Relatively weakly open sets in closed balls of Banach spaces, and real JB^*-triples of finite rank

Julio Becerra Guerrero, Ginés López Pérez, Antonio M. Peralta, and A. Rodríguez-Palacios

ABSTRACT. We prove that, given a real JB^*-triple X, there exists a nonempty relatively weakly open subset of the closed unit ball of X with diameter less than 2 (if and) only if the Banach space of X is isomorphic to a Hilbert space. Moreover we give the structure of real JB^*-triples whose Banach spaces are isomorphic to Hilbert spaces. Such real JB^*-triples are also characterized in two different purely algebraic ways.

1. Introduction

In [**24**], O. Nygaard and D. Werner discover, perhaps by the first time, how some of the classical Banach spaces which fail to the Radon-Nikodym property actually fail to fulfil much weaker requirements. Indeed, it is proved in [**24**] that, if X is an infinite-dimensional uniform algebra, then X satisfies Property $\mathcal{P}$ which follows:

($\mathcal{P}$) Every nonempty relatively weakly open subset of the closed unit ball of X has diameter equal to 2.

This applies in particular to infinite-dimensional real or complex $C(\Omega)$-spaces. We remark that Property $\mathcal{P}$ is fulfilled by every Banach space satisfying the so-called Daugavet property [**31**, Lemma 3]. Other Banach spaces enjoying Property $\mathcal{P}$ are all infinite-dimensional complex C^*-algebras, and all JB-algebras whose Banach spaces are not isomorphic to Hilbert spaces [**3**]. The proof that infinite-dimensional complex C^*-algebras satisfy Property $\mathcal{P}$, given in [**3**], essentially relies on the more general fact, also shown in [**3**], that Property $\mathcal{P}$ is fulfilled by every complex JB^*-triple whose Banach space is not isomorphic to a Hilbert space. For JB-algebras and complex JB^*-triples the reader is referred to [**12**] and [**17**], respectively.

1991 *Mathematics Subject Classification.* Primary 46B04, 46B22, 46L05, 46L70.

Partially supported by Junta de Andalucía grant FQM 0199.

In the present paper we prove as main result that every real JB^*-triple X whose Banach space is not isomorphic to a Hilbert space satisfies Property $\mathcal{P}$ (Theorem 2.3). We note that the class of real JB^*-triples, introduced in [**14**], contains that of complex JB^*-triples (regarded as real Banach spaces), as well as that of JB-algebras. It is worth mentioning that the proof of Theorem 2.3 provided here is independent of the one given in [**3**] for complex JB^*-triples, and has the advantage that it works autonomously in all relevant subclasses of the class of real JB^*-triples. In fact the key tool in the proof of Theorem 2.3 is that every nonreflexive Banach space X such that X^* is an L-summand of X^{***} fulfils Property $\mathcal{P}$ (Proposition 2.1). Thus, since infinite-dimensional complex C^*-algebras have nonreflexive Banach spaces, and the dual of every complex C^*-algebra X is an L-summand of X^{***}, we are provided with a JB^*-triple-free proof of the main result in [**3**] that infinite-dimensional complex C^*-algebras X satisfy Property $\mathcal{P}$. By the way, as a consequence of Theorem 2.3, every infinite-dimensional real C^*-algebra fulfils Property $\mathcal{P}$ (Corollary 2.6).

Theorem 2.3 also contains a precise description of real JB^*-triples whose Banach spaces are isomorphic to Hilbert spaces. Indeed, such real JB^*-triples are nothing other than those which can be written as finite ℓ_∞-sums of simple JB^*-triples which are either finite-dimensional, infinite-dimensional generalized real spin factors, or of the form $\mathcal{L}(H, K)$ for some real, complex, or quaternionic Hilbert spaces H, K with $\dim(H) = \infty$ and $\dim(K) < \infty$. (Here, for H and K as above, $\mathcal{L}(H, K)$ means the real Banach space of all bounded linear operators from H to K.) We complete the description just reviewed by showing two different purely algebraic characterizations of real JB^*-triples whose Banach spaces are isomorphic to Hilbert spaces. Indeed, they are precisely the real JB^*-triples of finite rank (Theorem 3.1) as well as those real JB^*-triples such that all single-generated subtriples are finite-dimensional (Theorem 3.8).

To conclude this introduction, let us note that our results include the generalizations to real JB^*-triples of the facts proved by Bunce-Chu [**6**] and Kaup [**18**] concerning Radon-Nikodym Property and finite rank, respectively, on complex JB^*-triples.

2. The main result

Let X be a real or complex Banach space. We denote by S_X, B_X, and X^* the unit sphere, the closed unit ball, and the (topological) dual, respectively, of X. We denote by w the weak topology of X, and by w^* the weak* topology of X^*. We always see X as a subspace of its bidual X^{**} via the canonical injection, and, given a subspace P of X, we denote by P° the polar of P in X^*. Also, for a bounded linear operator T on X, we denote by T^* the transpose operator on X^*. An **L-projection** (respectively, **M-projection**) on X is a linear projection (say π) on X satisfying

$\|x\| = \|\pi(x)\| + \|x - \pi(x)\|$ (respectively, $\|x\| = \max\{\|\pi(x)\|, \|x - \pi(x)\|\}$) for every $x \in X$. A subspace P of X is said to be an ***L*-summand** (respectively, ***M*-summand**) of X if it is the range of an L-projection (respectively, M-projection) on X, and an ***M*-ideal** of X if P° is an L-summand of X^*. The Banach space X is said to be ***L*-embedded** (respectively, ***M*-embedded**) whenever X is an L-summand (respectively, an M-ideal) of X^{**}. According to [**13**, Proposition III.1.2], X is M-embedded if and only if the Dixmier projection on X^{***} is an L-projection. Consequently, if X is M-embedded, then X^* is L-embedded.

