On the Zelmanovian classification
of prime JB^*-triples.

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Dedicated to Professor Holger Petersson on the occasion of his 60^{th} birthday.

Introduction

Complex JB^*-triples are complex Banach spaces endowed with a triple product subjected to suitable conditions of algebraic and analytic nature (see Section 1 for the definition). They were introduced by W. Kaup in order to provide an algebraic setting for the study of bounded symmetric domains in complex Banach spaces. The open unit ball of every complex JB^*-triple is a bounded symmetric domain [Ka1], and every bounded symmetric domain in any complex Banach space is bi-holomorphically equivalent to the open unit ball of a suitable complex JB^*-triple [Ka3].

Fundamental examples of complex JB^*-triples are provided by complex C^*-algebras, with triple product defined by

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x). \tag{1}$$

A larger class of complex JB^*-triples consists of the so-called ternary rings of operators [Zet], which are nothing but norm-closed subspaces of complex C^*-algebras, closed under the associative triple product xy^*z. Ternary rings of operators are seen as complex JB^*-triples by symmetrizing their associative triple products in the outer variables. A still larger class is that of complex JC^*-triples, i.e. JB^*-subtriples of complex C^*-algebras. The classical structure theory for complex JB^*-triples consists of a precise classification of certain prime complex JB^*-triples (the so-called complex Cartan

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factors) and the fact that every complex JB^*-triple has a faithful family of Cartan factor representations. Complex Cartan factors come in six different types. Those of type **I** are prime ternary rings of operators, whereas the ones of type **II** and **III** are the hermitian parts of certain prime ternary rings of operators relative to suitable complex-linear involutions. Complex Cartan factors of type **IV** (called complex spin factors) are JC^*-triples of a very simple algebraic and analytic nature, but in general they are neither ternary rings of operators nor hermitian parts of ternary rings of operators. Complex Cartan factors of types **V** and **VI** are exceptional (i.e., they are not JC^*-triples). Exceptional complex Cartan factors are very scarce: exactly, there is a single member in each type.

Applying the techniques of E. Zel'manov in [Ze2, Ze3, Ze4] (see also [D'Am, D'AMc]), we prove a classification theorem for general prime complex JB^*-triples, which, roughly speaking, asserts that prime complex JB^*-triples, which are neither spin factors nor exceptional Cartan factors, are "essentially" either prime ternary rings of operators or hermitian parts of prime ternary rings of operators. More precisely, our theorem establishes that, if J is a prime complex JB^*-triple, and if J is neither a spin factor nor an exceptional Cartan factor, then J contains a nonzero closed triple ideal which is either a prime ternary ring of operators or the hermitian part of a prime ternary ring of operators relative to a linear involution. By noticing that the multiplier complex JB^*-triple $M(R)$ (in the sense of [BC1]) of any ternary ring of operators R is also a ternary ring of operators, to which every linear involution on R extends uniquely, it follows that, for J as above, we have one of the following possibilities:

i) $R \subseteq J \subseteq M(R)$

ii) $H(R, \tau) \subseteq J \subseteq H(M(R), \tau)$,

where in both cases R is a prime ternary ring of operators, in the second case τ is a linear involution on R and $H(R, \tau)$ stands for the hermitian part of R relative to τ, the right inclusions must be read as "J is a JB^*-subtriple of ...", and consequently the left inclusions read as "... is a closed triple ideal of J".

The result just reviewed arises in Theorem 8.2 in a lightly different formulation involving "matricially decomposed" complex C^*-algebras (see Section 5 for the definition) instead of ternary rings of operators. We have preferred such a reformulation because of the scarcity of a well-developed theory for ternary rings of operators. We note that, if $A = \sum_{i,j\in\{1,2\}} A_{ij}$ is a matricially decomposed complex C^*-algebra, then A_{12} is a ternary ring of operators, and that, conversely, it follows from [Zet] that every ternary ring of operators is of the form A_{12} for some matricially decomposed complex C^*-algebra A.

After some forerunners [BC2, CDRV, Dan, DaRu], real JB^*-triples have recently attracted the attention of several authors. Real JB^*-triples are defined as norm-closed real subtriples of complex JB^*-triples (or, equivalently, as real forms of complex JB^*-triples). They have been introduced and studied in the paper of J.M. Isidro, W. Kaup and A. Rodríguez [IKR], where, as main result, it is proved that bijective linear mappings between real JB^*-triples are isometric if and only if they preserve the cube mapping $x \mapsto \{xxx\}$. For further developments of the theory of real JB^*-triples the reader is referred to [CMR, EdRu, Ka4, MaPe].

As in the complex case, real C^*-algebras are real JB^*-triples under the triple product formally defined as in (1). It is also important for our approach the fact that, if A is a real C^*-algebra, then the self-adjoint part of A, A_{sa}, is a JB^*-subtriple of A, and hence a real JB^*-triple. Other relevant examples of real JB^*-triples are obtained from real C^*-algebras A with a $*$-involution τ, by considering the set $S(A,\tau)$ of all skew elements of A relative to τ.

We also apply zelmanovian techniques to obtain the corresponding classification theorem for prime real JB^*-triples (Theorem 8.4). Let us say that a real JB^*-triple is a generalized real Cartan factor if it is either a complex Cartan factor (regarded as a real JB^*-triple) or a real form of a complex Cartan factor (compare [Ka4, Lemma 4.5]). Our theorem asserts that, if J is a prime real JB^*-triple, and if J is neither a generalized real spin factor nor an exceptional generalized real Cartan factor, then one of the following possibilities hold for J:

i) There exists a prime real C^*-algebra A such that J can be regarded as a JB^*-subtriple of the multiplier real C^*-algebra $M(A)$ contained in $M(A)_{sa}$ and containing A_{sa} .

ii) There exists a prime real C^*-algebra A with $*$-involution τ such that J can be regarded as a JB^*-subtriple of the real C^*-algebra $M(A)$ contained in $S(M(A),\tau) \cap M(A)_{sa}$ and containing $S(A,\tau) \cap A_{sa}$.

Now, let us lightly comment on the techniques applied in the proof of our results. In Zel'manov's work, Jordan triples over a field $\mathbb{F}$ of characteristic different from 2 and 3 are defined as vector spaces over $\mathbb{F}$ endowed with a triple product which is $\mathbb{F}$-linear in each of its variables and satisfies the same main identity required for JB^*-triples. The Zel'manov classification of nondegenerate prime Jordan triples relies on an apparently ingenuous alternative, by considering three mutually excluding cases, namely, non i-special, Clifford, and hermitian. A Jordan triple T is said to be special if it can be seen as a subtriple of an associative algebra A endowed with the triple product

$$\{abc\} := \frac{1}{2}(abc + cba), \tag{2}$$

and i-special if it is the homomorphic image of a special Jordan triple. An i-special Jordan triple T over $\mathbb{F}$ is said to be Clifford or hermitian depending on whether or not all the identities collected in a certain ideal of the free special Jordan triple over $\mathbb{F}$ vanish on T. Roughly speaking, a part of Zel'manov's prime theorem for Jordan triples establishes the scarcity, up to suitable scalar extensions, of nondegenerate prime Jordan triples which are not of hermitian type. The remaining part of Zel'manov's theorem shows that nondegenerate prime Jordan triples of hermitian type over $\mathbb{F}$ are "essentially" of the form $H(A,*) \cap S(A,\tau)$ for some associative algebra A over $\mathbb{F}$ with two commuting $\mathbb{F}$-linear involutions $*$ and τ. Here $H(A,*) \cap S(A,\tau)$ is regarded as a subtriple of A with triple product defined by (2).

The conjugate linear behaviour of the triple product of a complex JB^*-triple in its middle variable becomes a first handicap in applying zelmanovian notions and techniques in our setting. Concerning notions, there are no problems: we see complex JB^*-triples as Jordan triples over $\mathbb{R}$, and consider separately the non i-special, hermitian, and Clifford cases. However, a verbatim application of zelmanovian techniques to prime complex JB^*-triples would provide in the best of cases only a determination of the real structure of such JB^*-triples (see for instance Theorem 5.3). To overcome this difficulty, we have designed different strategies, which are explained in what follows. Our determination of non i-special complex prime JB^*-triples (the first part of Theorem 2.4) actually avoids Zel'manov's prime theorem for Jordan triples, and only uses Zel'manov's prime theorem for Jordan algebras [Ze1] through its version for JB^*-algebras [FGR]. Concerning prime complex JB^*-triples of Clifford type, we start with a rather artisanal determination of complex Cartan factors of Clifford type (Proposition 6.1). Such a determination leads us to realize that Banach ultraproducts of arbitrary families of complex Cartan factors of Clifford type are Hilbert spaces up to equivalent renormings (Corollary 6.2). Then we replace algebraic ultraproducts with Banach ultraproducts in an argument in [Ze4, pp. 63-64] (see also [D'AMc]) to obtain that every prime complex JB^*-triple of Clifford type is in fact a complex Cartan factor (Proposition 7.3). The determination of Clifford and non i-special prime real JB^*-triples (second parts of Theorems 7.4 and 2.4, respectively) follows easily from that of complex ones, by applying classical theory.

In studying real or complex JB^*-triples of hermitian type, a new handicap arises. Indeed, in the zelmanovian theory, the associative envelopes for special Jordan triples are associative algebras regarded as Jordan triples under the triple product (2), whereas the natural associative envelopes for real (respectively, complex) JC^*-triples are real (respectively, complex) C^*-algebras regarded as JB^*-triples under the triple product (1). Concerning prime real JB^*-triples of hermitian type (Theorem 4.5), things are not too difficult because, if A is a real C^*-algebra, then the two triple products of A given by (1) and (2) coincide on the self-adjoint part A_{sa} of A, and moreover every real JC^*-triple can be represented into a real JB^*-triple of the form A_{sa} for some real C^*-algebra A [IKR, Corollary 2.4]. Then zelmanovian techniques apply almost verbatim. The proof of the structure theorem for prime complex JB^*-triples of hermitian type (Theorem 5.9) is more difficult. Following an idea of O. Loos in [Lo2, 2.9], when a complex JB^*-triple J is regarded as a real Jordan pair, such a real Jordan pair is in fact the realification of a Jordan pair (say V) over $\mathbb{C}$. In the case that J is prime and hermitian, the polarization of V (say T) is a Jordan triple over $\mathbb{C}$ of hermitian type, which can be represented into the secondary diagonal of a matricially decomposed complex C^*-algebra regarded as Jordan triple under the product (2). Then zelmanovian techniques successfully apply to T, providing enough information for J (see the proof of Proposition 5.6 for details).

To conclude this introduction, let us refer the reader to the papers [BoFe, FGS] where Zel'manov's prime theorem for Jordan triples is applied to determine the structure of nondegenerate prime complex Banach Jordan triples with nonzero socle.

1 Notation and preliminaries.

For Banach spaces E, F, $BL(E,F)$ will denote the Banach space of all bounded linear mappings from E into F. When $E = F$, $BL(E)$ will stand for the Banach algebra $BL(E,E)$. A *complex JB*-triple* is a complex Banach space $\mathcal{A}$ with a continuous triple product $\{\cdots\} : \mathcal{A}\times\mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$ which is linear and symmetric in the outer variables, and conjugate linear in the middle variable, and satisfies

(i) for all $x \in \mathcal{A}$, the mapping $a \mapsto \{xxa\}$ from $\mathcal{A}$ to $\mathcal{A}$ is a hermitian element (in the sense of [BoDu, Definition §10.12]) of $BL(\mathcal{A})$ and has nonnegative spectrum;

(ii) $\{\mathrm{ab}\{\mathrm{xyz}\}\} = \{\{\mathrm{abx}\}\mathrm{yz}\} - \{\mathrm{x}\{\mathrm{bay}\}\mathrm{z}\} + \{\mathrm{xy}\{\mathrm{abz}\}\}$ (the main identity);

(iii) $\|\{aaa\}\| = \|a\|^3$.

Let E be a complex Banach space. By a *conjugation* on E we mean an involutive conjugate linear isometry on E. For a conjugation σ on E, we put $E^\sigma := \{x \in E : \sigma(x) = x\}$. Subsets of E of the form E^σ will be called *real forms* of E.

We define *real JB^*-triples* as norm-closed real subspaces J of complex JB^*-triples satisfying $\{JJJ\} \subseteq J$. Since conjugations on complex JB^*-triples preserve triple products (see for instance [IKR, pp. 316-317]), real forms of complex JB^*-triples are real JB^*-triples. Actually, every real JB^*-triple is a real form of a complex JB^*-triple [IKR, Proposition 2.2]. We note that if $J = \mathcal{A}^\sigma$ for some complex JB^*-triple $\mathcal{A}$ with a conjugation σ, then, essentially, the complex JB^*-triple with conjugation $(\mathcal{A}, \sigma)$ is uniquely determined by the real JB^*-triple J. Indeed, as a complex vector space, $\mathcal{A}$ is nothing but $\mathbb{C} \otimes_{\mathbb{R}} J$, the triple product of $\mathcal{A}$ and the conjugation σ on $\mathcal{A}$ are settled by $\mathbb{R}$-linearity in their variables and the equalities

$$\{(\lambda_1 \otimes x_1)(\lambda_2 \otimes x_2)(\lambda_3 \otimes x_3)\} = \lambda_1 \overline{\lambda_2} \lambda_3 \otimes \{x_1 x_2 x_3\}$$

and

$$\sigma(\lambda \otimes x) = \overline{\lambda} \otimes x \ ,$$

respectively, and the norm on $\mathcal{A}$ is unique because triple isomorphisms between complex JB^*-triples are isometries.

