On the Zelmanovian classification of prime JB^*- and JBW^*-triples.

A. Moreno Galindo* and A. Rodríguez-Palacios*

Universidad de Granada, Facultad de Ciencias.
Departamento de Análisis Matemático, 18071-Granada (Spain)
agalindo@goliat.ugr.es, apalacio@goliat.ugr.es

Introduction.

In [1] we applied the techniques of E. Zel'manov in [2, 3, 4] to obtain classification theorems for real and complex prime JB^*-triples. In the present paper we refine the results obtained in [1], and prove relevant specializations of such results for JBW^*-factors (i.e., prime JBW^*-triples).

In Section 1, we collect the statement of the zelmanovian classification theorems for real and complex prime JB^*-triples (Theorems 1.7 and 1.5, respectively). In the real case, the result is formulated verbatim as in [1]. By the way, the version of the real result in [1] seems to us quite natural. Concerning the result in the complex case, in [1] we introduced matricial decompositions on C^*-algebras, and described those prime complex JB^*-triples which are neither exceptional Cartan factors nor spin factors in terms of matricially decomposed prime complex C^*-algebras. Now, in Proposition 1.3 we prove that matricial decompositions of a given C^*-algebra A are in a one-to-one correspondence with projections in the multiplier C^*-algebra $M(A)$ of A. Then, we reformulate the classification theorem for prime complex JB^*-triples in the following terms.

For a prime complex JB^*-triple J which is neither an exceptional Cartan factor nor a spin factor, we have one of the following possibilities:

(i) $eA(1-e) \subseteq J \subseteq eM(A)(1-e)$
(ii) $H(eAe^\tau, \tau) \subseteq J \subseteq H(eM(A)e^\tau, \tau)$,

where in both cases A is a prime complex C^*-algebra and e is a projection in $M(A)$, in the second case τ is a $*$-involution on A with $e+e^\tau = 1$, the right inclusions must be read as "J is a JB^*-subtriple of ...", and consequently the left inclusions read as " ... is a closed triple ideal of J". As usual, for a vector space X with a linear involution τ, $H(X, \tau)$ denotes the set of all τ-hermitian elements of X.

---

* Partially supported by DGICYT Grant PB95-1146 and Junta de Andalucía grant FQM 0199.
1991 AMS Subject Classification: 17C65, 46K70.

Section 1 also contains the "converse" of the classification theorems just commented (i.e., the fact that all JB^*-triples listed in those theorems are prime). In the real case this follows directly from Zel'manov's work (see Proposition 1.6). The complex case needs a proof, which is given in Propositions 1.1 and 1.2.

In Section 2 we review the main strategy applied in [1] to obtain Theorems 1.5 and 1.7. We regard both real and complex prime JB^*-triples as Jordan triples over $\mathbb{R}$, and consider separately the three mutually excluding zelmanovian cases, namelly "exceptional", "Clifford", and "hermitian". Exceptional (respectively, Clifford) real and complex prime JB^*-triples are described in Theorem 2.1 (respectively, Theorem 2.2). The structure of real (respectively, complex) hermitian prime JB^*-triples is given by Theorem 2.6 (respectively, 2.12). Then Theorems 1.5 and 1.7 follow easily from the partial results just mentioned.

Since both exceptional and Clifford prime JB^*-triples are in fact JBW^*-factors, their determination (given by Theorems 2.1 and 2.2) become crucial tools in the classification of JBW^*-factors to be made in Section 3. Other minor auxiliary results, proved in [1] to obtain Theorems 2.6 and 2.12, will be needed. Such results are also included in Section 2.

The concluding section (Section 3) contains the main result in this paper, namely the zelmanovian classification theorems for real and complex JBW^*-factors. The formulations of these theorems are the following.