PROPOSITION 2.1. *Let X be a Banach space such that X^* is L-embedded. If there exists a nonempty relatively w-open subset of B_X with diameter less than 2, then X is reflexive.*

PROOF. We have $X^{***} = (X^* \oplus N)_{\ell_1}$ for some subspace N of X^{***}, and hence $X^{****} = ((X^*)^\circ \oplus N^\circ)_{\ell_\infty}$. Assume that there is a nonempty relatively w-open subset U of B_X with $\operatorname{diam}(U) < 2$. Then U contains a set V of the form

$$\{x \in B_X : |f_i(x - x_0)| < 1 \ \ \forall i = 1, ..., n\},$$

for suitable $x_0 \in B_X$, $n \in \mathbb{N}$, and $f_1, ..., f_n \in X^*$. Put

$$V^{**} := \{z \in B_{X^{**}} : |f_i(z - x_0)| < 1 \ \ \forall i = 1, ..., n\}.$$

Since V^{**} is relatively w^*-open in $B_{X^{**}}$, and B_X is w^*-dense in $B_{X^{**}}$, the set V $(= V^{**} \cap B_X)$ is w^*-dense in V^{**}. Therefore $V - V$ is w^*-dense in $V^{**} - V^{**}$, and consequently , by the lower w^*-semicontinuity of the norm of X^{**}, we have $\operatorname{diam}(V^{**}) = \operatorname{diam}(V) \leq \operatorname{diam}(U) < 2$. In the same way, the set

$$V^{****} := \{\beta \in B_{X^{****}} : |f_i(\beta - x_0)| < 1 \ \ \forall i = 1, ..., n\}$$

has diameter less than 2. Write $x_0 = u + v$ with $(u, v) \in (X^*)^\circ \times N^\circ$. We claim that $B_{(X^*)^\circ} + v$ is contained in V^{****}. Indeed, for $\alpha \in B_{(X^*)^\circ}$, $\alpha + v$ belongs to $B_{X^{****}}$ because $X^{****} = ((X^*)^\circ \oplus N^\circ)_{\ell_\infty}$, and, on the other hand, for every $i = 1, ..., n$ we have $f_i(\alpha + v - x_0) = f_i(\alpha - u) = 0$ because $(\alpha - u, f_i)$ belongs to $(X^*)^\circ \times X^*$. Keeping in mind that $\operatorname{diam}(V^{****}) < 2$, it follows from the claim just shown that $\operatorname{diam}(B_{(X^*)^\circ}) = \operatorname{diam}(B_{(X^*)^\circ} + v) < 2$. Therefore X is reflexive. ■

We recall that a **complex JB^*-triple** is a complex Banach space X with a continuous triple product $\{...\} : X \times X \times X \to X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

(1) For all x in X, the mapping $y \to \{xxy\}$ from X to X is a hermitian operator on X and has nonnegative spectrum.
(2) The **main identity**

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

holds for all a, b, x, y, z in X.

(3) $\|\{xxx\}\| = \|x\|^3$ for every x in X.

Concerning Condition (1) above, we also recall that a bounded linear operator T on a complex Banach space X is said to be **hermitian** if $\|\exp(irT)\| = 1$ for every r in $\mathbb{R}$. Following [**14**], we define **real** JB^***-triples** as norm-closed real subtriples of complex JB^*-triples. Here, by a **subtriple** we mean a subspace which is closed under triple products of its elements. A **triple ideal** of a real or complex JB^*-triple X is a subspace M of X such that $\{XXM\}+\{XMX\} \subseteq M$. We say that the JB^*-triple X is **simple** if there are not triple ideals of X others than $\{0\}$ and X. **Real** JBW^***-triples** where first introduced as those real JB^*-triples which are dual Banach spaces in such a way that the triple product becomes separately w^*-continuous (see [**14**, Definition 4.1 and Theorem 4.4]). Later, it has been shown in [**22**] that the requirement of separate w^*-continuity of the triple product is superabundant. We will apply without notice that the bidual of every real JB^*-triple X is a JBW^*-triple under a suitable triple product which extends the one of X [**14**, Lemma 4.2].

The next proposition becomes the real variant of [**2**, Proposition 3.4].

PROPOSITION 2.2. *The predual of every real JBW^*-triple is L-embedded.*

PROOF. Let X be a real JBW^*-triple, and let X_* stand for the predual of X. For $x, y \in X$, let $L(x, y)$ and $Q(x, y)$ denote the operators on X defined by $L_{x,y}(z) := \{xyz\}$ and $Q_{x,y}(z) := \{xzy\}$, respectively. By standard theory of duality, the separate w^*-continuity of the triple product of X is equivalent to the inclusions $(L_{x,y})^*(X_*) \subseteq X_*$ and $(Q_{x,y})^*(X_*) \subseteq X_*$ for all $x, y \in X$. Therefore we have $(L_{x,y})^{**}((X_*)^\circ) \subseteq (X_*)^\circ$ and $(Q_{x,y})^{**}((X_*)^\circ) \subseteq (X_*)^\circ$ for all $x, y \in X$. Keeping in mind the separate w^*-continuity of the triple product of X^{**} and the w^*-density of X in X^{**}, the above inclusions read as $\{XX(X_*)^\circ\} \subseteq (X_*)^\circ$ and $\{X(X_*)^\circ X\} \subseteq (X_*)^\circ$, respectively. Applying again the separate w^*-continuity of the triple product of X^{**} and the w^*-density of X in X^{**}, we deduce

$$\{X^{**}X^{**}(X_*)^\circ\} + \{X^{**}(X_*)^\circ X^{**}\} \subseteq (X_*)^\circ.$$