The next crucial proposition follows from [IsKa, Proposition 1.2] and the above quoted fact that real JB^*-triples are real forms of complex JB^*-triples.

PROPOSITION 1.1. *Triple homomorphisms between real or complex JB^*-triples are contractive and have closed range. If moreover they are injective, then they are isometries.*

Closed linear subspaces B of a real or complex JB^*-triple A satisfying $\{BBB\} \subseteq B$ will be called *JB^*-subtriples* of A. The next result is folklore.

PROPOSITION 1.2. *Let A be a real or complex JB^*-triple, and let y be in A. Then there is a unique element z in A such that $\{zzz\} = y$.*

Proof. First assume that A is complex. For x in A, let us denote by A_x the JB^*-subtriple of A generated by x. Since A_y is of the form $C_0(L)$ for a suitable locally compact subset of $\mathbb{C}$ [Ka1], certainly there exists z in A_y satisfying $\{zzz\} = y$. Let w be in A such that $\{www\} = y$. Then we have $y \in A_y \subseteq A_w = C_0(L')$, so that, by the uniqueness of the solution of the equation $\{ttt\} = y$ in $C_0(L')$, we deduce $z = w$. Now assume that A is real. Then $A = \mathcal{A}^\sigma$ for some complex JB^*-triple with conjugation $(\mathcal{A}, \sigma)$. Taking z in $\mathcal{A}$ satisfying $\{zzz\} = y$, we have $\{\sigma(z)\sigma(z)\sigma(z)\} = \sigma(y) = y$. By the uniqueness, we obtain $\sigma(z) = z$, and hence z lies in A. ∎

A linear subspace I of a (real or complex) JB^*-triple A is a *triple ideal* of A if $\{AAI\} + \{AIA\} \subseteq I$.

COROLLARY 1.3. *Let A be a real or complex JB^*-triple.*

i) A closed subspace I of A is a triple ideal if (and only if) $\{AAI\} \subseteq I$.

ii) If J is a closed triple ideal of A, and K is a closed triple ideal of J, then K is a triple ideal of A.

Proof. For the complex case of assertion i), see [DiTi, Proposition 1.4]. Then the real case of that assertion follows from the complex one and the fact that real JB^*-triples are real forms of complex JB^*-triples.

To prove ii), let J be a closed triple ideal of A, and K a closed triple ideal of J. Take a, b in A and y in K. By Proposition 1.2 there is z in K such that $\{zzz\} = y$. Using the main identity, we obtain

$$\{aby\} = \{ab\{zzz\}\} = 2\{\{abz\}zz\} - \{z\{baz\}z\} \in \{JKK\} + \{KJK\} \subseteq K.$$

Therefore, by assertion i), K is a triple ideal of A. ∎

Assertion ii) in the above corollary is the natural variant for JB^*-triples of the a result for C^*-algebras, which is collected here for later reference.

PROPOSITION 1.4. *Each closed ideal K in a real or complex C^*-algebra A is $*$-invariant. A closed ideal I in K is an ideal in A.*

Proof. For the complex case see [KaRi, Corollary 4.2.10]. The real case follows from the complex one by considering the complexification of the ideals in the C^*-algebra complexification of A (see [Goo, p. 108]). ∎

A real or complex JB^*-triple A is said to be *prime* if it has no nonzero mutually orthogonal triple ideals, that is, if P, Q are triple ideals of A such that $\{PAQ\} = 0$ (equivalently $P \cap Q = 0$), then either $P = 0$ or $Q = 0$. The following result follows directly from Corollary 1.3.ii.

COROLLARY 1.5. *Every closed triple ideal of a prime real or complex JB^*-triple is a prime JB^*-triple.*

Real (respectively, complex) C^*-algebras are natural examples of real (respectively, complex) JB^*-triples under their natural norms and the triple product defined by

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x) \ .$$

JB^*-subtriples of real (respectively, complex) C^*-algebras are known under the name of real (respectively, complex) *JC^*-triples.*

2 Prime JB^*-triples which are not JC^*-triples.

In this section we determine those prime JB^*-triples which are not JC^*-triples. The key tools for such a determination are the Gelfand-Naimark theorem for complex JB^*-triples [FR2], the classification theorem for prime JB^*-algebras [FGR], and Loos' theory of finite dimensional JB^*-triples [Lo2]. We recall that a *JB^*-algebra* is a complete normed Jordan complex algebra A with a conjugate linear algebra involution $*$ satisfying $\|U_a(a^*)\| = \|a\|^3$ for every a in A, where, for a, b in A, $U_a(b) := 2a.(a.b) - a^2.b$. According to [BKU, You], every JB^*-algebra becomes a JB^*-triple under its natural norm and the triple product defined by

$$\{abc\} := a.(b^*.c) + c.(b^*.a) - (a.c).b^* \ .$$

Complex C^*-algebras become JB^*-algebras under their natural norms and involutions, whenever the Jordan product is defined by $a.b := \frac{1}{2}(ab+ba)$. Norm-closed $*$-invariant subalgebras of JB^*-algebras are called *JB^*-subalgebras*, and JB^*-subalgebras of complex C^*-algebras are known under the name of *JC^*-algebras.* Of course JC^*-algebras are natural examples of complex JC^*-triples.

PROPOSITION 2.1. *Let J be a real (respectively, complex) JB^*-triple. Then there exists a JB^*-algebra A containing J as a real (respectively,* *complex) JB^*-subtriple and with the property that every nonzero closed ideal of A has a nonzero intersection with J.*

Proof.- By [FR2, Corollary 2], there exists a JB^*-algebra B containing J as a real (respectively, complex) JB^*-subtriple. Consider the family $\mathcal{F}$ of all closed ideals of B which have zero intersection with J, ordered by inclusion. Let $\mathcal{C}$ be a totally ordered subset of $\mathcal{F}$. Then the closure of the union of the members of $\mathcal{C}$ is a closed ideal of B (say P). If I is in $\mathcal{C}$, then B/I is a JB^*-algebra in a natural manner [Wri], and the mapping $x \mapsto x+I$ from J to B/I is a one-to-one triple homomorphism, hence, by Proposition 1.1, the equality

$$\|x + I\| = \|x\|$$

holds for every x in J. Therefore we have $\|x - y\| \geq \|x\|$ for all x in J and y in P, and hence P is an upper bound of $\mathcal{C}$ in $\mathcal{F}$. Now that we know that $\mathcal{F}$ is inductive, we may take a maximal element Q in $\mathcal{F}$, and consider the JB^*-algebra $A := B/Q$ together with the embedding $x \mapsto x + Q$ from J to A. ■

Proposition 2.1 above leads directly to the following corollary.

COROLLARY 2.2. *Let J be a prime real (respectively, complex) JB^*-triple. Then there exists a prime JB^*-algebra A containing J as a real (respectively, complex) JB^*-subtriple.*

COROLLARY 2.3. *Let J be a prime real or complex JB^*-triple which is not a JC^*-triple. Then J is finite-dimensional.*

Proof.- Since J is prime, we can aplly Corollary 2.2 to find a prime JB^*-algebra A containing J as a real or complex JB^*-subtriple. Since J is not a JC^*-triple, A cannot be a JC^*-algebra. Now, looking at the list of prime JB^*-algebras provided by [FGR, Theorem 2.3], we realize that the unique prime JB^*-algebra which is not a JC^*-algebra is the 27-dimensional type **VI** Cartan factor $H_3(\mathbb{O}_{\mathbb{C}})$ of all hermitian 3×3-matrices over the complex Cayley algebra $\mathbb{O}_{\mathbb{C}}$. ■

After Corollary 2.3 above, the main goal in this section follows from the work of O. Loos on finite-dimensional JB^*-triples [Lo2]. In fact, via [Lo2, 2.9], finite-dimensional complex JB^*-triples can be recognized in Loos' book as the so-called "complex Jordan pairs with a positive hermitian involution".

Since every finite-dimensional JB^*-triple is a direct sum of simple ideals [Lo2, 4.10, 4.11, 11.4], it follows that finite-dimensional prime JB^*-triples are simple. Now, keeping in mind Corollary 2.3 and looking at the list of finite-dimensional simple JB^*-triples provided in [Lo2, 4.14] and [Lo2, 11.4] for the complex and real case, respectively, we get the following.

THEOREM 2.4. *The prime complex JB^*-triples which are not JC^*-triples are the type* $\mathbf{V}$ *complex Cartan factor $M_{1,2}(\mathbb{O}_{\mathbb{C}})$ of all 1×2-matrices over the complex Cayley algebra $\mathbb{O}_{\mathbb{C}}$, and the type* $\mathbf{VI}$ *complex Cartan factor $H_3(\mathbb{O}_{\mathbb{C}})$ of all hermitian 3×3-matrices over $\mathbb{O}_{\mathbb{C}}$. The prime real JB^*-triples which are not JC^*-triples are the two complex JB^*-triples above (regarded as real JB^*-triples) plus their real forms, namely the type $\mathbf{V}^{\mathbb{O}}$, $\mathbf{V}^{\mathbb{O}_0}$, $\mathbf{VI}^{\mathbb{O}}$, and $\mathbf{VI}^{\mathbb{O}_0}$ real Cartan factors. These are $M_{1,2}(\mathbb{O})$, $M_{1,2}(\mathbb{O}_0)$, $H_3(\mathbb{O})$, and $H_3(\mathbb{O}_0)$, respectively, where $\mathbb{O}$ is the real Cayley division algebra, and $\mathbb{O}_0$ the real split Cayley algebra.*

For details about the precise definition of the triple product on each of the JB^*-triples in Theorem 2.4 the reader is referred to [Lo2, 4.11, 2.9, 11.4].

3 Multipliers of real JB^*-triples of classical type.

The results in this section will be auxiliary tools for the classification of real JB^*-triples of hermitian type, which will be obtained in the next section. The theory in [BC1] about multipliers on complex JB^*-triples can be transferred to the setting of real JB^*-triples [CMR, Propositions 3 and 4], so that we have the following.

PROPOSITION 3.1. *Let J be a real (respectively, complex) JB^*-triple. Then J can be seen as a closed essential triple ideal of a real (respectively, complex) JB^*-triple $M(J)$ with the property that, if K is any real (respectively, complex) JB^*-triple containing J as a closed essential triple ideal, then K can be seen as a JB^*-subtriple of $M(J)$ containing J.*

Let J be a real or complex JB^*-triple. The JB^*-triple $M(J)$ in the above proposition is called the *multiplier JB^*-triple of J*. Actually, the bidual J'' of J is a JB^*-triple containing J as a JB^*-subtriple ([IKR, Lemma 4.2], [Din]), and $M(J)$ can be found as the JB^*-subtriple of J'' given by

$$M(J) = \{x \in J'' : \{xJJ\} \subseteq J\}$$

[CMR, BC1].

In Section 1, we introduced real C^*-algebras as relevant examples of real JB^*-triples. Now, we recall that *real C^*-algebras* can be defined as norm-closed $*$-invariant real subalgebras of complex C^*-algebras, and refer the reader to [CDRV, Goo, IsRo] for intrinsic characterizations of real C^*-algebras and background on their theory. If A is a real C^*-algebra regarded as a real JB^*-triple, then $M(A)$ is also a real C^*-algebra (again regarded as JB^*-triple). More precisely, we have the following.

PROPOSITION 3.2. *Let A be a real (respectively, complex) C^*-algebra. Then its second dual A'' is a real (respectively, complex) C^*-algebra containing A as a C^*-subalgebra, and $M(A)$ can be found as the C^*-subalgebra of A'' given by*

$$M(A) = \{x \in A'' : xA + Ax \subseteq A\} \ .$$

Proof. The first assertion is folklore in the complex case, and is proved in [CDRV, Theorem 1.6] in the real case. For the second assertion in the complex case the reader is referred to [BC1, p. 253] (actually, the proof given immediately below for the real case works verbatim in the complex setting).

Let A be a real C^*-algebra. Clearly the inclusion

$$M(A) \supseteq \{x \in A'' : xA + Ax \subseteq A\}$$

holds. To prove the converse inclusion, consider x in $M(A)$ and z in A. By Proposition 1.2 there exists y in A such that $z = \{yyy\}$, and we have

$$xz = x\{yyy\} = 2\{xy^*y\}y - y^*\{y^*xy^*\}^*.$$

Therefore xz lies in A. Analogously, zx belongs to A. ∎

For any involutive mapping τ on a vector space E, we put

$$H(E,\tau) := \{x \in E \ : \ \tau(x) = x\} \text{ and } S(E,\tau) := \{x \in E \ : \ \tau(x) = -x\}.$$

Let A be a real or complex C^*-algebra. As usual, we write A_{sa} instead of $H(A,*)$. By a *$*$-involution* on A we means a linear involution on A commuting with $*$. Let τ be a $*$-involution on A. Keeping in mind Proposition 3.2 and the fact that the C^*-involution and the product of A'' are w^*-continuous in their variables [CDRV, Theorem 1.6], it follows that the second transpose of τ, τ'', is a $*$-involution on A'' leaving invariant $M(A)$. Then the mapping

$x \mapsto \tau''(x)$ from $M(A)$ to $M(A)$ is a $*$-involution on $M(A)$ extending τ, and therefore it will be denoted by the same symbol τ. Now, when A is a real C^*-algebra, we can proceed to recognize the multiplier JB^*-triple of $S(A,\tau) \cap A_{sa}$ as a JB^*-subtriple of $M(A)$.

