x belongs to S_X (apply [**1**, Theorem 3.4] together with the Hahn-Banach separation theorem). Assume that $D(X,x)\cap X_*$ is $\sigma(X^*,X)$-dense in $D(X,x)$. Then $D(X,x)\cap X_*$ is nonempty, so $u := u(X,x)$ is a nonzero tripotent in X, and so, by Corollary 2.4, the norm of X is strongly subdifferentiable at u. Therefore $D(X,u)\cap X_*$ is $\sigma(X^*,X)$-dense in $D(X,u)$. Since

$$D(X,x)\cap X_* = D(X,u)\cap X_*,$$

it follows from our assumption that $D(X,x) = D(X,u)$. Now, applying again that the norm of X is strongly subdifferentiable at u, the subdifferentiability of the norm of X at x follows from the remarkable fact that the strong subdifferentiability of the norm of a Banach space at a point of its unit sphere depends only on the set of states of the point [**14**, Theorem 1.2 and Proposition 3.1]. ∎

The specialization of Theorem 2.5 for von Neumann algebras seems to be previously unknown, and therefore is emphasized in the next corollary.

Corollary 2.6. *Let X be a von Neumann algebra, and let x be in S_X. Then the norm of X is strongly subdifferentiable at x if and only if $D(X,x)\cap X_*$ is $\sigma(X^*,X)$-dense in $D(X,x)$.*

Let X be a Banach space. We recall that the set-valued function $v\to D(X,v)$ on S_X is called the **duality mapping** of X. Now, let u be in S_X, and let τ stand for either the weak or norm topology on X^* (which will be denoted by w or n, respectively). Following [**16**], we say that **the duality mapping of X is upper semicontinuous** $(n-\tau)$ **at** u if for every τ-neighborhood of zero (say B) in X^* there exists a norm-neighborhood of u (say C) in S_X such that $D(X,x) \subseteq D(X,u)+B$ whenever x belongs to C. According to [**17**, Corollary 4.4], *the strong subdifferentiability of the norm of X at u is equivalent to the upper semicontinuity $(n-n)$ of the duality mapping of X at u, and hence implies the upper semicontinuity $(n-w)$ of the duality mapping of X at u.* For the interest of the upper semicontinuity $(n-w)$ of the duality mapping in the general theory of Banach spaces, the reader is referred to [**16**], [**8**], and [**3**].

Let X be a JB^*-triple, and let x be in X. Denote by X_x the closed subtriple of X generated by x. It is well-known that there is a unique couple (S_x, ϕ_x), where S_x is a locally compact subset of $]0,\infty[$ such that $S_x \cup \{0\}$ is compact, and ϕ_x is a surjective triple isomorphism from X_x to the C^*-algebra $C_0(S_x)$ (of all complex-valued continuous functions on S_x vanishing at infinity) such that $\phi_x(x)$ is the inclusion mapping $S_x \hookrightarrow \mathbb{C}$ (see [**22**, 4.8], [**23**, 1.15], and [**15**]). We say that S_x is the **triple spectrum of** x. Since surjective triple isomorphisms between JB^*-triples are isometries (see again [**23**]), we have in fact $X_x = C_0(S_x)$ as JB^*-triples.

We recall that *the bidual X^{**} of every JB^*-triple X is a JBW^*-triple under a suitable triple product which extends the one of X* [**10**].

THEOREM 2.7. *Let X be a JB^*-triple, and let x be in S_X. Then the following assertions are equivalent:*

1. *The norm of X is strongly subdifferentiable at x.*
2. 1 *is an isolated point of the triple spectrum of x.*
3. *There exists a tripotent u in X satisfying $\{uux\} = u$, $\{uxu\} = u$, and $\|x-u\| < 1$.*
4. $u(X^{**}, x)$ *belongs to X.*
5. *The duality mapping of X is upper semicontinuous $n-w$ at x.*

PROOF. (1) $\Rightarrow$ (2).- Clearly 1 lies in S_x and is the unique element ω of S_x such that $x(\omega)=1$. Therefore the unit point measure δ_ω on S_x is the unique extreme point of $D(X_x, x)$, and hence, by the Krein-Milman theorem, X_x is smooth at x. This and the hereditary behaviour of the assumption (1) give that X_x is Fréchet-smooth at x. Now, it is folklore that 1 is an isolated point of S_x (see for example [**13**, Lemma 2.2]).

(2) $\Rightarrow$ (3).- Let u denote the characteristic function of the set $\{1\}$ on S_x. By the assumption (2), u belongs to X_x and is a tripotent in X satisfying $\{uux\} = u$, $\{uxu\} = u$, and $\|x-u\| < 1$.

(3) $\Rightarrow$ (4).- Let u be the tripotent in X given by the assumption (3), and let $P_0(u)$ denote the operator on X^{**} defined by

$$P_0(u)(y) := y - 2\{uuy\} + \{u\{uyu\}u\}$$

for every $y \in X^{**}$. Then we have $\{uxu\} = u$ and $\|P_0(u)(x)\| = \|x - u\| < 1$. Therefore, by [**12**, Lemma 3.4], we have $u(X^{**}, x) = u \in X$.