(Theorem 3.4) For a real JBW^*-factor J which is neither an exceptional generalized real Cartan factor nor a generalized real spin factor, we have one of the following possibilities:

(i) $J = A_{sa}$
(ii) $J = A_{sa} \cap S(A, \tau)$,

where in both cases A is a real W^*-factor, and in the second case τ is a $*$-involution on A. As usual, for a vector space X with a linear involution τ, $S(X, \tau)$ denotes the set of all τ-skew elements of X.

(Theorem 3.8) For a complex JBW^*-factor J which is neither an exceptional Cartan factor nor a spin factor, we have one of the following possibilities:

(i) $J = eA(1 - e)$
(ii) $J = H(eAe^\tau, \tau)$,

where in both cases A is a complex W^*-factor and e is a projection in A, and in the second case τ is a $*$-involution on A with $e + e^\tau = 1$.

An apparently different classification of complex JBW^*-factors can be obtained from the general structure theory of complex JBW^*-triples developed by G. Horn and E. Neher (see [5] and [6]). According to that theory, every complex JBW^*-factor J which is neither an exceptional

Cartan factor nor a spin factor must satisfy one of the following three assertions:

(a) There exist a complex W^*-factor B and a projection p in B such that $J = pB$.
(b) There exists a complex W^*-factor B with $*$-involution π such that $J = H(B, \pi)$.
(c) There exists a complex W^*-factor B with $*$-involution π such that $J = S(B, \pi)$.

We conclude Section 3 by showing that this last classification can be derived from Theorem 3.8 (see Claim 3.9 and Corollary 3.16).

1. Classification of prime JB^*-triples: the results.

We recall that a *complex JB*-triple* is a complex Banach space $\mathcal{A}$ with a continuous triple product $\{\cdots\} : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ which is linear and symmetric in the outer variables, and conjugate linear in the middle one, and satisfies

(i) for all $x \in \mathcal{A}$, the bounded linear operator $L_{x,x}$ on $\mathcal{A}$ defined by $L_{x,x}(a) := \{xxa\}$ is hermitian and has nonnegative spectrum;
(ii) $\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$ (the main identity);
(iii) $\|\{aaa\}\| = \|a\|^3$.

Complex JB^*-triples were introduced by W. Kaup in order to provide an algebraic setting for the study of bounded symmetric domains in complex Banach spaces. The open unit ball of every complex JB^*-triple is a bounded symmetric domain [7], and every bounded symmetric domain in any complex Banach space is bi-holomorphically equivalent to the open unit ball of a suitable complex JB^*-triple [8].

Real JB^*-triples are defined as norm-closed real subspaces J of complex JB^*-triples satisfying $\{JJJ\} \subseteq J$. They have been introduced and studied in the paper of J. M. Isidro, W. Kaup and A. Rodríguez [9], where, as main result, it is proved that surjective linear mappings between real JB^*-triples are isometric if and only if they preserve the cube mapping $x \mapsto \{xxx\}$. If $\mathcal{A}$ is a complex JB^*-triple and if σ is a *conjugation* (i.e., an involutive conjugate linear isometry) on $\mathcal{A}$, then the *real form* $\mathcal{A}^\sigma := \{a \in \mathcal{A} : \sigma(a) = a\}$ of $\mathcal{A}$ is a real JB^*-triple. In fact, every real JB^*-triple is a real form of a complex JB^*-triple [9, Proposition 2.2].

A linear subspace I of a (real or complex) JB^*-triple A is a JB^*-*subtriple* of A if it is closed in A and $\{III\} \subseteq I$, and a *triple ideal* of A if $\{AAI\} + \{AIA\} \subseteq I$. A JB^*-triple A is said to be *prime* if whenever P, Q are triple ideals of A with $\{PAQ\} = 0$ we have either $P = 0$ or $Q = 0$.