Therefore $(X_*)^\circ$ is a w^*-closed triple ideal of X^{**}, and hence we have $X^{**} = (X_*)^\circ \oplus P$ for a suitable closed triple ideal P of X^{**} [**14**, Lemma 4.3]. Now, the abstract ℓ_∞-product $((X_*)^\circ \times P)_{\ell_\infty}$ is a real JB^*-triple in a natural way, and the mapping $\Phi : (u, v) \to u+v$ from $((X_*)^\circ \times P)_{\ell_\infty}$ to X^{**} becomes a linear bijection preserving triple products. By [**14**, Theorem 4.8], Φ is an isometry, that is $X^{**} = ((X_*)^\circ \oplus P)_{\ell_\infty}$. Thus $(X_*)^\circ$ is an M-summand of X^{**}, and hence, by [**13**, Theorem I.1.9], X_* is an L-summand of X^*. ∎

Examples of real JB^*-triples are the spaces $\mathcal{L}(H, K)$, for arbitrary real, complex, or quaternionic Hilbert spaces H and K, under the triple product $\{xyz\} := \frac{1}{2}(xy^*z + zy^*x)$. The above examples become particular cases of those arising by considering either the so-called complex Cartan factors (regarded as real JB^*-triples) or real forms of complex Cartan factors [**19**].

We recall that **real forms** of a complex Banach space X are defined as the real closed subspaces of X of the form $X^\tau := \{x \in X : \tau(x) = x\}$, for some conjugation (i.e., conjugate-linear isometry of period two) on X. We note that, if X is a complex JB^*-triple, then every real form of X is a real JB^*-triple (since conjugations on X preserve triple products [**17**]). Among complex Cartan factors, the so-called **complex spin factors** become specially relevant for our present approach. They are built from an arbitrary complex Hilbert space $(H, (\cdot|\cdot))$ of hilbertian dimension ≥ 3, by taking a conjugation σ on H, and then by defining the triple product and the norm by

$$\{xyz\} := (x|y)z + (z|y)x - (x|\sigma(z))\sigma(y)$$

and

$$\|x\|^2 := (x|x) + \sqrt{(x|x)^2 - |(x|\sigma(x))|^2},$$

respectively, for all x, y, z in H. Following [**23**], we say that a real JB^*-triple is a **generalized real spin factor** if it is either a complex spin factor (regarded as a real JB^*-triple) or a real form of a complex spin factor.

THEOREM 2.3. *Let X be a real JB^*-triple. Then the following assertions are equivalent:*

(1) *There exists a relatively w-open subset of B_X with diameter less than two.*
(2) *The Banach space of X is reflexive.*
(3) *X is a finite ℓ_∞-sum of closed simple triple ideals which are either finite-dimensional, infinite-dimensional generalized real spin factors, or of the form $\mathcal{L}(H, K)$ for some real, complex, or quaternionic Hilbert spaces H, K with $\dim(H) = \infty$ and $\dim(K) < \infty$.*
(4) *The Banach space of X is isomorphic to a Hilbert space.*
(5) *X has the Radon-Nikodym property.*

PROOF. (1) $\Rightarrow$ (2).- By Proposition 2.2, X^* is L-embedded. Then Assertion (2) follows from the assumption (1) and Proposition 2.1.

(2) $\Rightarrow$ (3).- By [**14**, Proposition 2.2], there exists a complex JB^*-triple Y, and a conjugation τ on Y such that $X = Y^\tau$. By the assumption (2), the Banach space of X (and hence that of Y) is reflexive. By the concluding part of the proof of [**6**, Proposition 4.5], we have $Y = (\oplus_{i=1}^n Y_i)_{\ell_\infty}$ where $\{Y_i\}_{i=1,\dots,n}$ is the family of all minimal triple ideals of Y, and moreover, for $i = 1, ..., n$, Y_i is either finite-dimensional, an infinite-dimensional complex spin factor, or of the form $\mathcal{L}(H, K)$ for suitable complex Hilbert spaces H, K with $\dim(H) = \infty$ and $\dim(K) < \infty$. Since τ preserves the triple product of Y, and is of period two, the set $\{1, ...n\}$ must be the disjoint union of three subsets A, B, C such that Y_i is τ-invariant whenever i belongs to A, and for each $i \in B$ there is a unique $j \in C$ with $\tau(Y_i) = Y_j$. It follows that, putting $Z_j := (Y_j \oplus \tau(Y_j))_{\ell_\infty}$ whenever j lies in B, each Z_j is τ-invariant, and we have

$$X = ((\oplus_{i\in A} Y_i^\tau)_{\ell_\infty} \oplus (\oplus_{j\in B} Z_j^\tau)_{\ell_\infty})_{\ell_\infty}.$$

Since, for $j \in B$, the mapping $y_j \to y_j + \tau(y_j)$ from Y_j (regarded as a real JB^*-triple) to Z_j^τ is a surjective linear isometry preserving triple products, we can write

$$X = ((\oplus_{i\in A} Y_i^\tau)_{\ell_\infty} \oplus (\oplus_{j\in B} Y_j)_{\ell_\infty})_{\ell_\infty}.$$

Now, to conclude the proof it is enough to show that, if there exists $i \in A$ such that Y_i is of the form $\mathcal{L}(H,K)$ for suitable complex Hilbert spaces H, K with $\dim(H) = \infty$ and $\dim(K) < \infty$, then Y_i^τ is of the form $\mathcal{L}(H', K')$ for suitable real or quaternionic Hilbert spaces H', K' with $\dim(H') = \infty$ and $\dim(K') < \infty$. But this follows from [**19**, Theorem 4.1].