PROPOSITION 3.3. *Let A be a real C^*-algebra with $*$-involution τ such that A is generated (as a real C^*-algebra) by $S(A,\tau) \cap A_{sa}$. Then*

$$M[S(A,\tau) \cap A_{sa}] = S(M(A),\tau) \cap M(A)_{sa} \ .$$

Proof. Put $J := S(A,\tau) \cap A_{sa}$. For any subspace E of A, denote by E^{oo} the bipolar of E in A''. Through the natural identification of J'' with J^{oo}, we can see $M(J)$ as a JB^*-subtriple of A'' contained in J^{oo}. Now, the inclusion

$$M(J) \supseteq S(M(A),\tau) \cap M(A)_{sa}$$

is clear. To prove the converse inclusion, let x be in $M(J)$. For $y \in J$ we have

$$xy^3 = 2\{xyy\}y - y\{yxy\} \in A.$$

In this way, since every self-adjoint element in a real C^*-algebra has a unique self-adjoint cube root (a particular case of Proposition 1.2), we easily obtain $xJ \subseteq A$. Analogously, we have $Jx \subseteq A$. Therefore, the set $\{a \in A : xa, ax \in A\}$ is a self-adjoint closed subalgebra of A containing J, hence $xA + Ax \subseteq A$ (because J generates A as C^*-algebra). It follows that x lies in $M(A)$. Now that we know that $M(J)$ is contained in $M(A)$, the inclusion

$$M(J) \subseteq S(M(A),\tau) \cap M(A)_{sa}$$

follows from the equality $J^{oo} = S(A'',\tau'') \cap (A'')_{sa}$. ∎

Arguing as in the proof of the above proposition we find the following.

PROPOSITION 3.4. *Let A be a real C^*-algebra generated (as C^*-algebra) by A_{sa}. Then $M(A_{sa}) = M(A)_{sa}$.*

4 Prime real JB^*-triples of hermitian type.

In this section we determine the real JB^*-triples which are of "hermitian type" in the sense of Zel'manov. Before to introduce zelmanovian concepts and techniques, let us formulate and prove some lemmas.

LEMMA 4.1. *Let J be a real JC^*-triple. Then there exists a real C^*-algebra A with $*$-involution τ such that J is a JB^*-subtriple of A contained in $S(A,\tau)\cap A_{sa}$.*

Proof.- By [IKR, Corollary 2.4] there exists a real C^*-algebra B such that J is a JB^*-subtriple of B contained in B_{sa} . Now it is enough to consider the real C^*-algebra $A := B \oplus B$, the $*$-involution τ on A given by $\tau(x,y) := (y^*, x^*)$, and the embedding $x \mapsto (x, -x)$ from J to A. ■

An associative algebra A with involution τ is said to be *τ-prime* if whenever P, Q are τ-invariant ideals of A with $PQ = 0$ we have either $P = 0$ or $Q = 0$.

LEMMA 4.2. *Let A be a real C^*-algebra with a $*$-involution τ such that A is generated as a closed ideal by $S(A,\tau)\cap A_{sa}$. Then every nonzero τ-invariant closed ideal of A meets $S(A,\tau)\cap A_{sa}$. Therefore, if the JB^*-triple $S(A,\tau)\cap A_{sa}$ is prime, then A is τ-prime.*

Proof. Let P be a τ-invariant closed ideal of A satisfying $P\cap S(A,\tau)\cap A_{sa} = 0$. Let P^{oo} be the bipolar of P in A''. Then P^{oo} is a w^*-closed ideal of A''. Moreover, since the equality $P \cap S(A,\tau)\cap A_{sa} = 0$ can be read as $(1+*)(1-\tau)(P) = 0$ (by Proposition 1.4), we have $(1+*)(1-\tau'')(P^{oo}) = 0$, where now $*$ denote the C^*-involution of A''. On the other hand, by [CDRV, Proposition 2.10], there exists a central self-adjoint idempotent e in A'' such that $P^{oo} = A''e$. Since P^{oo} is τ''-invariant, we have $\tau''(e) = e$. It follows that the operator R_e of multiplication by e on A'' commutes with $*$ and τ''. Now, we have

$$\begin{aligned} R_e(S(A'',\tau'')\cap A''_{sa}) &= R_e(1+*)(1-\tau'')(A'') \\ &= (1+*)(1-\tau'')R_e(A'') \\ &= (1+*)(1-\tau'')(P^{oo}) \\ &= 0\ , \end{aligned}$$

hence $S(A'',\tau'')\cap A''_{sa} \subseteq Ker(R_e)$. Since $Ker(R_e)$ is a w^*-closed ideal of A'' and A'' is generated by $S(A'',\tau'')\cap A''_{sa}$ as a w^*-closed ideal (because

A is generated by $A_{sa} \cap S(A, \tau)$ as a norm-closed ideal), we actually have $P \subseteq P^{oo} = R_e(A'') = 0$.

Now, if the JB^*-triple $S(A, \tau) \cap A_{sa}$ is prime, and if P and Q are τ-invariant closed ideals of A such that $PQ = 0$, then $P \cap S(A, \tau) \cap A_{sa}$ and $Q \cap S(A, \tau) \cap A_{sa}$ are mutually orthogonal triple ideals of $S(A, \tau) \cap A_{sa}$, so

$$P \cap S(A, \tau) \cap A_{sa} = 0,$$

say, and so $P = 0$ by the first paragraph of the proof. ∎

The proof of the following lemma is much easier.

LEMMA 4.3. *Let A be a real C^*-algebra. Then every nonzero closed ideal of A meets A_{sa}. Therefore, if the JB^*-triple A_{sa} is prime, then A is prime.*

Proof. Let P be a closed ideal of A such that $P \cap A_{sa} = 0$. Since P is $*$-invariant, P is contained in $S(A, *)$. Therefore P is an associative anticommutative algebra, and hence $P^3 = 0$. This implies $P = 0$ by the semiprimeness of A. ∎

Now we pass to explain those zelmanovian techniques which are needed for our purpose.

Let $\mathbb{F}$ be a field of characteristic different from 2 and 3. By a *Jordan triple* over $\mathbb{F}$ we mean a vector space T over $\mathbb{F}$ together with a trilinear triple product $\{\cdots\} : T \times T \times T \longrightarrow T$ satisfying

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

for all a, b, x, y, z in T. According to [Lo2, 2.8], "this definition may be extended to arbitrary base fields but then the identity above turns out to be too weak to develop a satisfactory theory, and has to be replaced by more complicated identities (cf. [Lo1, 1.2]). As long as the base field has characteristic different from 2 and 3, however, the identity above is sufficient (cf. [Lo1, 2.2])". Forgetting the analytic conditions, and restricting the scalars in the complex case, every real or complex JB^*-triple becomes a Jordan triple over $\mathbb{R}$. Any associative algebra A can be seen as a Jordan triple under the triple product $\{abc\} := \frac{1}{2}(abc + cba)$. If moreover A has two commuting linear involutions $*, \tau$, then $H(A, *) \cap S(A, \tau)$ is a Jordan subtriple of A. A Jordan triple is called *special* whenever it is (isomorphic to) a Jordan subtriple of some associative algebra, and *i-special* if it is the homomorphic image of a special Jordan triple. Thanks to Lemma 4.1 (or even its forerunner [IKR, Corollary 2.4]), every JC^*-triple is a special Jordan triple over $\mathbb{R}$.

From now on, let X denote an infinite set of indeterminates. We consider the free associative algebra $\mathcal{A}(X)$ over $\mathbb{F}$. $\mathcal{A}(X)$ has two natural linear involutions, namely the involution $*$ leaving the elements of the set X fixed, and the one τ which maps each element x in X into $-x$. The Jordan subtriple of $\mathcal{A}(X)$ generated by X is called the *free special Jordan triple* on the set of free generators X, and is denoted by $\mathcal{ST} = \mathcal{ST}(X)$. Clearly, the inclusion $\mathcal{ST} \subseteq H(\mathcal{A}(X), *) \cap S(\mathcal{A}(X), \tau)$ holds.

It follows from the universal property of $\mathcal{A}(X)$ that, if A is an associative algebra with two commuting involutions $*, \tau$, and if T is a Jordan subtriple of A contained in $H(A, *) \cap S(A, \tau)$ then every map $\phi : X \to T$ extends to a unique associative $*$-τ-homomorphism $\hat{\phi} : \mathcal{A}(X) \to A$ such that $\hat{\phi}(\mathcal{ST}) \subseteq T$. Since every special Jordan triple can be regarded as a Jordan subtriple of a suitable associative algebra with two commuting linear involutions $(A, *, \tau)$ contained in $H(A, *) \cap S(A, \tau)$, it follows that, given a special Jordan triple T, every mapping from X to T extends uniquely to a triple-homomorphism from $\mathcal{ST}$ to T. Keeping in mind the definition of i-special Jordan triples, the above universal property remains true in the more general case that T is a i-special Jordan triple. In this way, we can consider valuations of elements of $\mathcal{ST}$ in any i-special Jordan triple.

For elements $a_1, ..., a_n$ in an associative algebra A, we define the *n-tad* $\{a_1 ... a_n\}_n$ as the element of A given by

$$\{a_1 ... a_n\} = \frac{1}{2}(a_1 ... a_n + a_n ... a_1) \ .$$

An ideal $\mathcal{I}$ of $\mathcal{ST}$ is called *formal* if

$$p(x_1, ..., x_n) \in \mathcal{I} \Rightarrow p(\sigma(x_1), ..., \sigma(x_n)) \in \mathcal{I}$$

for all permutations σ of X, and *hermitian* if it is formal and "n-tad closed" in $\mathcal{A}(X)$ for all odd $n \geq 5$, i.e.,

$$\{\mathcal{I} ... \mathcal{I}\}_n \subseteq \mathcal{I} \text{ for all odd } n \geq 5 \ .$$

Now, if A is an associative algebra with two commuting involutions $*$ and τ, if T is a Jordan subtriple of A contained in $H(A, *) \cap S(A, \tau)$, and if $\mathcal{I}$ is a hermitian ideal of $\mathcal{ST}$, then $\{a_1 ... a_n\}$ lies in $\mathcal{I}(T)$ whenever n is any odd natural number and $a_1, ..., a_n$ are in $\mathcal{I}(T)$.

LEMMA 4.4. *Let B be an associative algebra with two commuting involutions $*, \tau$, and T a Jordan subtriple of B contained in $H(B,*) \cap S(B,\tau)$. If $\mathcal{I}$ is a hermitian ideal of $\mathcal{ST}$ satisfying $\mathcal{I}(T) \neq 0$, then the subalgebra C of B generated by $\mathcal{I}(T)$ is $*$-τ-invariant and we have $\mathcal{I}(T) = H(C,*) \cap S(C,\tau)$.*

Proof. By the previous comments, $\mathcal{I}(T)$ is an n-tad closed ideal of T and the inclusion $\mathcal{I}(T) \subset H(B,*) \cap S(B,\tau)$ holds. From that inclusion we deduce that the subalgebra C of B generated by $\mathcal{I}(T)$ is $*$-τ-invariant and that $\mathcal{I}(T)$ is contained in $H(C,*) \cap S(C,\tau)$. Let z be in $H(C,*) \cap S(C,\tau)$. Since z is in C, $z = \sum_{n=1}^{k} b_n^1 ... b_n^{i_n}$ for suitable $k \in \mathbb{N}$ and $b_n^i \in \mathcal{I}(T)$ $(n = 1, ..., k, i = 1, ..., i_n)$. Moreover, we have

$$\sum_{n=1}^{k} b_n^1 ... b_n^{i_n} = z = z^* = \sum_{n=1}^{k} b_n^{i_n} ... b_n^1$$

and hence

$$z = \sum_{n=1}^{k} \frac{1}{2}(b_n^1 ... b_n^{i_n} + b_n^{i_n} ... b_n^1) \ .$$

Analogously, the equality $z = -z^\tau$ implies

$$z = \sum_{n=1}^{k} \frac{1}{2}(b_n^1 ... b_n^{i_n} - (-1)^{i_n} b_n^{i_n} ... b_n^1) \ .$$

It follows

$$z = \sum_{\substack{n=1 \\ n \text{ odd}}}^{k} \frac{1}{2}(b_n^1 ... b_n^{i_n} + b_n^{i_n} ... b_n^1) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{k} \{b_n^1 ... b_n^{i_n}\} \in \mathcal{I}(T) \ . \quad \blacksquare$$

One of the key tools in Zel'manov's work is the discovery of a precise hermitian ideal $\mathcal{G}$ (see [Ze3, p. 730]) in $\mathcal{ST}$ with the property that the behaviour of i-special prime "nondegenerate" Jordan triples drastically differs depending on whether or not $\mathcal{G}$ vanishes on them. By a Jordan triple *of hermitian type* we mean an i-special Jordan triple T satisfying $\mathcal{G}(T) \neq 0$.

Now, the structure of prime real JB^*-triples of hermitian type is given by the theorem which follows.

THEOREM 4.5. *Let J be a prime real JB^*-triple of hermitian type. Then one of the following assertions is true for J :*

i) There exists a prime real C^*-algebra A such that J can be regarded as a JB^*-subtriple of the real C^*-algebra $M(A)$ contained in $M(A)_{sa}$ and containing A_{sa} .

ii) There exists a prime real C^*-algebra A with $*$-involution τ such that J can be regarded as a JB^*-subtriple of the real C^*-algebra $M(A)$ contained in $S(M(A),\tau)\cap M(A)_{sa}$ and containing $S(A,\tau)\cap A_{sa}$.