(4) $\Rightarrow$ (1).- Note that, for y in B_X, the equality

$$D(X^{**}, y) \cap X^* = D(X, y)$$

holds. Therefore, putting $u := u(X^{**}, x)$, the assumption (4) and the definition of the support of x in X^{**} give $D(X, x) = D(X, u)$ (which implies $u \neq 0$). Now, since the norm of X is strongly subdifferentiable at u (by Corollary 2.4), the subdifferentiability of the norm of X at x follows from [**14**, Theorem 1.2 and Proposition 3.1].

(1) $\Rightarrow$ (5).- As commented before, this implication is true at the general level of Banach spaces.

(5) $\Rightarrow$ (1).- By [**16**, Theorem 3.1], the assumption (5) is equivalent to the $\sigma(X^{***}, X^{**})$-density of $D(X, x)$ in $D(X^{**}, x)$. Since

$$D(X^{**}, x) \cap X^* = D(X, x),$$

Theorem 2.5 applies giving that the norm of X^{**} (and hence that of X) is strongly subdifferentiable at x. ∎

Theorem 2.7 above becomes an appropriate version for JB^*-triples of [**9**, Theorem 1], where the subdifferentiability of the norm of a C^*-algebra at a point of its unit sphere is characterized in several ways. However, in [**9**] the following immediate consequence of Theorem 2.7 is not noticed.

Corollary 2.8. *Let X be a C^*-algebra, and let x be in S_X. Then the norm of X is strongly subdifferentiable at x if (and only if) the duality mapping of X is upper semicontinuous $n - w$ at x.*

The equivalence (1) $\Longleftrightarrow$ (2) in Theorem 2.7 has the following remarkable consequence.

Corollary 2.9. *Let X be a JB^*-triple, let Y be a closed subtriple of X, and let y be in S_Y. Then the norm of X is strongly subdifferentiable at y if (and only if) the norm of Y is strongly subdifferentiable at y.*

The main result of [**26**] asserts that, if X is a von Neumann algebra, and if x belongs to S_X, then X is Fréchet-smooth at x if (and only if) X is smooth at x. With the help of Corollary 2.9 above, we can easily generalize such a result to the case that X is a JBW^*-triple.

Corollary 2.10. *Let X be a JBW^*-triple, and let x be in S_X. Then X is Fréchet-smooth at x if (and only if) X is smooth at x.*

Proof. Let Y denote the smallest $\sigma(X, X_*)$-closed subtriple of X containing x. Then Y is linearly isomorphic to a von Neumann algebra [**21**, p. 122]. Assume that X is smooth at x. Then Y is also smooth at x, so that, by [**26**, Theorem], Y is Fréchet-smooth at x. Therefore, by Corollary 2.9, the norm of X is strongly subdifferentiable at x. Finally, applying again that X is smooth at x, the Fréchet-smmothness of X at x follows. ∎

Following [**15**, p. 79], we say that a tripotent u of a JB^*-triple X is a **minimal tripotent of** X if $u \neq 0$ and $\{u\{uXu\}u\} = \mathbb{C}u$. When X is actually a JBW^*-triple, it is well-known that minimal tripotents of X just defined are nothing but those tripotents in X which are minimal relative to the order on the set of all tripotents of X defined by $u \leq v$ if and only if $\{v\{vuv\}v\} = u$. Thus, an easy consequence of [**12**, Theorem 4.4] is that *a JB^*-triple X is smooth at a point x of its unit sphere if and only if $u(X^{**}, x)$ is a minimal tripotent of X^{**}* [**13**, Theorem 3.1.(i)].

COROLLARY 2.11. *Let X be a JB^*-triple, and let x be in S_X. Then X is Fréchet-smooth at x if and only if $u(X^{**}, x)$ lies in X and is a minimal tripotent of X.*

PROOF. It is enough to put together the equivalence (1) $\Longleftrightarrow$ (4) in Theorem 2.7, the characterization of the smoothness of X at x commented above, and the fact that a tripotent u in X is a minimal tripotent of X (if and) only if it is a minimal tripotent of X^{**} (by the $\sigma(X^{**}, X^*)$-density of X in X^{**} and the separate $\sigma(X^{**}, X^*)$-continuity of the triple product of X^{**}). ∎

Now we are going to determine those JB^*-triples whose norms are strongly subdifferentiable at every point of their unit spheres. It will turn out that the class of such JB^*-triples was well-understood in the literature, and hence abundant different characterizations of its members were known (see Remark 2.13 below). We recall that a closed subspace Y of a Banach space X is said to be an M**-ideal** of X if there is a linear projection π on X^* satisfying $\pi(X^*) = Y^\circ$ (the polar of Y in X^*) and $\|f\| = \|\pi(f)\| + \|f - \pi(f)\|$ for every $f \in X^*$. We also recall that a Banach space is called M**-embedded** whenever it is an M-ideal of its bidual. **Ideals** of a JB^*-triple X are defined as those subspaces Y of X such that $\{YXX\} + \{XYX\} \subseteq Y$.