(3) $\Rightarrow$ (4).- This implication is clear.

(4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1).- These implications are true even if X is only assumed to be an arbitrary Banach space. Indeed, Banach spaces isomorphic to Hilbert spaces are reflexive, reflexive Banach spaces have the Radon-Nikodym property, the Radon-Nikodym property implies the existence of "slices" of the unit closed ball with arbitrarily small diameter, and such slices are w-open relative to the closed unit ball. ∎

For the determination of finite-dimensional simple real JB^*-triples (including finite-dimensional simple complex JB^*-triples) and of real forms of complex spin factors, the reader is referred to [**21**] and [**19**], respectively.

The next corollary follows straightforwardly from Propositions 2.2 and 2.1, and Theorem 2.3. The Banach space appearing in it could be considered as an arbitrary "nonassociative Lindenstrauss space".

Corollary 2.4. *Let X be a real Banach space such that X^{**} is a real JB^*-triple (for some triple product). Then there exists a nonempty relatively weakly open subset of the closed unit ball of X with diameter less than 2 (if and) only if X is isomorphic to a Hilbert space.*

Let X be a Banach space. For u in S_X, we define the **roughness of** X **at** u, $\eta(X, u)$, by the equality

$$\eta(X,u) \;:=\; \limsup_{\|h\|\to 0} \frac{\|u+h\| + \|u-h\| - 2}{\|h\|} \;.$$

We remark that the absence of roughness of X at u (i.e., $\eta(X, u) = 0$) is nothing other than the Fréchet differentiability of the norm of X at u [**9**, Lemma I.1.3]. Given $\epsilon > 0$, the Banach space X is said to be ϵ**-rough** if, for every u in S_X, we have $\eta(X, u) \geq \epsilon$. We say that X is **rough** whenever it is ϵ-rough for some $\epsilon > 0$, and **extremely rough** whenever it is 2-rough.

Invoking the proof of [**9**, Proposition I.1.11], as is done in the proof of [**3**, Corollary 2.7], the following corollary follows from Theorem 2.3.

Corollary 2.5. *Let X be the predual of a real JBW^*-triple. Then X is extremely rough if (and only if) it is not isomorphic to a Hilbert space.*

Despite **real C^*-algebras** can be defined by different systems of intrinsic axioms (see [**15**] for a summary), we prefer to introduce them as the norm-closed self-adjoint real subalgebras of complex C^*-algebras. Since complex

C^*-algebras are complex JB^*-triples under the triple product

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x),$$

certainly real C^*-algebras are real JB^*-triples.

COROLLARY 2.6. *Let X be a real C^*-algebra. If there exists a non-empty relatively w-open subset of B_X with diameter less than two, then X is finite dimensional.*

PROOF. By [**27**, 4.1.13] and [**11**, 15.4], there exists a complex C^*-algebra Y with a conjugate-linear $*$-automorphism τ of period two such that $X = Y^\tau$. Assume that there is a non-empty relativey w-open subset of B_X with diameter less than two. Then, by Theorem 2.3, the Banach space of X (and hence that of Y) is reflexive. Finally, apply that complex C^*-algebras whose Banach spaces are reflexive actually are finite-dimensional [**30**]. ∎

Real W^*-algebras are usually defined as those real C^*-algebras which are dual Banach spaces in such a way that the product becomes separately w^*-continuous (see for instance [**7**]). Nevertheless, the requirement of separate w^*-continuity of the product is superabundant [**15**].

COROLLARY 2.7. *Let X be the predual of a real W^*-algebra. If X is not extremely rough, then X is finite-dimensional.*

3. Algebraic characterizations of real JB^*-triples whose Banach spaces are isomorphic to Hilbert spaces

In this section we are going to prove two purely algebraic characterizations of real JB^*-triples whose Banach spaces are isomorphic to Hilbert spaces. To this end, it seems to us convenient to introduce the appropriate concepts and basic results at a suitable level of generality. The relevant references in such a level are [**16**] and [**20**].

From now on, let $\mathbb{F}$ be an arbitrary field of characteristic different from 2 and 3. A **Jordan algebra** over $\mathbb{F}$ is a (possibly non associative) commutative algebra over $\mathbb{F}$ satisfying the identity $(xy)x^2 = x(yx^2)$. An element x in a Jordan algebra X with a unit $\mathbf{1}$ is said to be **invertible** if there exists $y \in X$ such that $xy = \mathbf{1}$ and $x^2y = x$. A **division Jordan algebra** is a nonzero unital Jordan algebra whose nonzero elements are invertible. A **Jordan triple** over $\mathbb{F}$ is a vector space (say X) over $\mathbb{F}$ endowed with a TRILINEAR triple product $\{...\} : X \times X \times X \to X$ satisfying the same main identity required in the definition of complex JB^*-triples. Now, let X be a Jordan triple over $\mathbb{F}$. A **tripotent** of X is an element $u \in X$ such that $\{uuu\} = \sigma u$ where $\sigma = \sigma(u) = \pm 1$. Those tripotents u in X satisfying $\sigma(u) = 1$ are called **positive**. Given a tripotent u in X, we have

$X = X_0(u) \oplus X_1(u) \oplus X_2(u)$, where, for $j \in \{0,1,2\}$, $X_j(u)$ denotes the eigenspace of the operator $x \to \sigma\{u,u,x\}$ corresponding to the eigenvalue $\frac{1}{2}j$. The space $X_2(u)$ becomes a Jordan algebra over $\mathbb{F}$ with unit u under the product $xy := \sigma\{xuy\}$, and moreover the operator $x \to \sigma\{uxu\}$ on $X_2(u)$ is a linear algebra involution. The tripotent u is said to be a **division tripotent** whenever $X_2(u)$ is a division Jordan algebra. Two tripotents u, v of X are called **orthogonal** if $u \in X_0(v)$, or equivalently $v \in X_0(u)$. By a **frame** in X we mean a family $\mathcal{E}$ of pairwise orthogonal division tripotents of X such that $\bigcap_{u\in\mathcal{E}} X_0(u) = 0$. We say that X is of **finite rank** if there exists a finite frame in X.