Proof.- Since J is i-special and all JB^*-triples listed in Theorem 2.4 are non i-special, it follows from that theorem that J is a JC^*-triple. Then, by Lemma 4.1, there exists a real C^*-algebra B with $*$-involution τ such that J is a JB^*-subtriple of B contained in $S(B,\tau)\cap B_{sa}$. Since $\mathcal{G}(J)\neq 0$, Lemma 4.4 gives us that, if C denotes the subalgebra of B generated by $\mathcal{G}(J))$, then $H(C,*)\cap S(C,\tau)$ is a nonzero ideal of J. Denote by A and I the norm closures of C and $H(C,*)\cap S(C,\tau)$ in B, respectively. Then A is a τ-invariant C^*-subalgebra of B, I is a nonzero closed triple ideal of J, A is generated by I as a C^*-algebra, and we have $I=S(A,\tau)\cap A_{sa}$. Since J is prime, I is an essential triple ideal of J, so that Propositions 3.1 and 3.3 allow us to see J as a JB^*-subtriple of the real C^*-algebra $M(A)$ contained in $S(M(A),\tau)\cap M(A)_{sa}$ and containing $S(A,\tau)\cap A_{sa}$. If A is prime, then we are in case ii). Assume from now on that A is nonprime. Since A is τ-prime (by Corollary 1.5 and Lemma 4.2), we can find a nonzero closed ideal P of A such that $P\cap\tau(P)=0$. But P is a prime algebra because, if R,S are closed ideals of P with $RS=O$, then $R+\tau(R), S+\tau(S)$ are τ-invariant ideals of A (by Proposition 1.4) satisfying $(R+\tau(R))(S+\tau(S))=0$ and therefore either $R+\tau(R)=0$ or $S+\tau(S)=0$. Now, P is a real C^*-algebra (by Proposition 1.4), and the mapping $\phi : x\mapsto x-\tau(x)$ from P_{sa} into $S(A,\tau)\cap A_{sa}$ is a one-to-one triple homomorphism whose range is a triple ideal of $S(A,\tau)\cap A_{sa}$. Finally, let us denote by $\mathtt{A}$ the closed subalgebra of P generated by P_{sa}. Then $\mathtt{A}$ is a real C^*-algebra, we have $\mathtt{A}_{sa}=P_{sa}$, and, keeping in mind Proposition 1.1, we can forget the embedding ϕ above and regard $\mathtt{A}_{sa}$ as a closed triple ideal of $S(A,\tau)\cap A_{sa}$. Since $S(A,\tau)\cap A_{sa}$ is a closed triple ideal of J, it follows from Corollary 1.3.ii that $\mathtt{A}_{sa}$ can be seen as a nonzero closed triple ideal of the prime real JB^*-triple J. By Corollary 1.5 $\mathtt{A}_{sa}$ is a prime JB^*-triple, and hence, by Lemma 4.3, $\mathtt{A}$ is a prime C^*-algebra. By Propositions 3.1 and 3.4, J can be regarded as a JB^*-subtriple of the real C^*-algebra $M(\mathtt{A})$ contained in $M(\mathtt{A})_{sa}$ and containing $\mathtt{A}_{sa}$. Therefore we are in case i) of the statement. ∎

5 Prime complex JB^*-triples of hermitian type.

In the first part of this section we prove that, when a prime complex JB^*-triple of hermitian type J is regarded as a real JB^*-triple, then only case *ii*) in Theorem 4.5 can occur for J. In the second part, we determine the complex structure of prime complex JB^*-triples of hermitian type.

For every complex Banach space E we denote by $E_{\mathbb{R}}$ the real Banach space obtained by restriction of the scalars to $\mathbb{R}$. Note that, if J is a complex JB^*-triple, then $J_{\mathbb{R}}$ is a real JB^*-triple in a natural way.

LEMMA 5.1. *Let J be a complex JB^*-triple, and P a closed triple ideal of $J_{\mathbb{R}}$. Then P is a triple ideal of J.*

Proof. Let x be in P. Then, by Proposition 1.2, there is y in P such that $x = \{yyy\}$. Therefore $\mathtt{i}x = \{yy(\mathtt{i}y)\}$ lies in P. ■

Recall that a *JB-algebra* is a complete normed real Jordan algebra B satisfying $\|x\|^2 \leq \|x^2 + y^2\|$ for all x, y in B. If A is a real C^*-algebra, then A_{sa} is a JB-algebra under the Jordan product $x.y := \frac{1}{2}(xy + yx)$.

LEMMA 5.2. *A nonzero JB-algebra cannot be linearly isometric to the realification of any complex Banach space.*

Proof. Let B a nonzero JB-algebra, and assume that, for some complex Banach space E, there exists a linear isometry φ from B onto $E_{\mathbb{R}}$. Since B'' is a JB-algebra with a unit [HaSt, Theorem 4.4.3 and Lemma 4.17], we may assume that B has a unit $\mathbf{1}$. Put $x := \varphi^{-1}(\mathtt{i}\varphi(\mathbf{1}))$. Then, for α in $\mathbb{R}$, we have $\varphi(\mathbf{1} + \alpha x) = (1 + \mathtt{i}\alpha)\varphi(\mathbf{1})$, and hence $\|\mathbf{1} + \alpha x\|^2 = 1 + \alpha^2$. Now, let ρ be in B' such that $\|\rho\| = \rho(\mathbf{1}) = 1$. Then, for every positive number α, we have

$$\pm\rho(x) = \frac{\rho(\mathbf{1} \pm \alpha x) - 1}{\alpha} \leq \frac{\|\mathbf{1} \pm \alpha x\| - 1}{\alpha} = \frac{\sqrt{1 + \alpha^2} - 1}{\alpha},$$

so $|\rho(x)| \leq \lim_{\alpha \to 0^+} \frac{\sqrt{1+\alpha^2}-1}{\alpha} = 0$, and so $\rho(x) = 0$. By [HaSt, Proposition 3.3.10, Lemma 1.2.2, and Lemma 1.2.3 (ii)], we have that $x = 0$, a contradiction. ■

THEOREM 5.3. *Let J be a prime complex JB^*-triple of hermitian type. Then there exists a prime real C^*-algebra A with $*$-involution τ such* *that $J_{\mathbb{R}}$ can be regarded as a JB^*-subtriple of the real C^*-algebra $M(A)$ contained in $S(M(A),\tau)\cap M(A)_{sa}$ and containing $S(A,\tau)\cap A_{sa}$.*

Proof. Clearly, $J_{\mathbb{R}}$ is a prime real JB^*-triple of hermitian type, so, in view of Theorem 4.5, it is enough to show that case *i)* in that theorem cannot occur for $J_{\mathbb{R}}$. If case *i)* in that theorem happened for $J_{\mathbb{R}}$, and if A is the real C^*-algebra arising there, then, by Lemma 5.1, A_{sa} would be a closed triple ideal of J and hence a complex Banach space. But this is not possible because A_{sa} is a JB-algebra and Lemma 5.2 applies. ■

Now, we proceed to deal with the determination of the complex structure of prime complex JB^*-triples of hermitian type.

LEMMA 5.4. *Let T be a i-special complex Jordan triple. Then $\mathcal{G}(T)$ is invariant under every conjugate linear automorphism of T.*

Proof. If $\mathcal{A}(X)$ is the free complex associative algebra on the infinite set X of indeterminates, then the vector space of $\mathcal{A}(X)$ is the free complex vector space on the set of all associative words with letters in X, so that we can define a mapping $\Phi:\mathcal{A}(X)\to\mathcal{A}(X)$ by $\Phi(\sum_i \lambda_i w_i) := \sum_i \overline{\lambda_i} w_i$ (here $\sum_i \lambda_i w_i$ is any finite linear combination of associative words w_i). Clearly, Φ is an involutive conjugate linear algebra automorphism of $\mathcal{A}(X)$ which fixes the elements of X. Therefore the free special complex Jordan triple $\mathcal{ST}(X)$ is invariant under Φ. Since the zelmanovian complex ideal $\mathcal{G}$ of $\mathcal{ST}(X)$ is generated (as ideal) by a set of elements of $\mathcal{ST}(X)$ whose expressions as linear combinations of associative words only involve real scalars, it follows that $\mathcal{G}$ is Φ-invariant. Now, if ϕ is a conjugate linear automorphism of T, if $p(x_1,...,x_n)$ is in $\mathcal{G}$, and if $p=\sum_k \lambda_k q_k$ where λ_k are complex numbers and q_k are Jordan triple monomials, then $\Phi(p)=\sum_k \overline{\lambda_k} q_k$ lies in $\mathcal{G}$, and hence, for $t_1,...t_n$ in T,

$$\phi[p(t_1,...,t_n)] = \sum_k \overline{\lambda_k}\phi[q_k(t_1,...,t_n)] = \sum_k \overline{\lambda_k} q_k(\phi(t_1),...,\phi(t_n))$$

belongs to $\mathcal{G}(T)$. ■

By a *matricial decomposition* of a C^*-algebra A we means a family $\{A_{ij}\}_{i,j\,\in\{1,2\}}$ of closed subspaces of A satisfying $A_{ij}^* = A_{j\,i}$ (i.e., $*$ is "odd-swapping" relative to the matricial decomposition), $A=\oplus_{i,j\,\in\{1,2\}} A_{ij}$, and $A_{ij}\ A_{kl} \subseteq \delta_{j\ k} A_{il}$ for all $i, j, k, l \in \{1, 2\}$. A *matricially decomposed* C^*-*algebra* will be a C^*-algebra endowed with a matricial decomposition. Given a $*$-involution τ on a matricially decomposed C^*-algebra A, we say that τ is *even-swapping* whenever the equalities $\tau(A_{11}) = A_{22}$ and $\tau(A_{12}) = A_{12}$ hold.

LEMMA 5.5. *Let J be a complex JC^*-triple. Then there exists a matricially decomposed complex C^*-algebra B with an even-swapping $*$-involution τ such that J can be seen as a JB^*-subtriple of B contained in $H(B, \tau) \cap B_{12}$.*

Proof. Let H be a complex Hilbert space such that J is a JB^*-subtriple of $BL(H)$. Put $A := BL(H)$, and consider the $*$-involution t on A defined by $t(a) := \sigma a^* \sigma$ for every a in A, where σ is a conjugation on H. Let s denote the involution on the C^*-algebra $M_2(\mathbb{C})$ given by $s \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} \lambda_{22} & \lambda_{12} \\ \lambda_{21} & \lambda_{11} \end{pmatrix}$, and e the exchange involution on the C^*-algebra $\mathbb{C}^2$. Now, consider the C^*-algebra $B := M_2(\mathbb{C}) \otimes \mathbb{C}^2 \otimes A$, and the $*$-involution τ on B given by $\tau := s \otimes e \otimes t$. Finally, let $\{u_{ij}\}_{i,j\ \in \{1,2\}}$ the usual family of matrix units for $M_2(\mathbb{C})$, and put $B_{ij} := u_{ij} \otimes \mathbb{C}^2 \otimes A$. Then $\{B_{ij}\}_{i,j\ \in \{1,2\}}$ is a matricial decomposition of B, τ is even-swapping relative to such a decomposition, and, via the one-to-one triple homomorphism

$$x \mapsto u_{12} \otimes [(1, 0) \otimes x + (0, 1) \otimes t(x)]$$

from J to B, we can see J as a JB^*-subtriple of B contained in $H(B, \tau) \cap B_{12}$. ■

PROPOSITION 5.6. *Let J be a complex JC^*-triple of hermitian type. Then J contains a nonzero closed triple ideal of the form $H(A, \tau) \cap A_{12}$, where A is a matricially decomposed C^*-algebra, τ is an even-swapping $*$-involution on A, and A is generated as C^*-algebra by $H(A, \tau) \cap A_{12}$.*

Proof. Let B and τ be the matricially decomposed complex C^*-algebra and the even-swapping $*$-involution on B, respectively, given by Lemma 5.5, so that we have

$$J \subseteq H(B, \tau) \cap B_{12}.$$

For $b = \sum b_{ij}$ with b_{ij} in B_{ij} , we write $\pi(b) = \tau(b_{11} + b_{22}) - \tau(b_{12} + b_{21})$, so that π becomes an even-swapping $*$-involution on B commuting with τ, and we have

$$H(B,\tau)\cap S(B,\pi)=H(B,\tau)\cap B_{12}+H(B,\tau)\cap B_{21}.$$

Put $T:=J+J^*$, and note that

$$T\subseteq H(B,\tau)\cap S(B,\pi).$$

Then T is a JB^*-subtriple of B, but for the moment this is not relevant for our argument. At this point, we emphasize that T is a Jordan subtriple of B (i.e., T is a subspace of B closed under the triple product $[abc]:=\frac{1}{2}(abc+cba)$). In fact, for x_1,y_1,x_2,y_2,x_3,y_3 in J, we have

$$[(x_1+y_1^*)(x_2+y_2^*)(x_3+y_3^*)]=\{x_1y_2x_3\}+\{y_1x_2y_3\}^*,$$

where $\{...\}$ is the triple product of the JB^*-triple J. Since the set

$$\{x+x^*:x\in J\}$$

is a copy of $J_{\mathbb{R}}$ contained in T, and $J_{\mathbb{R}}$ is of hermitian type, T (regarded as a Jordan triple) is also of hermitian type. By Lemma 4.4, the subalgebra C of B generated by $\mathcal{G}(T)$ is invariant under τ and π, and we have

$$\mathcal{G}(T)=H(C,\tau)\cap S(C,\pi).$$

Moreover, since T is a $*$-invariant subset of B, and the restriction of $*$ to T is a conjugate linear Jordan triple automorphism of T, it follows from Lemma 5.4 that $\mathcal{G}(T)$ is $*$-invariant. Since C is generated by $\mathcal{G}(T)$, we conclude that C is $*$-invariant.

