On Urbanik's axioms for absolute valued algebras with involution

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Abstract. We prove that, in Urbanik's definition [**9**] of absolute valued algebras with involution, the axiom $\|a^*\| = \|a\|$ (the unique one which relates the absolute value $\|\cdot\|$ and the involution $*$) is redundant.

1. Introduction

Absolute valued algebras with involution were introduced in Urbanik's early paper [**9**] and, since then, they have been studied in deep [**2, 4, 5**], and have found interesting applications [**1, 3, 6**]. In the present note, we deal with the definition itself of absolute valued algebras with involution in order to realize that there is some redundance in its axioms. To be precise, let us recall the appropriate notions.

By an absolute value on a real algebra A we mean a norm $\|\cdot\|$ on the vector space of A satisfying $\|ab\| = \|a\|\|b\|$ for all $a, b \in A$. By an absolute valued algebra we mean a (possibly nonassociative) real algebra $A \neq 0$ endowed with an absolute value. Following [**9**], by an involution on an absolute valued algebra A we mean a linear mapping $a \to a^*$ from A to A satisfying:

(i) $a^{**} = a$
(ii) $aa^* = a^*a$
(iii) $(ab)^* = b^*a^*$
(iv) $\|a^*\| = \|a\|$

for all $a, b \in A$.

The aim of this note is to show that Axiom (iv) above is a consequence of the remaining ones. Thus, we are going to prove the following.

Theorem 1.1. *Let A be an absolute valued algebra, and let $a \to a^*$ be a linear mapping from A to A satisfying Axioms* (i), (ii), *and* (iii) *above. Then we have $\|a^*\| = \|a\|$ for every $a \in A$.*

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We note that, by Proposition 1.1 of [**8**], Axioms (i) and (iii) imply Axiom (iv) whenever $*$ is continuous. Therefore we have the following:

Lemma 1.2. *Theorem 1.1 holds if A is finite-dimensional.*

Actually, by Theorem 4 of [**7**], Axioms (i) and (iii) imply Axiom (iv) whenever the absolute valued algebra A is complete. Anyway, this is a deeper result which will be not applied in this note.

In general, a real algebra A can admit more than one absolute value (see page 123 of [**8**]). However, this is not the case if A is finite-dimensional (a consequence of Proposition 1.1 of [**8**]) or, more generally, if A is a one-sided division algebra (Corollary 3.5 of [**8**]). Now, as a consequence of Theorem 1.1, we have the following.

Corollary 1.3. *Let A be a real algebra endowed with a linear mapping $a \to a^*$ from A to A satisfying Axioms* (i), (ii), *and* (iii) *above. Then there is at most one absolute value on A.*

Proof. Let $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ be absolute values on A. If $*$ is the identity mapping on A, then A is commutative, so finite-dimensional (by Theorem 3 of [**10**]), and so the result follows. Assume that $*$ is not the identity mapping on A. Then, by Theorem 1.1, $(A, \|\cdot\|, *)$ and $(A, |\!|\!|\cdot|\!|\!|, *)$ are absolute valued algebras with nontrivial involution. Therefore, as noticed in Proposition 3.3 of [**8**], Lemmas 1, 2, and 3 of [**9**] imply the existence of an idempotent $e \in A$ satisfying $a^*a = \|a\|^2 e$ and $a^*a = |\!|\!|a|\!|\!|^2 e$ for every $a \in A$. It follows that $\|\cdot\| = |\!|\!|\cdot|\!|\!|$. ■

2. Proof of Theorem 1.1

Throughout this section, A will denote an absolute valued algebra, and $a \to a^*$ will stand for a linear mapping from A to A satisfying Axioms (i), (ii), and (iii) in the introduction. Our goal is to show that $\|a^*\| = \|a\|$ for every $a \in A$.

Our argument begins by considering the subspaces X and Y of A defined by $X := \{a \in A : a^* = a\}$ and $Y := \{a \in A : a^* = -a\}$, and collecting (in Lemma 2.1 immediately below) some facts involving such subspaces, and which will be applied in what follows without notice. The last fact in Lemma 2.1 (the unique one which is not of straightforward verification) follows from the second one and Lemma 3 of [**10**].