Now, the first main result of this section reads as follows.

Theorem 3.1. *Let X be a real JB^*-triple. Then the Banach space of X is isomorphic to a Hilbert space if and only if X is of finite rank.*

The proof of Theorem 3.1 above needs further auxiliary notions and results. **Complex JB^*-algebras** are defined as those complete normed Jordan complex algebras X endowed with a conjugate-linear algebra involution $*$ satisfying $\|U_x(x^*)\| = \|x\|^3$ for every x in X, where, for x in X, the operator $U_x : X \to X$ is defined by $U_x(y) = 2x(xy) - x^2y$ [**33, 5, 32**]. We define **real JB^*-algebras** as the norm-closed self-adjoint real subalgebras of complex JB^*-algebras. Real JB^*-algebras were introduced by K. Alvermann [**1**] (under the name of J^*B-algebras), who provided a system of intrinsic axioms for them. **JB-algebras** are defined as those complete normed Jordan real algebras X satisfying $\| x \|^2 \leq \| x^2 + y^2 \|$ for all $x, y \in X$ [**12**]. Note that, although complex JB^*-triples are only real Jordan triples, if u is an (automatically positive) tripotent in a complex JB^*-triple X, then the Jordan algebra $X_2(u)$ is a complex algebra in a natural way, and the canonical involution of $X_2(u)$ becomes conjugate-linear, so that the norm of X converts $X_2(u)$ into a complex JB^*-algebra.

Proposition 3.2. *If K is a nonzero real Hilbert space, and if u is any element in S_K, then the Banach space of K, with product and involution defined by*

$$xy := (x|u)y + (y|u)x - (x|y)u$$

and

$$x^* := 2(x|u)u - x,$$

respectively, becomes a division real JB^*-algebra (denoted by $J(K,u)$), whose unit is precisely u. Moreover, there are no division real JB^*-algebras others than those constructed above. More precisely, for a nonzero unital real JB^*-algebra X, the following condition are equivalent:

(1) *X is a division Jordan algebra.*
(2) *The self-adjoint part X_{sa} of X reduces to $\mathbb{R}\mathbf{1}$.*
(3) *$X = J(K,u)$ for some nonzero real Hilbert space K and some $u \in S_K$.*

Proof. Let K be a nonzero real Hilbert space. If the dimension of K is 1 or 2, then $J(K,u)$ is $\mathbb{R}$ or $\mathbb{C}$, so that certainly $J(K,u)$ is a division real JB^*-algebra. Assume that $\dim(K) > 2$. Let H denote the hilbertian complexification of K, let σ stand for the natural conjugation

$$k_1 + ik_2 \to k_1 - ik_2$$

on H, and let Y be the complex spin factor built from (H,σ). Since u is a tripotent in Y with $Y_2(u) = Y$, Y becomes naturally a complex JB^*-algebra with unit u. Since σ is a conjugate linear $*$-automorphism of the complex JB^*-algebra Y, and $J(K,u) = Y^\sigma$, we deduce that $J(K,u)$ is a real JB^*-algebra. Moreover, $J(K,u)$ is a division Jordan algebra because, for $x \in J(K,u) \setminus \mathbb{R}u$, the subalgebra generated by x is a copy of $\mathbb{C}$.

Now, let X be a unital real JB^*-algebra. We have just proved that Condition (3) in the statement implies Condition (1).

(1) $\Rightarrow$ (2).- As a consequence of [**12**, Proposition 3.8.2], the self-adjoint part X_{sa} of X is a JB-algebra. Assume that (2) does not holds, so that there exists $x \in X_{sa} \setminus \mathbb{R}\mathbf{1}$. Then, since Jordan algebras are power associative [**12**, Lemma 2.4.5], the closed subalgebra of X_{sa} generated by x and $\mathbf{1}$ is a unital associative JB-algebra, and hence it is of the form $C^{\mathbb{R}}(\Omega)$ for some compact Hausdorff topological space Ω [**12**, Theorem 3.2.2], which must have at least two points. Take $y, z \in C^{\mathbb{R}}(\Omega) \setminus \{0\}$ such that $yz = 0$. Then we have $U_y(z) = 0$, which, in view of [**16**, Theorem I.13.(2)], implies that y is not invertible in X, and hence that X is not a Jordan division algebra.

(2) $\Rightarrow$ (3).- Let X_{sk} denote the skew part of X. Since the product of two skew elements of X is self-adjoint, the assumption (2) provides us with a symmetric bilinear form $(\cdot|\cdot)$ on X_{sk} satisfying $zt = -(z|t)\mathbf{1}$ for all $z, t \in X_{sk}$. Since $X = \mathbb{R}\mathbf{1} \oplus X_{sk}$, we can extend $(\cdot|\cdot)$ to a symmetric bilinear form on X by defining $(\mathbf{1}|\mathbf{1}) := 1$ and $(\mathbf{1}|z) := 0$ for every $z \in X_{sk}$. Then we have

$$xy := (x|\mathbf{1})y + (y|\mathbf{1})x - (x|y)\mathbf{1}$$

and

$$x^* := 2(x|\mathbf{1})\mathbf{1} - x$$

for all $x, y \in X$. Moreover, since, for $x \in X$, the (automatically self-adjoint) subalgebra of X generated by x is a real C^*-algebra, and the equality $x^*x = (x|x)\mathbf{1}$ holds, we deduce that $(\cdot|\cdot)$ is an inner product on X satisfying $(x|x) = \|x\|^2$. Thus the Banach space of X is a Hilbert space (say K), and we have $X = J(K,\mathbf{1})$. ∎