On the other hand, the decomposition $T=J\oplus J^*$ exhibits T as a "polarized" Jordan triple in the sense [D'Am, p. 229, (5.1)], and therefore $\mathcal{G}(T)$ inherits the polarization (see [D'Am, p. 231]). This means that

$$\mathcal{G}(T)=\mathcal{G}(T)\cap J+\mathcal{G}(T)\cap J^*.$$

Now, $\mathcal{G}(T)$ is contained in $B_{12}\cap C+B_{21}\cap C$ and hence in $\sum_{i,j\ \in\{1,2\}}B_{ij}\cap C$. Since the last sum is a subalgebra of B, and C is the subalgebra of B generated by $\mathcal{G}(T)$, it follows that

$$C=\sum_{i,j\ \in\{1,2\}}C_{ij}\,,$$

where $C_{ij}:=B_{ij}\ \cap C$.

To conclude the proof, we keep in mind that the sum $B = \sum_{i,j\ \in\{1,2\}} B_{ij}$ is topological to obtain that all properties proved for C pass to the closure of C (say A) in B. Therefore A is C^*-subalgebra of B, and inherits the matricial decomposition of B. Moreover, denoting by K the closure of $\mathcal{G}(T)$ in B, K is an ideal of the Jordan triple T satisfying

$$K = H(A,\tau) \cap S(A,\pi) = H(A,\tau) \cap A_{12} + H(A,\tau) \cap A_{21}.$$

Then, clearly, $H(A,\tau) \cap A_{12}$ is a closed triple ideal of J. ■

The structure theorem for prime complex JB^*-triples of hermitian type will follow from the above proposition and some general results on matricially decomposed C^*-algebras.

PROPOSITION 5.7. *For a matricially decomposed C^*-algebra A the following assertions hold:*

i) All closed ideals of A inherit the matricial decomposition.

ii) If A is generated as a closed ideal by A_{12}, then every nonzero closed ideal of A meets A_{12}.

iii) If A has an even-swapping $*$-involution τ, and if A is generated as a closed ideal by $H(A,\tau) \cap A_{12}$, then every nonzero τ-invariant closed ideal of A meets $H(A,\tau) \cap A_{12}$.

Proof. *i)* Let i,j be in $\{1,2\}$. Denote by p_{ij} the (continuous) linear projection on A with range A_{ij} and kernel $\sum_{(k,l)\neq(i,j)} A_{kl}$, and notice that, for a,b,c in A, the equality

$$p_{ij}\ (ap_{ij}\ (b)c) = p_{ii}(a)bp_{jj}\ (c)$$

follows straightforwardly from the multiplication rules of the matricial decomposition. Now, if P is a closed ideal of A, and if x is in P, it is enough to take an approximate unit $\{e^\lambda\}_{\lambda\in\Lambda}$ for A, to conclude that

$$p_{ij}\ (x) = \lim_\lambda \lim_\mu p_{ij}\ (e^\lambda p_{ij}\ (x)e^\mu) = \lim_\lambda \lim_\mu p_{ii}(e^\lambda)xp_{jj}\ (e^\mu)$$

lies in P.

ii) Let P be a closed ideal of A satisfying $P \cap A_{12} = 0$. By the $*$-invariance of P, we have also that $P \cap A_{21} = 0$. Hence, keeping in mind assertion *i)*, we deduce $p_{12}(P) = p_{21}(P) = 0$, and therefore $A_{12}p_{22}(P) = 0$. Since obviously $A_{12}p_{11}(P) = 0$, we obtain that $A_{12}P = 0$. Now, A_{12} is contained in the left annihilator $Lann(P)$ of P. Since A is generated as a closed ideal by A_{12} and $Lann(P)$ is a closed ideal of A, we conclude that $A \subseteq Lann(P)$ and therefore $P = 0$.

iii) Let P be a τ-invariant closed ideal of A satisfying $P \cap H(A,\tau) \cap A_{12} = 0$. Let P^{oo} be the bipolar of P in A''. Since τ and p_{12} commute (because τ is even-swapping), the equality $P \cap H(A,\tau) \cap A_{12} = 0$ can be read as $(1+\tau)p_{12}(P) = 0$, and hence we have $(1+\tau'')p''_{12}(P^{oo}) = 0$. Now, write $P^{oo} = A''e$ for a suitable central idempotent e in A''. Since the family $\{p''_{ij}(A'')\}_{i,j\in\{1,2\}}$ is a matricial decomposition of A'', and $A''e$ and $A''(1-e)$ are closed ideals of A'', we can apply assertion *i)* to obtain that, for $i,j \in \{1,2\}$ and $\alpha \in A''$, we have

$$p''_{ij}(\alpha e) = p''_{ij}(\alpha e)e \quad \text{and} \quad p''_{ij}(\alpha(1-e)) = p''_{ij}(\alpha(1-e))(1-e),$$

so

$$p''_{ij}R_e = R_e p''_{ij}R_e \quad \text{and} \quad p''_{ij}(1-R_e) = (1-R_e)p''_{ij}(1-R_e),$$

and so R_e and p_{ij} commute. It follows

$$\begin{aligned} R_e(H(A'',\tau'') \cap A''_{12}) &= R_e(1+\tau'')p''_{12}(A'') \\ &= (1+\tau'')p''_{12}R_e(A'') \\ &= (1+\tau'')p''_{12}(P^{oo}) \\ &= 0\ . \end{aligned}$$

Finally, arguing as in the conclusion of the proof of Lemma 4.2, we obtain $P = 0$. ■

PROPOSITION 5.8. *For a matricially decomposed C^*-algebra A the following assertions hold:*

i) The matricial decomposition of A can be uniquely extended to a matricial decomposition of the multiplier C^*-algebra $M(A)$ of A.

ii) If A has an even-swapping $*$-involution τ, and if A is generated as C^*-algebra by $H(A,\tau) \cap A_{12}$, then

$$M(H(A,\tau) \cap A_{12}) = H(M(A),\tau) \cap (M(A))_{12}.$$

iii) If A is generated as C^*-algebra by A_{12}, then

$$M(A_{12}) = (M(A))_{12}.$$

Proof. i) Let i, j be in $\{1,2\}$. With the notation in the proof of Proposition 5.7, for $\alpha, \beta \in A''$ we have

$$p''_{ij}(\beta)\alpha = (p''_{i1} + p''_{i2})[\beta(p''_{j\,1} + p''_{j\,2})(\alpha)]$$

and

$$\alpha p''_{ij}(\beta) = (p''_{1j} + p''_{2j})[(p''_{1i} + p''_{2i})(\alpha)\beta].$$

Then, for $x \in M(A)$ and $a \in A$ we obtain

$$p''_{ij}(x)a = (p_{i1} + p_{i2})[x(p_{j\,1} + p_{j\,2})(a)]$$

and

$$ap''_{ij}(x) = (p_{1j} + p_{2j})[(p_{1i} + p_{2i})(a)x],$$

hence $p''_{ij}(x)$ belongs to $M(A)$. Therefore $M(A)$ inherits the natural matricial decomposition of A'', which of course extends that of A. If $\{q_{ij}\}_{i,j\,\in\{1,2\}}$ is the family of projections on $M(A)$ corresponding to any matricial decomposition of $M(A)$ extending that of A, then, for $x \in M(A)$, $a \in A$ and $i, j \in \{1,2\}$ we have

$$q_{ij}(x)a = (p_{i1} + p_{i2})[x(p_{j\,1} + p_{j\,2})(a)],$$

so $[p''_{ij}(x) - q_{ij}(x)]A = 0$, and so $q_{ij} = p''_{ij\ |M(A)}$.

ii) Put $J := H(A,\tau) \cap A_{12}$. Through the natural identification of J'' with J^{oo}, we can regard $M(J)$ as a JB^*-subtriple of A'' contained in J^{oo}. Then, the inclusion

$$M(J) \supseteq H(M(A),\tau) \cap (M(A))_{12}$$

is clear. Let x be in $M(J)$ and z be in J. Since τ and p_{12} commute, we have $J^{oo} = H(A'',\tau'') \cap (A'')_{12}$, and hence, clearly, $xz = 0 = zx$. Moreover, there exists y in J such that $z = \{yyy\}$, and consequently, from the equalities

$$xz^* = x\{yyy\}^* = 2\{xyy\}y^* - y\{yxy\}^*$$

and

$$z^*x = \{yyy\}^*x = 2y^*\{xyy\} - \{yxy\}^*y$$

it follows that xz^*, z^*x belong to A. Now, the set

$$B := \{a \in A : xa, xa^*, ax, a^*x \in A\}$$

is a $*$-invariant closed subalgebra of A containing J, so $B = A$ (because A is generated by J as a C^*-algebra), and so, x lies in $M(A)$. Since x is an arbitrary element of $M(J)$, we deduce

$$M(J) \subseteq H(M(A), \tau) \cap (M(A))_{12}.$$

iii). Keeping in mind that $(A_{12})^{oo} = (A'')_{12}$, the argument is similar to that in the proof of assertion *ii).* ■

Now we are ready to formulate and prove the main result in this section.

THEOREM 5.9. *Let J be a prime complex JB^*-triple of hermitian type. Then one of the following assertions is true for J :*

i) There exists a matricially decomposed prime complex C^*-algebra A such that J can be regarded as a JB^*-subtriple of the complex C^*-algebra $M(A)$ contained in $(M(A))_{12}$ and containing A_{12} .

ii) There exists a matricially decomposed prime complex C^*-algebra A with an even-swapping $*$-involution τ such that J can be regarded as a JB^*-subtriple of the matricially decomposed complex C^*-algebra $M(A)$ contained in $H(M(A), \tau) \cap (M(A))_{12}$ and containing $H(A, \tau) \cap A_{12}$.

Proof.- Since J is i-special and all JB^*-triples listed in Theorem 2.4 are non i-special, it follows from that theorem that J is a JC^*-triple. Then, by Proposition 5.6, J contains a nonzero closed triple ideal of the form $H(A, \tau) \cap A_{12}$, where A is a matricially decomposed complex C^*-algebra, τ is an even-swapping $*$-involution on A, and A is generated as C^*-algebra by $H(A, \tau) \cap A_{12}$. Since J is prime, $H(A, \tau) \cap A_{12}$ is an essential triple ideal of J, so that Propositions 3.1 and 5.8.ii allow us to see J as a JB^*-subtriple of the matricially decomposed complex C^*-algebra $M(A)$ contained in $H(M(A), \tau) \cap (M(A))_{12}$ and containing $H(A, \tau) \cap A_{12}$. If A is prime, then we are in case ii). Assume from now on that A is nonprime. Since A is τ-prime (by Corollary 1.5 and Proposition 5.7.iii), we can find a nonzero closed ideal P of A such that $P \cap \tau(P) = 0$. As in the proof of Theorem 4.5, such an ideal P is a prime algebra. By Proposition 5.7.i, P inherits the matricial decomposition of A. Now, P is a matricially decomposed complex C^*-algebra, and the mapping $\phi : x \mapsto x + \tau(x)$ from P_{12} into $H(A, \tau) \cap A_{12}$ is a one-to-one triple homomorphism whose range is a triple ideal of $H(A,\tau)\cap A_{12}$. Let us denote by $\mathtt{A}$ the C^*-subalgebra of P generated by P_{12}. Since $\sum_{i,j\in\{1,2\}} P_{ij}\cap\mathtt{A}$ is a C^*-subalgebra of P containing P_{12} we deduce $\mathtt{A}=\sum_{i,j\in\{1,2\}} P_{ij}\cap\mathtt{A}$, and hence $\mathtt{A}$ inherits the matricial decomposition of P. In this way, $\mathtt{A}$ is a matricially decomposed complex C^*-algebra satisfying $\mathtt{A}_{12}=P_{12}$. Moreover, keeping in mind Proposition 1.1, we can forget the embedding ϕ above and regard $\mathtt{A}_{12}$ as a closed triple ideal of $H(A,\tau)\cap A_{12}$. Since $H(A,\tau)\cap A_{12}$ is a closed triple ideal of J, it follows from Corollary 1.3.ii that $\mathtt{A}_{12}$ can be seen as a nonzero closed triple ideal of the prime complex JB^*-triple J. By Corollary 1.5, $\mathtt{A}_{12}$ is a prime JB^*-triple, and hence, in view of Proposition 5.7.ii, $\mathtt{A}$ is a prime C^*-algebra. By Propositions 3.1 and 5.8.iii, J can be regarded as a JB^*-subtriple of the matricially decomposed complex C^*-algebra $M(\mathtt{A})$ contained in $(M(\mathtt{A}))_{12}$ and containing $\mathtt{A}_{12}$. Therefore J is in case i). ■

REMARK 5.10. In our approach to the structure of prime complex JB^*-triples of hermitian type, the JC^*-triples of the form A_{12}, where A is a matricially decomposed complex C^*-algebra, have become crucial. It is worth mentioning that such JC^*-triples are "more" than JB^*-subtriples of C^*-algebras. Actually, if A is a matricially decomposed complex C^*-algebra, then A_{12} is a "ternary ring of operators", in the sense of H. Zettl [Zet]. This means that A_{12} is a norm-closed subspace of a complex C^*-algebra closed under the associative triple product of the second kind xy^*z. Of course, the JB^*-triple structure of a ternary ring of operators is obtained by symmetrizing its associative triple product in the outer variables. Keeping in mind this fact, it turns out that, philosophically, Theorem 5.9 is close to the Zel'manov-type theorem for Jordan triples of hermitian type proved by A. D'Amour [D'Am, Theorem 4.1]. However, we note that associative triple products arising in D'Amour's theorem are of first kind and linear in the middle variable, whereas associative triple products of ternary rings of operators are of second kind and conjugate linear in the middle variable.

