Relatively weakly open sets in closed balls of C^*-algebras

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Abstract. Let A be an infinite-dimensional C^*-algebra. We prove that every nonempty relatively weakly open subset of the closed unit ball B_A of A has diameter equal to 2. This implies that B_A is not dentable, and that there is no any point of continuity for the identity mapping $(B_A, \text{weak}) \to (B_A, \text{norm})$.

1. Introduction

Despite the importance of the Radon-Nikodym property in the theory of Banach spaces, many classical Banach spaces fail to enjoy that property. For instance, this is the case of infinite-dimensional $C(K)$-spaces and C^*-algebras. In this paper, we show that infinite-dimensional $C(K)$-spaces and C^*-algebras actually fail to fulfill most consequences of the Radon-Nikodym property, like the dentability of closed balls, or the existence of points of weak-norm continuity for the identity mapping on closed balls. In fact we prove that, if A is an infinite-dimensional C^*-algebra, then every nonempty relatively weakly open subset of the closed unit ball of A has diameter equal to 2 (Theorem 2.5). Our proof starts by revisiting the $C(K)$-space case, where the result is known [**17**], and passes through the consideration of some mathematical objects, called JB^*-triples (see [**15**] and [**16**]), which become natural generalizations of C^*-algebras. In the result obtained for JB^*-triples, the infinite dimensionality must be replaced with the non hilbertizability (Proposition 2.4). We consider also the case of JB-algebras [**13**], where a similar result to that obtained for JB^*-triples is proven (Theorem 3.3).

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2. The main result

Throughout this paper $\mathbb{K}$ will mean the field of real or complex numbers. Let X be a Banach space X over $\mathbb{K}$. We denote by S_X, B_X, and X^* the unit sphere, the closed unit ball, and the (topological) dual, respectively, of X. We denote by n (respectively, w) the norm (respectively, weak) topology on X, and by w^* the weak* topology of X^*. If Y is another Banach space, $\mathcal{L}(X,Y)$ will stand for the space of all bounded linear operators from X to Y.

Given a locally compact Hausdorff topological space Ω, we denote by $C_0^{\mathbb{K}}(\Omega)$ the Banach space of all $\mathbb{K}$-valued continuous functions on Ω vanishing at infinity. When Ω is actually compact, we write as usual $C^{\mathbb{K}}(\Omega)$ instead of $C_0^{\mathbb{K}}(\Omega)$. The following lemma is a consequence of [**17**, Theorem 2]. For the sake of completeness, we give here a direct proof.

Lemma 2.1. *Let Ω be a compact Hausdorff topological space, and let X stand for $C^{\mathbb{K}}(\Omega)$. If there exists a nonempty relatively w-open subset of B_X with diameter less than 2, then Ω is finite.*

Proof. Assume that Ω is infinite. Let U be a nonempty relatively w-open subset of B_X. We are going to show that $\operatorname{diam}(U) = 2$. Since X is infinite-dimensional, there must exist some x in $U \cap S_X$.

Consider first the case that the set of isolated points of Ω is infinite. Take a sequence $\{t_n\}_{n\in\mathbb{N}}$ of pair-wise different isolated points of Ω, and, for n in $\mathbb{N}$, define $x_n \in X$ by $x_n(t_n) = 1$, $x_n(t_{n+1}) = -1$, and $x_n(t) = x(t)$ whenever t belongs to $\Omega\setminus\{t_n, t_{n+1}\}$. Since $\{x_n\}_{n\in\mathbb{N}}$ lies in B_X and converges pointwise to x, it follows from [**10**, Theorem VII.1] that $\{x_n\}_{n\in\mathbb{N}}$ actually converges weakly to x. Since U is a relatively w-neighbourhood of x, we have that x_n and x_{n+1} belong to U for n big enough, and therefore $\operatorname{diam}(U) = 2$.