We note that, if K is a real Hilbert space, and if u, v are in S_K, then we have $J(K,u) = J(K,v)$ structurally. Indeed, taking a surjective linear isometry T on K with $T(u) = v$, T becomes an isometric $*$-isomorphism from $J(K,u)$ onto $J(K,v)$. It it also worth mentioning that the algebras $J(K,u)$ have appeared in the literature as the solutions to other problems different from the characterization of division real JB^*-algebras given by Proposition 3.2. Thus, they are the unique norm-unital complete normed commutative real algebras whose Banach spaces are smooth at their unit [**28**], as well as the unique norm-unital complete normed commutative real algebras X such that the group of all surjective linear isometries on X acts transitively on S_X [**4**]. Moreover, they also become the natural parameters in the construction of all one-sided division complete absolute-valued real algebras [**29**].

Let u be a tripotent in a real JB^*-triple X. Then the Jordan algebra $X_2(u)$, endowed with its natural involution, is a real JB^*-algebra. Indeed, taking a complex JB^*-triple Y containing X as a closed real subtriple, $X_2(u)$ becomes a closed self-adjoint real subalgebra of the complex JB^*-algebra $Y_2(u)$. We denote by $X^1(u)$ the JB-algebra of all self-adjoint elements of the real JB^*-algebra $X_2(u)$. With the convention of symbols just made, the following corollary follows from Proposition 3.2.

COROLLARY 3.3. *Let u be a tripotent in a real JB^*-triple X. Then u is a division tripotent if and only if $X^1(u) = \mathbb{R}u$.*

COROLLARY 3.4. *Let u be a tripotent in a complex JB^*-triple X. Then u is a division tripotent if and only if $X_2(u) = \mathbb{C}u$.*

PROOF. We have $X_2(u) = X^1(u) \oplus iX^1(u)$, and Corollary 3.3 applies. ∎

Thus, in view of Corollary 3.3 (respectively, 3.4), division tripotents in a real (respectively, complex) JB^*-triple X coincide with the so-called in the literature **minimal tripotents** of X (see for instance [**10**] and [**18**]), which are defined as those tripotents u of X satisfying $X^1(u) = \mathbb{R}u$ (respectively, $X_2(u) = \mathbb{C}u$). The name "minimal" is in agreement with the fact that, in the case that X is actually a real or complex JBW^*-triple, minimal tripotents of X are precisely those tripotents of X which are minimal relative to the order defined in the set of all tripotents of X by $u \leq v$ if and only if $v = u+w$ for some tripotent w orthogonal to u [**26**, Proposition 2.2]. Now, by arguing as in the proof of [**26**, Lemma 3.2], we obtain the following corollary.

COROLLARY 3.5. *Let Y be a complex JB^*-triple with conjugation τ, and let u be a division tripotent of Y^τ. Then u is the sum of at most two orthogonal division tripotents of Y.*

Proof of Theorem 3.1.- Assume that the real JB^*-triple X is of finite rank, so that there exists a finite frame $\{u_1, ..., u_n\}$ in X. Then, taking a complex JB^*-triple Y with conjugation τ such that $X = Y^\tau$, Corollary 3.5 gives (up to a rearangement, if necessary) the existence of $1 \leq m \leq n$ and division tripotents $v_1, ...v_m, w_1, ..., w_m$ of Y such that $u_i = v_i + w_i$ and v_i is orthogonal to w_i for $i = 1, ..., m$, whereas u_j is a division tripotent of Y whenever $j = m+1, ..., n$. Now it is easily seen that $\{v_1, ...v_m, w_1, ..., w_m, u_{m+1}, ..., u_n\}$ is a frame in Y, and hence the complex JB^*-triple Y is of finite rank. By [**18**, Theorem 4.10], Y is a finite ℓ_∞-sum of simple triple ideals which are either finite-dimensional, infinite-dimensional complex spin factors, or of the form $\mathcal{L}(H, K)$ for complex Hilbert spaces H, K with $\dim(H) = \infty$ and $\dim(K) < \infty$. It follows that the Banach space of Y (and hence that of X) is isomorphic to a Hilbert space.

To conclude the proof it is enough to show that X is of finite rank whenever the Banach space of X is isomorphic to a Hilbert space. But, in view of the implication (4) $\Rightarrow$ (3) in Theorem 2.3, we are reduced to the case that X is either finite-dimensional, an infinite-dimensional generalized real spin factor, or of the form $\mathcal{L}(H, K)$ for some real, complex, or quaternionic Hilbert spaces H, K with $\dim(H) = \infty$ and $\dim(K) < \infty$. In all these cases, we realize that X is of finite rank by a direct inspection (see [**19**, Table 1 and Proposition 5.8.(ii)]). ∎

Let X be a Jordan triple over a field $\mathbb{F}$ of characteristic different from 2 and 3. We say that X is **algebraic** if all single-generated subtriples of X are finite-dimensional over $\mathbb{F}$. If in fact there exists $m \in \mathbb{N}$ such that all single-generated subtriples of X have dimension $\leq m$, then we say that X is of **bounded degree**, and the minimum such an m will be called the **degree** of X. We define inductively the odd powers of an element $x \in X$ by $x^1 := x$ and $x^{2n+1} := \{x^{2n-1}xx\}$.