6 Complex Cartan factors of Clifford type.

A Jordan triple T is said to be *of Clifford type* if it is i-special and satisfies $\mathcal{G}(T)=0$, where $\mathcal{G}$ is the zelmanovian ideal of $\mathcal{ST}$ introduced in Section 4. In the present section we establish the foundations for the earlier determination of all prime JB^*-triple of Clifford type. More precisely, we obtain here the list of complex Cartan factors which are of Clifford type.

Real (respectively, complex) JBW^**-triples* were defined as those real (respectively, complex) JB^*-triples which are Banach dual spaces in such a way that the triple product becomes separately w^*-continuous. Actually, the requirement of separate w^*-continuity of the triple product has been shown to be redundant [MaPe] (respectively, [BaTi]). Prime JBW^*-triples are called JBW^**-factors*. The so-called "atomic" complex JBW^*-factors are specially well-understood. According to [FR1], if J is a complex JBW^*-factor, and if Δ denotes the set of all extreme points of the closed unit ball of the predual J_* of J, then either Δ is empty or J_* is the closed linear hull of Δ. In the last case, the complex JBW^*-factor J is called *atomic*. By the main result of G. Horn in his Thesis (see the more available reference [Hor]), the atomic complex JBW^*-factors are nothing but those previously known under the name of (complex) *Cartan factors*. These come in six types as follows.

$\mathbf{I}_{n,m}$:= $BL(H,K)$, where H,K are complex Hilbert spaces of hilbertian dimension n,m, respectively, with $1 \leq n \leq m$, and the triple product is defined by

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x)$$

for all x,y,z in $BL(H,K)$.

$\mathbf{II}_n$:= $\{x \in BL(H) : \sigma x^* \sigma = -x\}$ as JB^*-subtriple of $BL(H)$, where H is a complex Hilbert space of hilbertian dimension $n \geq 5$ and σ is a conjugation on H.

$\mathbf{III}_n$:= $\{x \in BL(H) : \sigma x^* \sigma = x\}$ as JB^*-subtriple of $BL(H)$, where H is a complex Hilbert space of hilbertian dimension $n \geq 2$ and σ is a conjugation on H.

$\mathbf{IV}_n$:= H, where H is a complex Hilbert space of hilbertian dimension $n \geq 5$, σ is a conjugation on H, and the triple product and the norm are given by

$$\{xyz\} := (x|y)z + (z|y)x - (x|\sigma(z))\sigma(y)$$

and

$$\|x\|^2 := (x|x) + \sqrt{(x|x)^2 - |(x|\sigma(x))|^2},$$

respectively, for all x,y,z in H.

$\mathbf{V}$:= $M_{12}(\mathbb{O}_{\mathbb{C}})$ the 1×2-matrices over the complex Cayley numbers $\mathbb{O}_{\mathbb{C}}$.

$\mathbf{VI}$:= $H_3(\mathbb{O}_{\mathbb{C}})$ the hermitian 3×3-matrices over $\mathbb{O}_{\mathbb{C}}$.

Sometimes, it is convenient to consider complex Cartan factors of type $\mathbf{I}_{n,m}$ for arbitrary cardinal numbers $n, m \geq 1$, $\mathbf{II}_n$ for $n \geq 2$, $\mathbf{III}_n$ for $n \geq 1$ and $\mathbf{IV}_n$ for $n \geq 3$. The stronger requirements on n, m above have been imposed in order to have that every Cartan factor occurs (up to isomorphism) in the list precisely once [Ka2, p. 475].

The selection (among the complex Cartan factors above) of those which are of Clifford type will be made by considering their associated Jordan pairs and then applying the determination of prime nondegenerate Jordan pairs of Clifford type provided in [D'AMc]. A *Jordan pair* over a field $\mathbb{F}$ of characteristic different from 2 and 3 is a couple (V^+, V^-) of vector spaces over $\mathbb{F}$ together with trilinear mappings

$$[...]_\varepsilon : V^\varepsilon \times V^{-\varepsilon} \times V^\varepsilon \longrightarrow V^\varepsilon \quad (\varepsilon = \pm)$$

satisfying

$$[ab[xyz]_\varepsilon]_\varepsilon = [[abx]_\varepsilon yz]_\varepsilon - [x[bay]_{-\varepsilon}z]_\varepsilon + [xy[abz]_\varepsilon]_\varepsilon$$

for all $a, x, z \in V^\varepsilon$ and $b, y \in V^{-\varepsilon}$. Every Jordan triple T over $\mathbb{F}$ has an associated Jordan pair over $\mathbb{F}$, namely the pair (V^+, V^-), where $V^\varepsilon = T$ and $[...]_\varepsilon = \{...\}$ $(\varepsilon = \pm)$. In [D'AMc], a suitable notion of Jordan pair of Clifford type is given in such a way that a Jordan triple is of Clifford type (in the sense introduced at the beginning of this section) if and only if its associated Jordan pair is of Clifford type. Although complex JB^*-triples are only real Jordan triples, the real Jordan pair associated to a complex JB^*-triple actually is the realification of a complex Jordan pair. Indeed, if J is a complex JB^*-triple, and if $\overline{J}$ denotes the complex vector space obtained from that of J by replacing the complex structure with the conjugate one, then the realifications of J and $\overline{J}$ coincide, and, taking $V^+ = J$ and $V^- = \overline{J}$, the mappings $(x, y, z) \mapsto [xyz]_\varepsilon = \{xyz\}$ from $V^\varepsilon \times V^{-\varepsilon} \times V^\varepsilon$ to V^ε $(\varepsilon = \pm)$ become complex-linear in each of their variables.

PROPOSITION 6.1. *The complex Cartan factors of Clifford type are the following:*

1. *Those of type $\mathbf{I}_{n,m}$ for $n = 1, 2$ and $n \leq m$,*
2. *The one of type $\mathbf{II}_5$,*
3. *The one of type $\mathbf{III}_2$,*
4. *All type $\mathbf{IV}_n$ complex Cartan factors, for $n \geq 5$.*

Proof. First we show that complex Cartan factors not listed in the proposition are not of Clifford type.

Let $J = BL(H, K)$ be a type $\mathbf{I}_{n,m}$ complex Cartan factor with $3 \leq n \leq m$. Take 3-dimensional subspaces H_1 and K_1 of H and K, respectively. Then, denoting by $H_1^{\perp}$ the orthogonal complement of H_1 in H, the set

$$J_1 := \{x \in BL(H, K) : x(H_1^{\perp}) = 0, x(H) \subseteq K_1\}$$

is a JB^*-subtriple of J isomorphic to the type $\mathbf{I}_{3,3}$ Cartan factor. But it easily seen that the complex Jordan pair associated to the type $\mathbf{I}_{3,3}$ Cartan factor is isomorphic to $(M_3(\mathbb{C}), M_3(\mathbb{C}))$ (where $M_n(\mathbb{C})$ denotes $n \times n$-complex matrices) with trilinear mappings $[xyz]_\varepsilon = \frac{1}{2}(xyz + zyx)$. Since this Jordan pair is not of Clifford type [D'AMc], it follows that J_1 (and hence J) is not of Clifford type.

Let $J = \{x \in BL(H) : \sigma x^* \sigma = -x\}$ be a type $\mathbf{II}_n$ complex Cartan factor with $n \geq 6$. Take a σ-invariant 6-dimensional subspace H_1 of H. Then the set

$$J_1 := \{x \in J : x(H_1^{\perp}) = 0, x(H) \subseteq H_1\}$$

is a JB^*-subtriple of J isomorphic to the type $\mathbf{II}_6$ Cartan factor. Since the complex Jordan pair associated to J_1 is isomorphic to the pair $(A_6(\mathbb{C}), A_6(\mathbb{C}))$ (where $A_n(\mathbb{C}) := \{x \in M_n(\mathbb{C}) : x^t = -x\}$, t the transpose involution) with trilinear mappings $[xyz]_\varepsilon = -\frac{1}{2}(xyz + zyx)$, and this Jordan pair is not of Clifford type [D'AMc], we obtain that J is not of Clifford type.

Let $J = \{x \in BL(H) : \sigma x^* \sigma = x\}$ be a type $\mathbf{III}_n$ complex Cartan factor with $n \geq 3$. Arguing as in the previous case, we realize that J contains as a JB^*-subtriple a copy of the type $\mathbf{III}_3$ Cartan factor, whose associated complex Jordan pair is isomorphic to $(H_3(\mathbb{C}), H_3(\mathbb{C}))$ (where $H_n(\mathbb{C}) := \{x \in M_n(\mathbb{C}) : x^t = x\}$) with trilinear mappings $[xyz]_\varepsilon = \frac{1}{2}(xyz + zyx)$. As above, the fact that J is not of Clifford type follows from [D'AMc].

Since the type $\mathbf{V}$ and $\mathbf{VI}$ complex Cartan factors are not i-special, the first part of the proof is concluded.

Now, we prove that all complex Cartan factors listed in the proposition are of Clifford type.

Let $J = BL(H, K)$ be a type $\mathbf{I}_{n,m}$ complex Cartan factor with $n = 1, 2$ and $n \leq m$. Let $p(x_1, ..., x_r)$ be in the zelmanovian ideal $\mathcal{G}$, and $y_1, ..., y_r$ be elements of J. Take a subspace K_1 of K containing $y_1(H) + ... + y_r(H)$ and having finite dimension $q \geq n$. Then

$$J_1 := \{x \in BL(H, K) : x(H) \subseteq K_1\}$$

is a JB^*-subtriple of J isomorphic to the type $\mathbf{I}_{n,q}$ Cartan factor and contains $\{y_1, ..., y_r\}$. Since the complex Jordan pair associated to the type $\mathbf{I}_{n,q}$

Cartan factor is isomorphic to $(M_{n,q}(\mathbb{C}), M_{q,n}(\mathbb{C}))$ (where $M_{i,j}(\mathbb{C})$ denotes $i \times j$-complex matrices) with trilinear mappings $[xyz]_\varepsilon = \frac{1}{2}(xyz + zyx)$ and this Jordan pair is of Clifford type [D'AMc], it follows that J_1 is of Clifford type, and therefore $p(y_1, ..., y_r) = 0$.

Since the complex Jordan pair associated to the type $\mathbf{II}_5$ Cartan factor is isomorphic to $(A_5(\mathbb{C}), A_5(\mathbb{C}))$, and this Jordan pair is of Clifford type [D'AMc], such a Cartan factor is of Clifford type.

The remaining cases listed in the proposition can be treated in a unified way because, according to [Lo2, 4.18], the type $\mathbf{III}_2$ Cartan factor is isomorphic to the type $\mathbf{IV}_3$ Cartan factor. If J is a type $\mathbf{IV}_n$ complex Cartan factor ($n \geq 3$) then the complex Jordan pair associated to J is isomorphic to (H, H), where H is a complex Hilbert space of hilbertian dimension n, with trilinear mappings

$$[xyz]_\varepsilon := (x|\sigma(y))z + (z|\sigma(y))x - (x|\sigma(z))y,$$

where σ is a conjugation on H. Since the mapping $(x, y) \mapsto (x|\sigma(y))$ from $H \times H$ to $\mathbb{C}$ is a nondegenerate symmetric bilinear form, the above Jordan pair is of Clifford type [D'AMc], hence J is of Clifford type. ∎

We say that a Banach space E is *hilbertizable* if there are positive constants m, M, and an inner product $(.|.)$ on E such that $m\|x\|^2 \leq (x|x) \leq M\|x\|^2$ for every x in E. A family $\{E_i\}_{i\in I}$ of Banach spaces is said to be *uniformly hilbertizable* if there are positive constants m, M, and inner products $(.|.)_i$ on E_i $(i \in I)$ satisfying $m\|x_i\|^2 \leq (x_i|x_i)_i \leq M\|x_i\|^2$ for every i in I and all x_i in E_i.

COROLLARY 6.2. *Every family of complex Cartan factors of Clifford type is uniformly hilbertizable.*

Proof. Let $J = BL(H, K)$ be a type $\mathbf{I}_{n,m}$ complex Cartan factor with $n \leq m$ and n finite. Take an orthonormal basis $\{\eta_1, ..., \eta_n\}$ of H. Then the mapping $(x, y) \mapsto (x|y) := \frac{1}{n}\sum_{i=1}^n (x(\eta_i)|y(\eta_i))$ from $J \times J$ to $\mathbb{C}$ is an inner product on J satisfying $(x|x) \leq \|x\|^2$ for all x in J. Moreover, for x in J and $\eta = \sum_{i=1}^n \lambda_i\eta_i$ in H, we have

$$\|x(\eta)\|^2 \leq \left(\sum_{i=1}^n |\lambda_i|\, \|x(\eta_i)\|\right)^2 \leq \left(\sum_{i=1}^n |\lambda_i|^2\right)\left(\sum_{i=1}^n \|x(\eta_i)\|^2\right) = n\|\eta\|^2(x|x),$$

and hence $\frac{1}{n}\|x\|^2 \leq (x|x)$.