Lemma 2.1. *We have:*

(1) $A = X \oplus Y$.
(2) *Elements of X commute with those of Y.*
(3) $XY \subseteq Y$.
(4) *For $a \in X \cup Y$ we have $a^2 \in X$.*
(5) *For $(x, y) \in X \times Y$, $Lin\{x, y\}$ (with the restriction of the absolute value of A) is a Hilbert space.*

Here, Lin means linear hull in A.

Now, for $(x, y) \in X \times Y$ we put

$$f(x, y) := \frac{\|x+y\|^2 - \|x\|^2 - \|y\|^2}{2}.$$

We will show in Lemma 2.3 below that the mapping $f : (x, y) \to f(x, y)$ from $X \times Y$ to $\mathbb{R}$ is bilinear. Anyway, by noticing that, for $(x, y) \in X \times Y$, $f(x, y)$ is nothing other than the inner product of x and y in the Hilbert space $Lin\{x, y\}$, we have clearly $f(x, \alpha y) = f(\alpha x, y) = \alpha f(x, y)$ for every $\alpha \in \mathbb{R}$, and in particular $f(x, -y) = -f(x, y)$. This last equality is enough to straightforwardly deduce, from the definition of f, the following.

LEMMA 2.2. *The following assertions are equivalent:*

(1) $\|a^*\| = \|a\|$ *for every* $a \in A$.
(2) $f(x, y) = 0$ *for every* $(x, y) \in X \times Y$.

LEMMA 2.3. *The mapping* $f : X \times Y \to \mathbb{R}$ *is bilinear, and satisfies* $f(x^2, xy) = \|x\|^2 f(x, y)$ *for every* $(x, y) \in X \times Y$.

PROOF. For every norm-one element u in a real normed space E, denote by $D(E, u)$ the set of all norm-one continuous linear functionals ϕ on E such that $\phi(u) = 1$, and note that, thanks to the Hahn-Banach theorem, $D(E, u)$ is nonempty, as well as that, if the norm of E derives from an inner product $(\cdot|\cdot)$, then $D(E, u)$ reduces to the singleton consisting of the mapping $z \to (z|u)$ from E to $\mathbb{R}$. Now, let x be a fixed norm-one element of X. Take ϕ in $D(A, x)$, and note that, for each $y \in Y$ the restriction of ϕ to $Lin\{x, y\}$ belongs to $D(Lin\{x, y\}, x)$. It follows that $f(x, y) = \phi(y)$ for every $y \in Y$. This shows the linearity of f in its second variable. The linearity of f in its firs variable is proved in a similar way. Finally, for $(x, y) \in X \times Y$ we have

$$f(x^2, xy) = \frac{\|x(x+y)\|^2 - \|x^2\|^2 - \|xy\|^2}{2}$$

$$= \|x\|^2 \frac{\|x+y\|^2 - \|x\|^2 - \|y\|^2}{2} = \|x\|^2 f(x, y).$$

■

From now on, we argue by contradiction, and hence WE ASSUME THAT THE DESIRED EQUALITY $\|a^*\| = \|a\|$ FAILS FOR SOME $a \in A$. Then, by Lemma 2.2, we have the following.

CLAIM 2.4. *The mapping* $f : X \times Y$ *is not identically zero*

Therefore, since f is bilinear (by Lemma 2.3), the set

$$Z := \{x \in X : f(x, Y) = 0\}$$

is a proper subspace of X, and so $D := X \setminus Z$ is a dense subset of X. We have the following.