LEMMA 3.6. *Let X be a Jordan triple of degree m over a field $\mathbb{F}$ of characteristic different from 2 and 3 containing at least $2m+3$ elements. Then every family of nonzero pairwise orthogonal positive tripotents of X has at most m elements.*

PROOF. Assume on the contrary that we can find nonzero pairwise orthogonal positive tripotents $u_1, ..., u_{m+1}$ in X. Since $\mathbb{F}$ has at least $2m+3$ elements, we can also find nonzero elements $a_1, ..., a_{m+1}$ in $\mathbb{F}$ such that $a_1^2, ..., a_{m+1}^2$ are pairwise different. Put $x := \sum_{i=1}^{m+1} a_i u_i$. Since for $j = 1, ..., m+1$ we have $x^{2j-1} = \sum_{i=1}^{m+1} a_i^{2j-1} u_i$, and the matrix $(a_i^{2j-1})_{i,j=1,...,m+1}$ is invertible, the subtriple of X generated by x contains $\{u_1, ..., u_{m+1}\}$, and hence is of dimension $\geq m+1$. This contradicts that X is of degree m. ∎

We note that Jordan triples are "power-associative". Indeed, for an element x in a Jordan triple X, we have $\{x^{2i-1}x^{2j-1}x^{2k-1}\} = x^{2(i+j+k)-3}$ for all $i, j, k \in \mathbb{N}$, and therefore the subtriple of X generated by x is equal to the linear hull of the set of all odd powers of x. Keeping in mind this fact, the proof of [**8**, Theorem 1], originally made in the setting of complete normed power-associative algebras, works almost verbatim in the setting of complete normed Jordan triples. Thus we have the following lemma.

LEMMA 3.7. *Let X be an algebraic Jordan triple over $\mathbb{R}$ endowed with a complete norm making the triple product continuous. Then X is of bounded degree.*

Now we can prove the concluding main result of the paper.

Theorem 3.8. *Let X be a real JB^*-triple. Then the Banach space of X is isomorphic to a Hilbert space if and only if X is algebraic.*

Proof. Assume that X is algebraic. By Lemma 3.7, X is of bounded degree. Let m denote the degree of X. We claim that X^{**} is also of bounded degree (equal to m). To prove the claim we invoke the so-called strong* (in short s^*) topology of X^{**} [**25**, Section 4]. The strong* topology of X^{**} is compatible with the duality (X^{**}, X^*) [**25**, Corollary 9], and makes the triple product of X^{**} jointly continuous on bounded subsets of X^{**} [**25**, Theorem 9]. Let x be in $B_{X^{**}}$. Since B_X is w^*-dense in $B_{X^{**}}$, it is also s^*-dense in $B_{X^{**}}$, and hence there is a net $\{x_\lambda\}_{\lambda\in\Lambda}$ in B_X s^*-convergent to x. Since X is of degree m, for $\lambda \in \Lambda$ there exist $a_{0\lambda}, a_{1\lambda}, ..., a_{m\lambda} \in \mathbb{R}$ such that $|a_{0\lambda}|+|a_{1\lambda}|+...+|a_{m\lambda}| = 1$ and $a_{0\lambda}x_\lambda+a_{1\lambda}x_\lambda^3+...+a_{m\lambda}x_\lambda^{2m+1} = 0$. Taking a cluster point $(a_0, a_1, ..., a_m)$ of the net $\{(a_{0\lambda}, a_{1\lambda}, ..., a_{m\lambda})\}_{\lambda\in\Lambda}$ in $\mathbb{R}^{m+1}$, and applying the joint s^*-continuity of the triple product of X^{**} on bounded sets, we obtain $|a_0|+|a_1|+...+|a_m| = 1$ and $a_0x+a_1x^3+...+a_mx^{2m+1} = 0$. Therefore, the subtriple of X^{**} generated by x has dimension $\leq m$. Since x is arbitrary in $B_{X^{**}}$, and X^{**} contains X, and X is of degree m, we deduce that X^{**} is also of degree m. Now that the claim is proved, to show that the Banach space of X is isomorphic to a Hilbert space we can assume that X is in fact a JBW^*-triple of degree m. Then, invoking the Krein-Milman theorem, and applying [**14**, Lemma 3.3], we are provided with a tripotent u in X such that $X_0(u) = 0$. Let $u_1, ..., u_k$ be nonzero pairwise orthogonal tripotents of X such that $u = u_1 + ... + u_k$. According to Lemma 3.6, we must have $k \leq m$. Therefore, we may choose the family $\{u_1, ..., u_k\}$ above of maximum length, and then each u_i becomes a minimal (equivalently, division) tripotent of X (see the comments after Corollary 3.4). Now, since $u = u_1 + ... + u_k$ and $X_0(u) = 0$, we have $\cap_{i=1}^k X_0(u_i) = 0$, so $\{u_1, ..., u_k\}$ is a frame in X, and so X is of finite rank. By Theorem 3.1, the Banach space of X is isomorphic to a Hilbert space.

Now assume that the Banach space of X is isomorphic to a Hilbert space. In view of the implication (4) ⇒ (3) of Theorem 2.3, to show that X is algebraic we can additionally assume that X is either a generalized real spin factor or of the form $\mathcal{L}(H, K)$ for some real, complex, or quaternionic Hilbert spaces H, K with $\dim(K) < \infty$. Then we realize that X is algebraic by a direct inspection. ∎

References

[1] K. ALVERMANN, Real normed Jordan algebras with involution. *Arch. Math.* **47** (1986), 135-150.

[2] T. J. BARTON and R. M. TIMONEY, Weak*-continuity of Jordan triple products and applications. *Math. Scand.* **59** (1986), 177-191.