Now consider the case that x attains its norm at an accumulation point t_0 of Ω. There is no loss of generality if we assume $x(t_0) = 1$. Let $\varepsilon > 0$. Then the set

$$\omega_0 := \{t \in \Omega : |x(t) - 1| < \varepsilon\}$$

is open in Ω and contains t_0. Since t_0 is an accumulation point of Ω, there is an infinite sequence $\{\omega_n\}_{n\in\mathbb{N}}$ of nonempty pair-wise disjoint open subsets of ω_0. For $n \in \mathbb{N}$, take $t_n \in \omega_n$, and apply Uryson's lemma to pick x_n in X with $-\mathbf{1} \leq x_n \leq \mathbf{1}$, $x_n(t_n) = -1$, and $x_n(t) = 1$ whenever $t \in \Omega \setminus \omega_n$. Since $\{x_n\}_{n\in\mathbb{N}}$ converges pointwise to $\mathbf{1}$, $\{xx_n\}_{n\in\mathbb{N}}$ converges pointwise to x, and hence $\{xx_n\}_{n\in\mathbb{N}}$ actually converges weakly to x because $\{xx_n\}_{n\in\mathbb{N}}$ lies in B_X. Let m be in $\mathbb{N}$ such that xx_m belongs to U. Then, since t_m lies in ω_0, we have

$$\operatorname{diam}(U) \geq \|x - xx_m\| \geq |x(t_m)(1 - x_m(t_m))| = 2|x(t_m)| \geq 2(1-\varepsilon).$$

By the arbitraryness of ε, we deduce $\operatorname{diam}(U) = 2$.

To conclude the proof, consider the case that the set of isolated points of Ω is finite. Then we can write $X = (Y \times Z)_\infty$, where $Y = (\mathbb{K}^p)_\infty$ for some nonnegative integer number p, and $Z = C^{\mathbb{K}}(K)$ for some perfect compact Hausdorff topological space K. As a consequence, we have $(B_X, w) = (B_Y, w) \times (B_Z, w)$. Since the coordinate projection $\pi_Z : (B_X, w) \to (B_Z, w)$ is open, it follows from the preceding paragraph that $\mathrm{diam}(\pi_Z(U)) = 2$. Since π_Z is contractive, we deduce $\mathrm{diam}(U) = 2$. ■

LEMMA 2.2. *Let Ω be a locally compact Hausdorff topological space, and let X stand for $C_0^{\mathbb{K}}(\Omega)$. If there exists a nonempty relatively w-open subset of B_X with diameter less than 2, then Ω is finite.*

PROOF. Assume that there is a nonempty relatively w-open subset U of B_X with $\mathrm{diam}(U) < 2$. Then U contains a set V of the form

$$\{x \in B_X : |f_i(x - x_0)| < 1 \ \forall i = 1, ..., n\},$$

for suitable $x_0 \in B_X$, $n \in \mathbb{N}$, and $f_1, ..., f_n \in X^*$. Put

$$V^{**} := \{z \in B_{X^{**}} : |f_i(z - x_0)| < 1 \ \forall i = 1, ..., n\}.$$

Since V^{**} is relatively w^*-open in $B_{X^{**}}$, and B_X is w^*-dense in $B_{X^{**}}$, V $(= V^{**} \cap B_X)$ is w^*-dense in V^{**}. Therefore $V - V$ is w^*-dense in $V^{**} - V^{**}$, and consequently , by the lower semicontinuity of the norm of X^{**}, we have $\mathrm{diam}(V^{**}) = \mathrm{diam}(V) \leq \mathrm{diam}(U) < 2$. Since $X^{**} = C^{\mathbb{K}}(K)$ for a suitable compact Hausdorff topological space K, and V^{**} is a nonempty relatively w-open subset of X^{**}, it follows from Lemma 2.1 that K is finite. Now $X = X^{**}$ is finite-dimensional, and hence Ω is finite. ■

The proof of Lemma 2.2 actually shows that, *if X is an infinite-dimensional real or complex Lindenstrauss space, then every nonempty relatively w-open subset of B_X has diameter equal to* 2.

We recall that a JB^***-triple** is a complex Banach space J with a continuous triple product $\{...\} : J \times J \times J \to J$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

(1) For all x in J, the mapping $y \to \{xxy\}$ from J to J is a hermitian operator on J and has nonnegative spectrum.
(2) The **main identity**

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

holds for all a, b, x, y, z in J.
(3) $\|\{xxx\}\| = \|x\|^3$ for every x in J.

Concerning Condition 1 above, we also recall that a bounded linear operator T on a complex Banach space X is said to be **hermitian** if $\|\exp(irT)\| = 1$ for every r in $\mathbb{R}$.