Claim 2.5. *X satisfies the parallelogram law.*

Proof. Let u and v be in D and X, respectively. Choose $y \in Y$ with $f(u, y) = 1$, and put $t_0 := -f(v, y)$. Keeping in mind that f is bilinear, and that $f(u^2, uy) \neq 0$ (both facts assured by Lemma 2.3), we can consider the third-degree polynomial function $p : \mathbb{R} \to \mathbb{R}$ defined by

$$p(t) := f((tu + v)^2, (tu + v)y),$$

which, again by Lemma 2.3, satisfies

$$p(t) = \|tu + v\|^2 (t - t_0)$$

for every $t \in \mathbb{R}$. Since t_0 is a root of p, there exist $\alpha, \beta, \delta \in \mathbb{R}$ satisfying

$$p(t) = (\alpha t^2 + \beta t + \gamma)(t - t_0)$$

for every $t \in \mathbb{R}$. It follows that $\|tu + v\|^2 = \alpha t^2 + \beta t + \gamma$ for every $t \in \mathbb{R}$ or, equivalently, $\|ru + sv\|^2 = \alpha r^2 + \beta rs + \gamma s^2$ for all $r, s \in \mathbb{R}$. Now, the equality $\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$ is of straightforward verification. Since u and v are arbitrary in D and X, respectively, and D is dense in X, the result follows. ∎

For every normed space E, we denote by S_E the unit sphere of E, i.e., the set of all norm-one elements in E.

Claim 2.6. *Let (x, y) be in $S_X \times S_Y$ such that $f(x, y) = 0$. Then the equality $x^2 = -y^2$ holds.*

Proof. By the definition of f, we have

$$\|x^2 - y^2\| = \|(x + y)(x - y)\| = \|x + y\|\|x - y\| = \sqrt{2}\sqrt{2} = 2.$$

Therefore, since x^2 and y^2 lie in X, and X satisfies the parallelogram law (by Claim 2.5), we deduce $x^2 = -y^2$, as desired. ∎

Claim 2.7. *There exists $e \in S_X$ such that we have $x^2 = e = -y^2$ whenever (x, y) is in $S_X \times S_Y$.*

Proof. By Lemma 1.2, A is infinite-dimensional. Therefore, since $A = X \oplus Y$, and X can be linearly imbedded into Y (by means of the mapping $x \to xy$, where y is any fixed nonzero element of Y), Y is infinite-dimensional. Let u, v be in S_X. Since the dimension of Y is greater than two, there exists $y_0 \in S_Y$ such that $f(u, y_0) = f(v, y_0) = 0$. It follows from Claim 2.6 that $u^2 = -y_0^2 = v^2$. This shows that the mapping $x \to x^2$ is constant on S_X. We denote by e the constant value of such a mapping, so that it only remains to show that $y^2 = -e$ for every $y \in S_Y$. To this end, we distingue two cases.

Case 1: The dimension of X is one.- Then we have $X = \mathbb{R}e$, and hence $y^2 = \pm e$ for every $y \in S_Y$ (by Lemma 2.1.(4)). On the other hand, since the dimension of Y is greater than one, S_Y is connected, and, as we have seen above, there exists $y_0 \in S_Y$ with $y_0^2 = -e$. It follows that $y^2 = -e$ for every $y \in S_Y$, as desired.

Case 2: The dimension of X is greater than one.- Then, for $y \in S_Y$ there exists $x \in S_X$ with $f(x, y) = 0$, so that it is enough to apply Claim 2.6 and the definition of e to obtain $y^2 = -e$. ∎

Now the contradiction is coming. Indeed, by Claim 2.7, for every $(x, y) \in X \times Y$ we have $x^2 = \|x\|^2 e$ and $y^2 = -\|y\|^2 e$, so

$$(x + y)(x - y) = (\|x\|^2 + \|y\|^2)e,$$

and so

$$\begin{aligned}&(\|x\|^2 + \|y\|^2)^2 - 4f(x, y)^2 \\ &= [\|x\|^2 + \|y\|^2 + 2f(x, y)][\|x\|^2 + \|y\|^2 - 2f(x, y)] = \|x + y\|^2\|x - y\|^2 \\ &= \|(x + y)(x - y)\|^2 = (\|x\|^2 + \|y\|^2)^2,\end{aligned}$$

which implies $f(x, y) = 0$, contrarily to Claim 2.4.

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