[3] J. BECERRA, G. LÓPEZ, and A. RODRÍGUEZ, Relatively weakly open sets in closed balls of C^*-algebras. *J. London Math. Soc.* (to appear).

[4] J. BECERRA and A. RODRÍGUEZ, Isometric reflections on Banach spaces after a paper of A. Skorik and M. Zaidenberg. *Rocky Mountain J. Math.* **30** (2000), 63-83.

[5] R. B. BRAUN, W. KAUP, and H. UPMEIER, A holomorphic characterization of Jordan C^*-algebras. *Math. Z.* **161** (1978), 277-290.

[6] L. J. BUNCE and C. H. CHU, Compact operations, multipliers and Radon-Nikodym property in JB^*-triples. *Pacific J. Math.* **153** (1992), 249-265.

[7] C. H. CHU, T. DANG, B. RUSSO, and B. VENTURA, Surjective isometries of real C^*-algebras. *J. London Math. Soc.* **47** (1993), 87-110.

[8] B. CUARTERO and J. E. GALÉ, Bounded degree of algebraic topological algebras. *Commun. Algebra* **22** (1994), 329-337.

[9] R. DEVILLE, G. GODEFROY and V. ZIZLER, *Smoothness and renormings in Banach spaces.* Pitman Monographs and Surveys in Pure and Applied Math. **64**, 1993.

[10] Y. FRIEDMAN and B. RUSSO, Structure of the predual of a JBW^*-triple. *J. Reine Angew. Math.* **356** (1985), 67-89.

[11] K. R. GOODERAL, *Notes on real and complex C^*-algebras.* Shiva Math. Series **5**, Devon, 1982.

[12] H. HANCHE-OLSEN and E. STORMER, *Jordan operator algebras.* Monograph Stud. Math. **21**, Pitman, 1984.

[13] P. HARMAND, D. WERNER, and W. WERNER, *M-ideals in Banach spaces and Banach algebras.* Lecture Notes in Mathematics **1547**, Springer-Verlag, Berlin, 1993.

[14] J. M. ISIDRO, W. KAUP, and A. RODRÍGUEZ, On real forms of JB^*-triples. *Manuscripta Math.* **86** (1995), 311-335.

[15] J. M. ISIDRO and A. RODRÍGUEZ, On the definition of real W*-algebras. *Proc. Amer. Math. Soc.* **124** (1996), 3407-3410.

[16] N. JACOBSON, *Structure and representations of Jordan algebras.* Amer. Math. Soc. Coll. Publ. **39**, Providence, Rhode Island 1968.

[17] W. KAUP, A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces. *Math. Z.* **183** (1983), 503-529.

[18] W. KAUP, Uber die Klassifikation der symmetrischen hermiteschen Mannigfaltigkeiten unendlicher Dimension. I. *Math. Ann* **257** (1981), 463-486.

[19] W. KAUP, On real Cartan factors. *Manuscripta Math.* **92** (1997), 191-222.

[20] O. LOOS, *Jordan pairs.* Lecture Notes in Mathematics **460**, Springer-Verlag, Berlin, 1975.

[21] O. LOOS, *Bounded symmetric domains and Jordan pairs.* Mathemathical Lectures, Irvine, University of California at Irvine, 1977.

[22] J. MARTÍNEZ and A. M. PERALTA, Separate weak*-continuity of the triple product in dual real JB^*-triples. *Math. Z.* **234**, 635-646.

[23] A. MORENO and A. RODRÍGUEZ, On the Zel'manovian classification of prime JB^*-triples. *J. Algebra* **226** (2000), 577-613.

[24] O. NYGAARD and D. WERNER, Slices in the unit ball of a uniform algebra. *Archiv Math.* **76** (2001), 441-444.

[25] A. M. PERALTA and A. RODRÍGUEZ, Grothendiekc's inequalities for real and complex JBW^*-triples. *Proc. London Math. Soc.* **83** (2001), 605-625.

[26] A. M. PERALTA and L. L. STACHÓ, Atomic decomposition of real JBW^*-triples. *Quart. J. Math.* **52** (2001), 79-87.

[27] C. E. RICKART, *General theory of Banach algebras.* Kreiger, New York, 1974.

[28] A. RODRÍGUEZ, Nonassociative normed algebras spanned by hermitian elements. *Proc. London Math. Soc.* **47** (1983), 258-274.

[29] A. RODRÍGUEZ, One-sided division absolute valued algebras. *Publ. Mat.* **36** (1992), 925-954.
[30] S. SAKAI, Weakly compact operators on operator algebras. *Pacific J. Math.* **14** (1964), 659-664.
[31] R. V. SHVYDKOY, Geometric aspects of the Daugavet property. *J. Funct. Anal.* **176** (2000), 198-212.
[32] M. A. YOUNGSON, Non unital Banach Jordan algebras and C^*-triple systems. *Proc. Edinburgh Math. Soc.* **24** (1981), 19-31.
[33] J. D. M. WRIGHT, Jordan C^*-algebras. *Michigan Math. J.* **24** (1977), 291-302.

Universidad de Granada, Facultad de Ciencias. Departamento de Matemática Aplicada, 18071-Granada (Spain)
E-mail address: juliobg@ugr.es

Universidad de Granada, Facultad de Ciencias. Departamento de Análisis Matemático, 18071-Granada (Spain)
E-mail address: glopezp@ugr.es

Universidad de Granada, Facultad de Ciencias. Departamento de Análisis Matemático, 18071-Granada (Spain)
E-mail address: aperalta@ugr.es

Universidad de Granada, Facultad de Ciencias. Departamento de Análisis Matemático, 18071-Granada (Spain)
E-mail address: apalacio@ugr.es