JB^*-triples are of capital importance in the study of bounded symmetric domains in complex Banach spaces. Indeed, open balls in JB^*-triples are bounded symmetric domains, and every symmetric domain in a complex Banach space is biholomorphically equivalent to the the open unit ball of a suitable JB^*-triple (see [**15**] and [**16**]). Examples of JB^*-triples are all C^* algebras under the triple product

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x), \tag{2.1}$$

the spaces $\mathcal{L}(H_1, H_2)$ for arbitrary complex Hilbert spaces H_1 and H_2 (with triple product formally defined as in (2.1)), and the so-called **spin factors**. These are constructed from an arbitrary complex Hilbert space $(H, (\cdot|\cdot))$ of hilbertian dimension ≥ 3, by taking a conjugate-linear involutive isometry σ on H, and then by defining the triple product and the norm by

$$\{xyz\} := (x|y)z + (z|y)x - (x|\sigma(z))\sigma(y)$$

and

$$\|x\|^2 := (x|x) + \sqrt{(x|x)^2 - |(x|\sigma(x))|^2},$$

respectively, for all x, y, z in H.

Let J be a JB^*-triple. A **subtriple** (respectively, **triple ideal**) of J is a subspace M of J such that $\{MMM\} \subseteq M$ (respectively, $\{MJJ\} + \{JMJ\} \subseteq M$). We say that J is **simple** if there are not triple ideals of J other than $\{0\}$ and J. If x is in J, and if the subtriple of J generated by x is finite-dimensional, then we say that x is an **algebraic element of** J. The JB^***-triple** J is said to be **algebraic** if every element of J is algebraic.

The proof of the next lemma involves minor changes on that of [**7**, Theorem 1] or [**8**, Theorem D], and hence is omitted. Because of the conjugate-linear behaviour of triple products of JB^*-triples in the middle variable, and the essentially linear nature of the arguments in [**7**] and [**8**], the application of such arguments in our setting needs a formal consideration of "real algebraic" elements, being sure that "real algebraic" elements and (complex) algebraic elements coincide, thanks to the fact that $\mathbb{C}$ is finite-dimensional over $\mathbb{R}$.

LEMMA 2.3. *Let J be a JB^*-triple. If there exists a nonempty open subset of J consisting only of algebraic elements of J, then J is algebraic.*

PROPOSITION 2.4. *Let J be a JB^*-triple such that there exists a nonempty relatively w-open subset of B_J with diameter less than 2. Then J is a finite ℓ_∞-sum of closed simple triple ideals which are either finite-dimensional, spin factors, or of the form $\mathcal{L}(H_1, H_2)$ for suitable complex Hilbert spaces H_1 and H_2 with* $\dim(H_2) < \infty$. *Consequently, the Banach space of J is hilbertizable.*

PROOF. Let U be a nonempty relatively w-open subset of B_J with $\text{diam}(U) < 2$. Let x be in J. Denote by J_x the closed subtriple of J generated by x. It is well-known that there is a unique locally compact subset S_x of $]0, \infty[$ and a surjective triple isomorphism $\phi_x : J_x \to C_0^{\mathbb{C}}(S_x)$ such that $S_x \cup \{0\}$ is compact and $\phi_x(x)$ is the inclusion mapping $S_x \to \mathbb{C}$ (see [**15**, 4.8], [**16**, 1.15], and [**12**]). Since surjective triple isomorphisms between

JB^*-triples are isometries (see again [**16**]), we can write $J_x = C_0^{\mathbb{C}}(S_x)$, in the sense of the isometric theory of Banach spaces. Now assume that x actually lies in U. Then $U \cap J_x$ is a nonempty relatively w-open subset of B_{J_x} with $\operatorname{diam}(U \cap J_x) < 2$. It follows from Lemma 2.2 that S_x is finite, and hence x is an algebraic element of J. Since x is arbitrary in U, and U has nonempty n-interior in J, it follows from lemma 2.3 that J is algebraic. Now, since S_x is finite for every x in X, the result follows easily from the results in [**6**] (see the concluding part of the proof of [**6**, Proposition 4.5] for details). ∎

Since C^*-algebras are JB^*-triples, and they are finite-dimensional whenever their Banach spaces are reflexive [**21**], our main result follows straightforwardly from Proposition 2.4.