Let J be a type $\mathbf{IV}_n$ complex Cartan factor with $n \geq 3$. Then there exist a complex Hilbert space $(H, (.|.))$ of hilbertian dimension n and a conjugation σ on H such that $J = H$ as complex vector spaces and the equality

$$\|x\|^2 := (x|x) + \sqrt{(x|x)^2 - |(x|\sigma(x))|^2}$$

holds for all x in H. Therefore we have $\frac{1}{2}\|x\|^2 \leq (x|x) \leq \|x\|^2$ for all x in J.

It follows from the above and Proposition 6.1 that every complex Cartan factor J of Clifford type has an inner product $(.|.)$ satisfying $\frac{1}{5}\|x\|^2 \leq (x|x) \leq \|x\|^2$ for all x in J. ∎

7 Prime JB^*-triples of Clifford type.

In this section we conclude the determination of prime JB^*-triple of Clifford type. Such a determination will be obtained by combining the results in the previous section with the technology of Banach ultraproducts [Hei].

It is well known that the ℓ_∞-sum of any family of complex JB^*-triples, endowed with the triple product defined point-wise, is a JB^*-triple. Then the following lemma is a direct consequence of Proposition 1.1.

LEMMA 7.1. *Let J be a complex JB^*-triple, I a non-empty set, and, for each i in I, let ϕ_i be a triple homomorphism from J into a complex JB^*-triple J_i. If $\cap_{i\in I}$ Ker $(\phi_i) = 0$, then*

$$\|x\| = \ sup \ \{\|\phi_i(x)\| : i \in I\}.$$

Let $\mathcal{U}$ be an ultrafilter on a nonempty set I, and $\{E_i\}_{i\in I}$ a family of Banach spaces. We can consider the Banach space $\oplus_{i\in I}^{\ell_\infty} E_i$, ℓ_∞-sum of that family, and the closed subspace $N_\mathcal{U}$ of $\oplus_{i\in I}^{\ell_\infty} E_i$ given by

$$N_\mathcal{U} := \left\{ \{x_i\}_{i\in I} \in \oplus_{i\in I}^{\ell_\infty} E_i : \lim_{\mathcal{U}} \|x_i\| = 0 \right\} .$$

The (Banach) *ultraproduct* $(E_i)_\mathcal{U}$ of the family of Banach spaces $\{E_i\}_{i\in I}$ relative to the ultrafilter $\mathcal{U}$ is defined as the quotient Banach space $\oplus_{i\in I}^{\ell_\infty} E_i / N_\mathcal{U}$.

If we denote by (x_i) the element of $(E_i)_{\mathcal{U}}$ containing a given element $\{x_i\}$ of $\oplus_{i\in I}^{\ell_\infty} E_i$, then it is easily checked that the equality $\|(x_i)\| = \lim_{\mathcal{U}} \|x_i\|$ holds. We note that, if the family of Banach spaces $\{E_i\}_{i\in I}$ is uniformly hilbertizable, then the ultraproduct $(E_i)_{\mathcal{U}}$ is a hilbertizable Banach space. Indeed, if m, M are positive constants and $(.|.)_i$ is an inner products on E_i $(i \in I)$ satisfying $m\|x_i\|^2 \leq (x_i|x_i)_i \leq M\|x_i\|^2$ for every i in I and all x_i in E_i, then $((x_i)|(y_i)) \mapsto ((x_i)|(y_i)) := \lim_{\mathcal{U}}(x_i|y_i)$ is a well defined inner product on $(E_i)_{\mathcal{U}}$ satisfying $m\|(x_i)\|^2 \leq ((x_i)|(x_i)) \leq M\|(x_i)\|^2$ for all (x_i) in $(E_i)_{\mathcal{U}}$. We note also that, if for every i in I, E_i is a complex JB^*-triple, then $(E_i)_{\mathcal{U}}$ is a complex JB^*-triple under the triple product $\{(x_i)(y_i)(z_i)\} := (\{x_i y_i z_i\})$ [Din].

LEMMA 7.2. *Let J be a prime complex JB^*-triple, I a non-empty set, and, for each i in I, let ϕ_i be a triple homomorphism from J into a complex JB^*-triple J_i. Assume that $\cap_{i\in I}$ Ker $(\phi_i) = 0$. Then there exists an ultrafilter $\mathcal{U}$ on I such that the triple homomorphism $\phi : x \mapsto (\phi_i(x))$ from J to $(J_i)_{\mathcal{U}}$ is injective.*

Proof. For i in I, put $P_i :=$ Ker (ϕ_i), and, for x in $J\setminus\{0\}$, write $I_x := \{i \in I : x \notin P_i\}$. Then it follows easily from the primeness of J and the equality $\cap_{i\in I} P_i = 0$ that $\mathcal{B} := \{I_x : x \in J\setminus\{0\}\}$ is a filter basis on I. Take an ultrafilter $\mathcal{U}$ on I containing $\mathcal{B}$. Suppose that the mapping $\phi : x \mapsto (\phi_i(x))$ from J to $(J_i)_{\mathcal{U}}$ is not injective. Then there exists x in J satisfying $\|x\| = 1$ and $\lim_{\mathcal{U}} \phi_i(x) = 0$. Therefore $I' := \{i \in I : \|\phi_i(x)\| < \frac{1}{2}\}$ is an element of $\mathcal{U}$. But, by the definition of I' and Lemma 7.1, we must have $\cap_{i\in I'} P_i \neq 0$. Then, taking a nonzero element y in $\cap_{i\in I'} P_i$, we have $I' \cap I_y = \emptyset$. Since I' and I_y are elements of $\mathcal{U}$, this is a contradiction. ∎

PROPOSITION 7.3. *Every prime complex JB^*-triple of Clifford type is a complex Cartan factor.*

Proof. Recall that a *factor representation* of a JB^*-triple J is a w^*-dense range triple homomorphism from J into some JBW^*-factor. First we prove that, if J is a complex JB^*-triple of Clifford type, and if ϕ is a representation of J into a JBW^*-factor Z, then Z is also of Clifford type. To this end, we invoke the so-called "strong* topology" of an arbitrary complex JBW^*-triple Z. Referring the reader to [BaFr] for the definition of such a topology, we only recall here that the strong* topology is a Hausdorff vector space topology, that, if Z' is a w^*-dense JB^*-subtriple of Z, then the closed unit ball of Z' is strong*-dense in the closed unit ball of Z [BaFr, Corollary 3.3], and that the triple product of Z is jointly strong*-continuous on bounded subsets of Z [Rod].

Let J be a complex JB^*-triple of Clifford type, and ϕ a representation of J into a JBW^*-factor Z. By Proposition 1.1, $\phi(J)$ is a JB^*-subtriple of Z and clearly $\phi(J)$ is of Clifford type. If Z were not i-special, then it would be finite-dimensional (by Corollary 2.3), so $\phi(J) = Z$ would be not i-special, which is a contradiction. Let $p(x_1, ..., x_r)$ be in the zelmanovian ideal $\mathcal{G}$, and $z_1, ..., z_r$ be elements of Z. Put $M := \max\{\|z_1\|, ..., \|z_r\|\}$. Then, there exist nets $\{y_i^\lambda\}_{\lambda\in\Lambda}$ in $\phi(J)$, with $\|y_i^\lambda\| \leq M$ for all λ in Λ, strong*-convergent to z_i ($i \in \{1, 2, ..., r\}$). Since the triple product is jointly strong*-continuous on bounded sets, we have that $\{p(y_1^\lambda, ..., y_r^\lambda)\}_\lambda$ strong*-converges to $p(z_1, ..., z_r)$. Since $p(y_1^\lambda, ..., y_r^\lambda) = 0$ for all λ in Λ, we obtain $p(z_1, ..., z_r) = 0$. Therefore Z is of Clifford type.

Now, let J be a prime complex JB^*-triple of Clifford type. Then there exists a faithful family of Cartan factor representations of J [FR2] (say, $\{\phi_i : J \to J_i\}_{i\in I}$). By the above paragraph, for i in I, J_i is a complex Cartan factor of Clifford type. Let $\mathcal{U}$ be the ultrafilter on I whose existence is provided by Lemma 7.2. It follows from Corollary 6.2 that $(J_i)_\mathcal{U}$ is a hilbertizable Banach space. Since, by Lemma 7.2 and Proposition 1.1, the mapping $x \mapsto (\phi_i(x))$ from J to $(J_i)_\mathcal{U}$ is a linear isometry, J is also a hilbertizable Banach space. As a consequence, the Banach space of J is reflexive, hence it is a dual Banach space whose predual has extreme points in its closed unit ball. Since J is prime, it follows that J is an atomic JBW^*-factor, hence a Cartan factor. ∎

THEOREM 7.4. *The prime complex JB^*-triples of Clifford type are the type $\mathbf{I}_{n,m}$ ($n = 1, 2,\ n \leq m$), $\mathbf{II}_5$, $\mathbf{III}_2$, and $\mathbf{IV}_n$ ($n \geq 5$) complex Cartan factors. The prime real JB^*-triples of Clifford type are the complex JB^*-triples just listed (regarded as real JB^*-triples) plus their real forms, namely the type $\mathbf{I}_{n,m}^{\mathbb{R}}$ ($n = 1, 2,\ n \leq m$), $\mathbf{I}_{2,2q}^{\mathbb{H}}$ ($1 \leq q$), $\mathbf{I}_{n,n}^{\mathbb{C}}$ ($n = 1, 2$), $\mathbf{II}_5^{\mathbb{R}}$, $\mathbf{III}_2^{\mathbb{R}}$, $\mathbf{III}_2^{\mathbb{H}}$ and $\mathbf{IV}_n^{r,s}$ ($n \geq 5,\ r \geq s \geq 0,\ r + s = n$) real Cartan factors (cf.* [Ka4, Theorem 4.1]*).*

Proof. The assertion concerning prime complex JB^*-triples follows directly from Proposition 7.3 and Proposition 6.1. On the other hand it is clear that all prime complex JB^*-triples of Clifford type (regarded as real JB^*-triple), as well as their real forms, are prime real JB^*-triples of Clifford type. Let J be a prime real JB^*-triple of Clifford type. First assume that J is a real form of a prime complex JB^*-triple: $J = \mathcal{A}^\sigma$ for some prime complex JB^*-triple $\mathcal{A}$ and some conjugation σ on $\mathcal{A}$. Then the complex

Jordan pair associated to $\mathcal{A}$ is isomorphic to the complexification of the real Jordan pair associated to J, and hence it is of Clifford type [D'AMc]. Therefore the prime complex JB^*-triple $\mathcal{A}$ is of Clifford type, and hence J is a real form of some prime complex JB^*-triple of Clifford type. Now, assume that J is not a real form of a prime complex JB^*-triple. Then, by [CMR, Theorem 2.1], there exists a prime complex JB^*-triple $\mathcal{A}$ such that J can be seen as a real JB^*-subtriple of the multiplier JB^*-triple $M(\mathcal{A})$ of $\mathcal{A}$ containing $\mathcal{A}$. Since $\mathcal{A}$ is contained in J, and J is of Clifford type, $\mathcal{A}$ is also of Clifford type, hence a hilbertizable Banach space (by Corollary 6.2). Therefore we have

$$\mathcal{A} \subseteq J \subseteq M(\mathcal{A}) \subseteq \mathcal{A}'' = \mathcal{A},$$

so $J = \mathcal{A}$ is a prime complex JB^*-triple of Clifford type regarded as a real JB^*-triple. ∎

For details about the meaning of the different types of real Cartan factors the reader is referred to [Ka4, Theorem 4.1]. The definition of real Cartan factors, as well as the precise meaning of some of their types, will be given in the next section.

8 Concluding results.

The aim of this section is to summarize the results previously obtained for real (respectively, complex) prime JB^*-triples in a single statement. Simultaneously, in the formulation of such a summarized version of the zelmanovian classification of real (respectively, complex) prime JB^*-triples, we will avoid any reference to the technical classification of prime Jordan triples in the cases exceptional, hermitian, and Clifford. We also include in this section the classification of real (respectively, complex) topologically simple JB^*-triples.

By a real (respectively, complex) *W^*-algebra* we mean a real (respectively, complex) C^*-algebra which is a dual Banach space. A real (respectively, complex) *W^*-factor* will be a real (respectively, complex) prime W^*-algebra.

LEMMA 8.1. *If J is a type $\mathbf{I}_{n,m}$ $(1 \leq n \leq m)$ complex Cartan factor, then there is a matricially decomposed W^*-factor A such that $J = A_{12}$. If*

J is either a type $\mathbf{II}_n$ ($n \geq 2$) or type $\mathbf{III}_n$ ($n \geq 1$) complex Cartan factor, then there is a matricially decomposed W^*-factor A with an even-swapping $*$-involution τ such that $J = H(A, \tau) \cap A_{12}$.