THEOREM 2.12. *Let X be a JB^* triple. Then the following assertions are equivalent:*

1. *The norm of X is strongly subdifferentiable at every point of S_X.*
2. *For every $x \in X$, the triple spectrum of x is discrete.*
3. *X is an ideal of X^{**}.*
4. *X is M-embedded.*
5. *The identity mapping on B_{X^*} is continuous $\sigma(X^*, X) - \sigma(X^*, X^{**})$ at every point of S_{X^*}.*
6. *For every $x \in S_X$, the equality $D(X^{**}, x) = D(X, x)$ holds.*
7. *The duality mapping of X is upper semicontinuous $n - w$ at every point of S_X.*

PROOF. (1) $\Rightarrow$ (2).- Assume that Assertion (2) does not holds, so that there exists $x \in S_X$ such that S_x is not discrete. Take a non isolated point α of S_x, and consider the function y on S_x defined by $y(\omega) := \frac{\omega}{\omega + |\omega - \alpha|}$. Then y is a positive norm-one element of $C_0(S_x)$, and α is the unique element ω

of S_x such that $y(\omega) = 1$. Since α is not an isolated point of S_x, it follows from [**13**, Lemma 2.2] that $C_0(S_x)$ is smooth but not Fréchet smooth at y. Therefore the norm of $C_0(S_x)$ is not strongly subdifferentiable at y. Since $C_0(S_x)$ is linearly isometric to $X_x \subseteq X$, and strong subdifferentiability of the norm behaves hereditarily, Assertion (1) fails.

(2) $\Rightarrow$ (3).- By [**7**, Proposition 4.5.(ii), Theorem 3.4, and Lemma 3.3.(iii)].

(3) $\Rightarrow$ (4).- By [**2**, Theorem 3.2].

(4) $\Rightarrow$ (5).- By [**18**, Corollary III.2.15].

(5) $\Rightarrow$ (6).- For every Banach space Z, let $J_Z : Z \to Z^{**}$ stand for the canonical injection, and let $\Pi_Z := J_{Z^*} \circ (J_Z)^* : Z^{***} \to Z^{***}$ be the Dixmier projection on Z^{***}. Let x be in S_X, and let ϕ be in $D(X^{**}, x)$. Then $\Pi_X(\phi)$ is a norm preserving linear extension of $(J_X)^*(\phi) \in D(X, x)$ to X^{**}. But, by the assumption (5) and [**18**, Lemma 2.14], such an extension is unique. Therefore we have $\Pi_X(\phi) = \phi$, and hence ϕ lies in $D(X^{**}, x) \cap X^* = D(X, x)$.

(6) $\Rightarrow$ (7).- By [**16**, Theorem 3.1].

(7) $\Rightarrow$ (1).- By Theorem 2.7. ■

Remark 2.13. (a).- As can be deduced from references [**7**] and [**2**] applied in the above proof, the equivalences (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) in Theorem 2.12 are previously known. Actually, as commented before, JB^*-triples fulfilling some (hence all) of Conditions (1) to (7) of Theorem 2.12 can be characterized in several other ways. In order to review such other characterizations, let us recall some definitions. Let X be a JB^*-triple. X is said to be **weakly compact** if, for every x in X, the operator $y \to \{xyx\}$ from X to X is weakly compact. Subspaces Y of X such that $\{YXY\} \subseteq Y$ are called **inner ideals** of X. The **socle** of X is defined as the sum of all inner ideals of X. The concept of a **modular annihilator** JB^*-triple is quite involved and, consequently, we limit ourselves to refer the interested reader to [**5**, pp. 392 and 383] for the definition. Nevertheless, we point out that such a concept is of a purely algebraic nature and works in a setting much larger than that of JB^*-triples. **Elementary** JB^*-triples can be defined as those JB^*-triples X with no nonzero proper closed ideals and such that X is the closed linear hull of its minimal tripotents [**7**, Lemma 3.3.(iii)]. We remark that elementary JB^*-triples can be perfectly described (see either [**6**, p. 330], [**7**, p. 250] or [**5**, p. 394]) and that, when X is in fact a C^*-algebra, then X is elementary if and only if X is the C^*-algebra of all compact operators on some complex Hilbert space. Now, it follows from [**7**] and [**5**] that, *for a JB^*-triple X, each of Conditions* (1) *to* (7) *in Theorem 2.12 is equivalent to any of the following:*

(8) *X is weakly compact.*
(9) *X has dense socle.*
(10) *X is modular annihilator.*
(11) *X is an inner ideal of X^{**}.*
(12) *X is the c_0-sum of a suitable family of elementary JB^*-triples.*

(b).- As can be deduced from the proof of Theorem 2.12, the implications (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are valid for every Banach space X.

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