THEOREM 2.5. *Let A be a C^*-algebra such that there exists a nonempty relatively w-open subset of B_A with diameter less than 2. Then A is finite-dimensional.*

Of course, the existence of a nonempty relatively w-open subsets of the closed unit ball with diameter less than 2, required in Theorem 2.5 (respectively, Proposition 2.4), is a condition much weaker than that of finite dimensionality (respectively, hilbertizability) arising in the conclusion of that result. Therefore we can obtain many other intermediate characterizations of the finite dimensionality (respectively, the hilbertizability) of C^*-algebras (respectively, JB^*-triples) in terms of the geometry of their closed unit balls. We do not state explicitly such characterizations, and only note as a hint that, for a Banach space X, each of the conditions 1 to 10 which follow implies the subsequent one:

(1) Finite-dimensionality
(2) Hilbertizability
(3) Superreflexivity
(4) Reflexivity
(5) The Radon-Nikodym property
(6) "Abundance" of denting points of B_X
(7) Existence of denting points of B_X
(8) Existence of slices of B_X of arbitrarily small diameter
(9) Existence of nonempty relatively w-open subsets of B_X of arbitrarily small diameter
(10) Existence of nonempty relatively w-open subsets of B_X with diameter less than 2.

The implication $8 \Rightarrow 9$ above follows because, in fact, slices of B_X are nonempty relatively w-open subsets of B_X. We note in addition that denting points of B_X are points of $w - n$ continuity of the identity mapping on B_X, and that the mere existence of a point of $w - n$ continuity of the identity on B_X implies condition 9 above. We recall that a **slice** of the closed unit ball of the Banach space X is a set of the form

$$S(X, f, \alpha) := \{x \in B_X : \Re e(f(x)) > 1 - \alpha\}$$

for some f in S_{X^*} and $\alpha > 0$, that a **denting point** of B_X is an element of B_X such that there are slices of B_X of arbitrarily small diameter containing it, and that condition 8 above is usually called **dentability** of B_X.

For some of the consequences of Theorem 2.5 suggested above, we are provided with a proof not involving the theory of JB^*-triples nor the nonassociative techniques in [**7**] or [**8**]. This is the case of the following corollary.

COROLLARY 2.6. *Let A be a C^*-algebra such that there exists a slice in A with diameter less than 2. Then A is finite-dimensional.*

PROOF. Take a slice S of B_A with diameter less than 2. As in the proof of Lemma 2.2, we can produce a slice S^{**} of $B_{A^{**}}$ with $\operatorname{diam}(S^{**}) = \operatorname{diam}(S)$. Then, since A^{**} is a C^*-algebra with a unit, there is no loss of generality if we assume that A has a unit $\mathbf{1}$. Let us denote by U the set of all unitary elements of A. Since $B_A \setminus S$ is a convex, closed, and proper subset of B_A, and $\overline{co}\,(U) = B_A$ [**3**, Theorem 30.2], there must exist u in $U \cap S$. Then $S' := u^*S$ is a slice in A containing $\mathbf{1}$ and whose diameter is less than 2. Let h be a self-adjoint element of A, and let A_h denote the closed subalgebra of A generated by $\{h, \mathbf{1}\}$. Then we have $A_h = C^{\mathbb{C}}(\Omega_h)$, where Ω_h denotes the spectrum of h. Moreover, since $S' \cap A_h$ is nonempty, it is in fact a slice in A_h whose diameter is less than 2. According to Lemma 2.1, the spectrum of h is finite. Since h is an arbitrary self-adjoint element of A, Theorem 3.2.2 of [**1**] applies, giving that A is finite-dimensional. ∎

Let X be a Banach space. For u in S_X, we define the **roughness of X at** u, $\eta(X, u)$, by the equality

$$\eta(X, u) \;:=\; \limsup_{\|h\| \to 0} \frac{\|u + h\| + \|u - h\| - 2}{\|h\|}\,.$$

We remark that the absence of roughness of X at u (i.e., $\eta(X, u) = 0$) is nothing but the Fréchet differentiability of the norm of X at u [**9**, Lemma I.1.3]. Given $\epsilon > 0$, the Banach space X is said to be **ϵ-rough** if, for every u in S_X, we have $\eta(X, u) \geq \epsilon$. We say that X is **rough** whenever it is ϵ-rough for some $\epsilon > 0$, and **extremely rough** whenever it is 2-rough.

COROLLARY 2.7. *Let A be an infinite-dimensional von Neumann algebra. Then the predual of A is extremely rough.*

PROOF. Let A_* denote the predual of A, and let u be in S_{A_*}. By Corollary 2.6, for every $\alpha > 0$, the diameter of the slice $S(A, u, \alpha)$ is equal to 2. According to the proof of [**9**, Proposition I.1.11], this implies $\eta(A_*, u) = 2$. ∎

Corollary 2.7 is the unique predualitation we know of Theorem 2.5. A double predualization of that theorem (which in fact contains it) is really easy. Actually it is enough to invoke Theorem 2.5 after applying the argument in the proof of Lemma 2.2.