Proof. For $J = BL(H, K)$ a type $\mathbf{I}_{n,m}$ complex Cartan factor, take $A = BL(H \oplus K)$ with matricial decomposition given by

$$A_{11} := \{x \in A : x(K) \subseteq K, x(H) = 0\}, \quad A_{12} := \{x \in A : x(H) \subseteq K, x(K) = 0\},$$

$$A_{21} := \{x \in A : x(H) = 0, x(K) \subseteq H\}, \quad A_{22} := \{x \in A : x(K) = 0, x(H) \subseteq H\},$$

to obtain $J = A_{12}$. For $J = \{x \in BL(H) : \sigma x^* \sigma = \mp x\}$ a type $\mathbf{II}_n$ or $\mathbf{III}_n$ complex Cartan factor, take $A = M_2(\mathbb{C}) \otimes BL(H)$, the matricial decomposition of A given by $A_{ij} := u_{ij} \otimes BL(H)$ (where the u_{ij} are the usual matrix units for $M_2(\mathbb{C})$), and $\tau = \tau_{\mp} \otimes t$ (where $\tau_{\mp} \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} \lambda_{22} & \mp\lambda_{12} \\ \mp\lambda_{21} & \lambda_{11} \end{pmatrix}$, and $t(x) = \sigma x^* \sigma$), to get $J = H(A, \tau) \cap A_{12}$. ■

The type $\mathbf{IV}_n$ ($n \geq 3$) complex Cartan factors are usually called *complex spin factors*. The following zelmanovian classification theorem for prime complex JB^*-triples follows directly from the above lemma and Theorems 2.4, 5.9, and 7.4.

THEOREM 8.2. *If J is a prime complex JB^*-triple, then one of the following assertions hold for J:*

i) J is either the type $\mathbf{V}$ or the type $\mathbf{VI}$ complex Cartan factor.

ii) J is a complex spin factor.

iii) There exists a matricially decomposed prime complex C^*-algebra A such that J can be regarded as a JB^*-subtriple of the complex C^*-algebra $M(A)$ contained in $(M(A))_{12}$ and containing A_{12}.

iv) There exists a matricially decomposed prime complex C^*-algebra A with an even-swapping $*$-involution τ such that J can be regarded as a JB^*-subtriple of the matricially decomposed complex C^*-algebra $M(A)$ contained in $H(M(A), \tau) \cap (M(A))_{12}$ and containing $H(A, \tau) \cap A_{12}$.

Real Cartan factors are defined as the real forms of complex Cartan factors. Their classification is due to O. Loos [Lo2] in the finite dimensional case, and W. Kaup in the general case [Ka4]. They come in 12 different types:

$$\mathbf{I}^{\mathbb{R}}_{n,m}, \mathbf{I}^{\mathbb{H}}_{2p,2q}, \mathbf{I}^{\mathbb{C}}_{n,n}, \mathbf{II}^{\mathbb{R}}_{n}, \mathbf{II}^{\mathbb{H}}_{2p}, \mathbf{III}^{\mathbb{R}}_{n}, \mathbf{III}^{\mathbb{H}}_{2p}, \mathbf{IV}^{r,s}_{n}, \mathbf{V}^{\mathbb{O}}, \mathbf{V}^{\mathbb{O}_0}, \mathbf{VI}^{\mathbb{O}}, \mathbf{VI}^{\mathbb{O}_0}.$$

The notation has the property that, erasing the superscripts, we obtain a complex Cartan factor J such that the given real Cartan factor is one of the real forms of J.

For the moment, we are only interested in the precise meaning of type $\mathbf{I}^{\mathbb{R}}_{n,m}$, $\mathbf{I}^{\mathbb{H}}_{2p,2q}$, $\mathbf{I}^{\mathbb{C}}_{n,n}$, and $\mathbf{II}^{\mathbb{R}}_{n}$ real Cartan factors:

$\mathbf{I}^{\mathbb{R}}_{n,m}$:= $BL(H,K)$, where H,K are real Hilbert spaces of hilbertian dimension n,m, respectively, with $1 \leq n \leq m$, and the triple product is defined by

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x)$$

for all x,y,z in $BL(H,K)$.

$\mathbf{I}^{\mathbb{H}}_{2p,2q}$:= $BL(H,K)$, where H,K are quaternionic Hilbert spaces of hilbertian dimension p,q, respectively, with $1 \leq p \leq q$, and the triple product is formally defined as in the above case.

$\mathbf{I}^{\mathbb{C}}_{n,n}$:= $\{x \in BL(H) : x^* = x\}$, where H is a complex Hilbert space of hilbertian dimension n, as real JB^*-subtriple of $BL(H)$.

$\mathbf{II}^{\mathbb{R}}_{n}$:= $\{x \in BL(H) : \sigma x^* \sigma = -x\}$, where H is a real Hilbert space of hilbertian dimension $n \geq 2$, as JB^*-subtriple of $BL(H)$.

By a *generalized real Cartan factor* we mean either a complex Cartan factor (regarded as a real one) or a real Cartan factor. Generalized real Cartan factor can be intrinsically characterized (see [Ka4, Lemma 4.5]).

LEMMA 8.3. *Let J be a type $\mathbf{I}_{n,m}$, $\mathbf{II}_{n}$, $\mathbf{I}^{\mathbb{R}}_{n,m}$, $\mathbf{I}^{\mathbb{H}}_{2p,2q}$, or $\mathbf{II}^{\mathbb{R}}_{n}$ generalized real Cartan factor. Then there exists a real W^*-factor A with a $*$-involution τ such that $J = A_{sa} \cap S(A,\tau)$.*

Proof. Let H and K be Hilbert spaces over $\mathbb{F}$, where $\mathbb{F}$ is equal to either $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Consider the real W^*-factor $A := BL(H \oplus K)$, and the $*$-involution τ on A given by $\tau(a) := \rho a^* \rho$, where $\rho(h+k) := h-k$ for h in H and k in K. Then, denoting by p the orthogonal projection from $H \oplus K$ onto

H, the mapping $x \mapsto xp+x^*(1-p)$ from $BL(H,K)$ to $A_{sa} \cap S(A,\tau)$ becomes a surjective triple isomorphism. Depending on the choice of $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, the fact just shown proves the lemma for the cases that J is a type $\mathbf{I}^{\mathbb{R}}_{n,m}$, $\mathbf{I}_{n,m}$, or $\mathbf{I}^{\mathbb{H}}_{2p,2q}$, respectively, generalized real Cartan factor.

Let $J = \{x \in BL(H) : \sigma x^* \sigma = -x\}$ be a type $\mathbf{II}^{\mathbb{R}}_n$ real Cartan factor (where H is a real Hilbert space). Consider the real W^*-factor $A := BL(\mathbb{C} \otimes H)$, and the $*$-involution τ on A given by $\tau(a) := \sigma a^* \sigma$, where $\sigma(\lambda \otimes h) := \overline{\lambda} \otimes h$ for λ in $\mathbb{C}$ and h in H. Then the mapping $x \mapsto \mathtt{i}(Id_{\mathbb{C}} \otimes x)$ from J to $A_{sa} \cap S(A,\tau)$ is a surjective triple isomorphism.

Let $J = \{x \in BL(H) : \sigma x^* \sigma = -x\}$ be a type $\mathbf{II}_n$ generalized real Cartan factor (where now H is a complex Hilbert space). Choose a real Hilbert space E such that $H = E \otimes \mathbb{C}$, and a canonical basis $\{1, \mathtt{i}, \mathtt{j}, \mathtt{k}\}$ of $\mathbb{H}$. Consider the right quaternionic Hilbert space $P := E \otimes \mathbb{H}$, and the real-linear operator ρ on P given by $\rho := Id_E \otimes \eta$, where η is the involutive automorphism of $\mathbb{H}$ defined by $\eta(\lambda) := -\mathtt{i}\lambda\mathtt{i}$ for λ in $\mathbb{H}$. Note that, for p in P and λ in $\mathbb{H}$, the equality $\rho(p\lambda) = \rho(p)\eta(\lambda)$ holds. Now, take the real W^*-factor $A := BL(P)$, and the $*$-involution τ on A given by $\tau(a) := \rho a^* \rho$. Then, keeping in mind the natural identifications $BL(H) = BL(E) \otimes \mathbb{C}$ and $BL(P) = BL(E) \otimes \mathbb{H}$, and the one-to-one triple homomorphism κ from $\mathbb{C}_{\mathbb{R}}$ to $\mathbb{H}$ given by $\kappa(\alpha + \mathtt{i}\beta) := \mathtt{k}(\alpha + \mathtt{i}\beta)$ for α, β in $\mathbb{R}$, the restriction of $Id_{BL(E)} \otimes \kappa$ to J becomes a triple isomorphism from J onto $A_{sa} \cap S(A,\tau)$.
■

Given a real JB^*-triple J, let us say that J is a *real spin factor* if it is a real form of a complex spin factor, and that J is a *generalized real spin factor* if it is either a complex spin factor (regarded as a real JB^*-triple) or a real spin factor. Now, recall that the type $\mathbf{III}_2$ complex Cartan factor is a complex spin factor (so that the type $\mathbf{III}_2$, $\mathbf{III}^{\mathbb{R}}_2$, and $\mathbf{III}^{\mathbb{H}}_2$ generalized real Cartan factors are generalized real spin factors). Recall also that the definition of the type $\mathbf{I}^{\mathbb{C}}_{n,n}$ real Cartan factors shows that such real Cartan factors are of the form A_{sa} for a suitable real W^*-factor A. With these facts in mind, the following zelmanovian classification theorem for prime real JB^*-triples follows directly from Lemma 8.3 and Theorems 2.4, 4.5, and 7.4.

THEOREM 8.4. *If J is a prime real JB^*-triple, then one of the following assertions hold for J:*

i) J is the type $\mathbf{V}$, $\mathbf{VI}$, $\mathbf{V}^{\mathbb{O}}$, $\mathbf{V}^{\mathbb{O}_0}$, $\mathbf{VI}^{\mathbb{O}}$, or $\mathbf{VI}^{\mathbb{O}_0}$ generalized real Cartan factor.

ii) J is a generalized real spin factor.

iii) There exists a prime real C^*-algebra A such that J can be regarded as a JB^*-subtriple of the real C^*-algebra $M(A)$ contained in $M(A)_{sa}$ and containing A_{sa} .

iv) There exists a prime real C^*-algebra A with $*$-involution τ such that J can be regarded as a JB^*-subtriple of the real C^*-algebra $M(A)$ contained in $S(M(A),\tau)\cap M(A)_{sa}$ and containing $S(A,\tau)\cap A_{sa}$.

A real or complex JB^*-triple is said to be *topologically simple* if it has no nonzero proper closed triple ideals. Clearly, topologically simple JB^*-triples are prime.

COROLLARY 8.5. *If J is a topologically simple complex JB^*-triple, then one of the following assertions hold for J:*

*i) J is either the type **V** or the type **VI** complex Cartan factor.*

ii) J is a complex spin factor.

iii) There exists a matricially decomposed topologically simple complex C^*-algebra A such that $J = A_{12}$.

iv) There exists a matricially decomposed topologically simple complex C^*-algebra A with an even-swapping $*$-involution τ such that $J = H(A,\tau)\cap A_{12}$.

Proof. If J is not a JC^*-triple, then, by Theorem 2.4, we are in case i). Assume that J is of Clifford type. Then, by Theorem 7.4, either J is a complex spin factor (and we are in case ii)) or J is a type $\mathbf{I}_{n,m}$ ($n = 1,2$, $n \leq m$) or $\mathbf{II}_5$ complex Cartan factor. In the case that J is a type $\mathbf{II}_5$ complex Cartan factor, by Lemma 8.1 and its proof, we have $J = H(A,\tau)\cap A_{12}$ for some matricially decomposed algebraically (hence topologically) simple complex C^*-algebra A with an even-swapping $*$-involution τ, and therefore we are in case iv). In the case that $J = BL(H,K)$ is a type $\mathbf{I}_{n,m}$ ($n = 1,2$, $n \leq m$) complex Cartan factor, the matricially decomposed complex W^*-factor $A = BL(H\oplus K)$ given by Lemma 8.1 and its proof, which satisfies $J = A_{12}$, need not be topologically simple. However, the topologically simple C^*-algebra $\mathtt{A} := \mathcal{K}(H\oplus K)$ of all compact operators on $H\oplus K$ inherits the matricial decomposition of A, and the equality $A_{12} = \mathtt{A}_{12}$ holds, which leads to case iii).

Finally, assume that J is of hermitian type. By the topological simplicity of J and Theorem 5.9, we have the following two possibilities for J:

1) There exists a matricially decomposed prime complex C^*-algebra A such that $J = A_{12}$.

2) There exists a matricially decomposed prime complex C^*-algebra A with an even-swapping $*$-involution τ such that $J = H(A, \tau) \cap A_{12}$.

Moreover, by the proof of Theorem 5.9, in both situations A is generated as C^*-algebra by J. Now, to conclude the proof it is enough to show that the topological simplicity of J implies that of A. In the case that J is in the situation 1), this follows from Proposition 5.7.ii. If J is in the situation 2), and if P is a nonzero closed ideal of A, then $P \cap \tau(P)$ is a nonzero (by primeness of A) τ-invariant closed ideal of A, so $P \cap \tau(P) = A$ (by Proposition 5.7.iii), and so $P = A$. ∎

Replacing in the above argument Proposition 5.7.ii and 5.7.iii, Theorem 5.9, and Lemma 8.1 with Lemmas 4.3 and 4.2, Theorem 4.5, and Lemma 8.3, respectively, we obtain the following.

COROLLARY 8.6. *If J is a topologically simple real JB^*-triple, then one of the following assertions hold for J:*

i) J is the type $\mathbf{V}$, $\mathbf{VI}$, $\mathbf{V}^{\mathbb{O}}$, $\mathbf{V}^{\mathbb{O}_0}$, $\mathbf{VI}^{\mathbb{O}}$, or $\mathbf{VI}^{\mathbb{O}_0}$ generalized real Cartan factor.

ii) J is a generalized real spin factor.

iii) There exists a topologically simple real C^*-algebra A such that $J = A_{sa}$.

iv) There exists a topologically simple real C^*-algebra A with $*$-involution τ such that $J = S(A, \tau) \cap A_{sa}$.

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