COROLLARY 2.8. *Let X be an infinite-dimensional complex Banach space such that X^{**} is a C^*-algebra. Then every nonempty relatively w-open subset of B_X has diameter equal to* 2.

3. Other related results

We begin this section by deriving again Theorem 2.5 from Proposition 2.4, but now throughout a slightly longer way. This way is quite natural because the class of "ternary" generalizations of C^*-algebras (namely, that of JB^*-triples) contains the one of "binary" generalizations of C^*-algebras (namely, that of non-commutative JB^*-algebras).

Non-commutative Jordan algebras are defined as those algebras A satisfying $(xy)x^2 = x(yx^2)$ and $(xy)x = x(yx)$, for all x, y in A. For an element x in a non-commutative Jordan algebra A, let us denote by U_x the mapping $y \to x(xy+yx)-x^2y$ from A to A. By a **non-commutative JB^*-algebra** we mean a complete normed non-commutative Jordan complex algebra A with a conjugate-linear algebra-involution $*$ satisfying $\|U_x(x^*)\| = \|x\|^3$ for every x in A. As an example of the relevance of non-commutative JB^*-algebras in the general non-associative setting, let us mention that they are the unique (possibly non associative) complete normed complex algebras having an approximate unit bounded by one and whose open unit balls are bounded symmetric domains [**14**, Theorem 3.3], whereas C^*-algebras are the unique associative complete normed complex algebras satisfying the same two properties as above [**14**, Corollary 3.4].

We recall that a complex algebra A is called **quadratic** if it has a unit $\mathbf{1}, A \neq \mathbb{C}\mathbf{1}$, and, for each x in A, there are elements $t(x)$ and $n(x)$ of $\mathbb{C}$ such that $x^2 - t(x)x + n(x)\mathbf{1} = 0$. All quadratic non-commutative JB^*-algebras are hilbertizable [**18**, Theorem 3.2]. On the other hand, non-commutative JB^*-algebras whose Banach spaces are reflexive are perfectly determined [**18**, Theorem 3.5]. Keeping in mind such a determination, and the fact already commented that non-commutative JB^*-algebras are JB^*-triples in a natural way (see [**4**] and [**22**]), the following result follows from Proposition 2.4.

PROPOSITION 3.1. *Let A be a non-commutative JB^*-algebra such that there exists a nonempty relatively w-open subset of B_A with diameter less than* 2. *Then A is a finite ℓ_∞-sum of closed simple ideals which are either finite-dimensional or quadratic. Consequently, the Banach space of A is hilbertizable.*

Alternative algebras are defined as those algebras A satisfying $x^2y = x(xy)$ and $yx^2 = (yx)x$ for all x, y in A. By Artin's theorem [**24**, Theorem 2.3.2], an algebra A is alternative (if and) only if, for all x, y in A, the subalgebra of A generated by $\{x, y\}$ is associative. By an **alternative C^*-algebra** we mean a complete normed alternative complex algebra (say A) with a conjugate-linear algebra-involution $*$ satisfying $\|x^*x\| = \|x\|^2$ for all x in A. Alternative C^*-algebras have also their own right to be considered as the non-associative counterparts of C^*-algebras. Indeed, Gelfand-Naimark axioms on a (possibly non associative) unital complex algebra A imply that A is alternative [**19**, Theorem 14]. Since alternative algebras are non-commutative Jordan algebras, and, for elements x, y in an alternative algebra, the equality $U_x(y) = xyx$ holds, it is not difficult to realize that alternative C^*-algebras become particular examples of non-commutative JB^*-algebras. Therefore, since simple quadratic alternative algebras are finite-dimensional [**24**, Theorems 2.3.4 and 2.2.1], the next result follows from Proposition 3.1.

Proposition 3.2. *Let A be an alternative C^*-algebra such that there exists a nonempty relatively w-open subset of B_A with diameter less than 2. Then A is finite-dimensional.*

Since associative algebras are alternative, Proposition 3.2 contains Theorem 2.5.

Our next and concluding goal in this section is to prove in the setting of JB-algebras similar results to that we have obtained for JB^*-triples, C^*-algebras, and non-commutative JB^*-algebras. **JB-algebras** are defined as those complete normed (commutative) Jordan real algebras A satisfying $\|x\|^2 \leq \|x^2 + y^2\|$ for all x, y in A. The basic reference for the theory of JB-algebras is the book of H. Hanche Olsen and E. Stormer [**13**]. By the main results in the papers of J. D. M. Wright [**23**] and M. A. Youngson [**22**], JB-algebras are in a bijective categorical correspondence with (commutative) JB^*-algebras. The correspondence is obtained by passing from each JB^*-algebra A to its self-adjoint part A_{sa}.

Among the examples of JB-algebras, we cite the self-adjoint parts of C^*-algebras under the Jordan product $x \circ y := \frac{1}{2}(xy + yx)$, and the so-called **$JB$-algebra spin factors**. These are the JB-algebras whose Banach spaces are of the form $(\mathbb{R} \oplus H)_{\ell_1}$, for an arbitrary real Hilbert space H of dimension ≥ 2, and whose products are defined by

$$(\lambda + x)(\mu + y) := (\lambda\mu + (x|y)) + (\lambda y + \mu x)$$

(see [**13**, Chapter 6]).

Theorem 3.3. *Let A be a JB-algebra such that there exists a nonempty relatively w-open subset of B_A with diameter less than 2. Then A is a finite ℓ_∞-sum of closed simple ideals which are either finite-dimensional, or JB-algebra spin factors. Consequently, the Banach space of A is hilbertizable.*

Proof. Let U be a nonempty relatively w-open subset of B_A with $\operatorname{diam}(U) < 2$. Let x be in A. Denote by A_x the closed subalgebra of A generated by x. Since Jordan algebras are power-associative [**13**, Lemma

2.4.5], if follows from [**13**, Theorem 3.2.2] that A_x is isometrically algebra-isomorphic to $C_0^{\mathbb{R}}(\Omega_x)$ for a suitable locally compact Hausdorff topological space Ω_x. Assume for the moment that x actually lies in U. Then $U \cap A_x$ is a nonempty relatively w-open subset of B_{A_x} with $\mathrm{diam}(U \cap A_x) < 2$, so that, by Lemma 2.2, Ω_x is finite, and hence A_x is finite-dimensional. Since x is arbitrary in U, and U has nonempty n-interior in A, it follows from either [**7**, Theorem 1] or [**8**, Theorem D] that in fact A_x is finite-dimensional for every x in A. Now, for every x in A, the set

$$\sigma(x) := \{\lambda \in \mathbb{R} : x - \lambda\mathbf{1} \text{ is not invertible in the unital hull of } A_x\}$$

is finite. Since, in particular, every nonzero point of $\sigma(x)$ is isolated for every $x \in A$, it follows from [**5**, Theorem 3.3 ($k \Rightarrow f$) and Corollary 1.4] and [**20**, Theorem 5.4] that A is a c_0-sum of topologically simple closed ideals which are either finite-dimensional, JB-algebra spin factors, or of the form $\mathcal{K}(H)_{sa}$ for some real, complex or quaternionic Hilbert space H. Since in fact $\sigma(x)$ is finite for every $x \in A$, the above c_0-sum must be finite, and the Hilbert spaces H parameterizing the ideals $\mathcal{K}(H)_{sa}$, eventually arising in such a sum, must be finite-dimensional. ∎

To conclude this paper, let us state the appropriate variants of Corollaries 2.7 and 2.8 in the setting of JB^*-triples and JB-algebras. We recall that the bidual of a JB^*-triple (respectively, JB-algebra) is a JB^*-triple (respectively, JB-algebra) in a natural way [**11**] (respectively, [**13**, Theorem 4.4.3]). Thus, JB^*-triples (respectively, JB-algebras) which are dual Banach spaces appear naturally. Such JB^*-triples (respectively, JB-algebras) are called JBW^***-triples** (respectively, JBW**-algebras**). We note that the predual of a JBW^*-triple (respectively, JBW-algebra) is unique [**2**] (respectively, [**13**, Theorem 4.4.16]).

Corollary 3.4. *Let X be a JBW^*-triple or a JBW-algebra. Then the predual of X is either hilbertizable or extremely rough.*

Note that

Hilbertizability ⇒ Superreflexivity ⇒ Reflexivity ⇒ Asplund

⇒ "Abundance" of points of Fréchet differentiability of the norm

⇒ Existence of points of Fréchet differentiability of the norm

⇒ Non roughness,

so that the hilbertizability is scandalously incompatible with the extreme roughness.

Corollary 3.5. *Let X be a complex (respectively, real) Banach space such that X^{**} is a JB^*-triple (respectively, JB-algebra). Then either X is hilbertizable or all nonempty relatively w-open subsets of B_X have diameter equal to 2.*

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