Certain prime complex JB^*-triples (the so-called complex Cartan factors) are well understood in the literature. (Real or complex) *JBW^*-triples* are defined as those JB^*-triples which are Banach dual spaces. From a given JBW^*-triple J we can obtain new JBW^*-triples by considering the so-called *JBW^*-subtriples* of J, namely the w^*-closed JB^*-subtriples of J. Prime JBW^*-triples are called *JBW^*-factors*. A *complex Cartan factor* is a complex JBW^*-factor such that the closed unit ball of its predual has some extreme point. Complex Cartan factors come in the following six types (see [5]).

$\mathbf{I}_{n,m}$ $:= BL(H,K)$ (the Banach space of all bounded linear mappings from H to K), where H, K are complex Hilbert spaces of hilbertian dimension n, m, respectively, with $1 \leq n \leq m$, and the triple product is defined by

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x)$$

for all x, y, z in $BL(H,K)$.

$\mathbf{II}_n$ $:= \{x \in BL(H) : \sigma x^* \sigma = -x\}$ as JBW^*-subtriple of $BL(H)$ $(:= BL(H,H))$, where H is a complex Hilbert space of hilbertian dimension $n \geq 5$ and σ is a conjugation on H.

$\mathbf{III}_n$ $:= \{x \in BL(H) : \sigma x^* \sigma = x\}$ as JBW^*-subtriple of $BL(H)$, where H is a complex Hilbert space of hilbertian dimension $n \geq 2$ and σ is a conjugation on H.

$\mathbf{IV}_n$ $:= H$, where H is a complex Hilbert space of hilbertian dimension $n \geq 3$, σ is a conjugation on H, and the triple product and the norm are given by

$$\{xyz\} := (x|y)z + (z|y)x - (x|\sigma(z))\sigma(y)$$

and

$$\|x\|^2 := (x|x) + \sqrt{(x|x)^2 - |(x|\sigma(x))|^2},$$

respectively, for all x, y, z in H.

$\mathbf{V}$ $:= M_{12}(\mathbb{O}_{\mathbb{C}})$ the 1×2-matrices over the complex Cayley numbers $\mathbb{O}_{\mathbb{C}}$.

$\mathbf{VI}$ $:= H_3(\mathbb{O}_{\mathbb{C}})$ the hermitian 3×3-matrices over $\mathbb{O}_{\mathbb{C}}$.

We recall that the type $\mathbf{IV}_n$ complex Cartan factors are usually called *complex spin factors*. For details about the precise definition of the triple product on the type $\mathbf{V}$ and $\mathbf{VI}$ Cartan factors the reader is referred to [10, Pag 4.12].

Other examples of prime complex JB^*-triples can be obtained from prime complex C^*-algebras by means of the constructive methods provided by Propositions 1.1 and 1.2 below.

By a *matricial decomposition* of a C^*-algebra A we mean a family $\{A_{ij}\}_{i,j\,\in\{1,2\}}$ of closed subspaces of A satisfying $A_{ij}^* = A_{ji}$, $A = \oplus_{i,j\in\{1,2\}} A_{ij}$, and $A_{ij}\ A_{kl} \subseteq \delta_{jk} A_{il}$ for all $i, j, k, l \in \{1,2\}$. A *matricially decomposed C^*-algebra* will be a C^*-algebra endowed with a matricial decomposition. From now on, for every C^*-algebra A, $M(A)$ will denote the C^*-algebra of multipliers of A. If A is a matricially decomposed C^*-algebra, then the matricial decomposition of A can be uniquely extended to a matricial decomposition of $M(A)$ [1, Proposition 5.8].

We recall that every (real or complex) C^*-algebra is a JB^*-triple under the triple product $\{abc\} := \frac{1}{2}(ab^*c + cb^*a)$. Moreover, if A is a matricially decomposed C^*-algebra, then A_{12} is a JB^*-subtriple of A.

Proposition 1.1. *Let A be a matricially decomposed prime C^*-algebra. Then every JB^*-subtriple of $M(A)$ contained in $(M(A))_{12}$ and containing A_{12} is a prime JB^*-triple.*

Proof. Let J be a JB^*-subtriple of $M(A)$ contained in $(M(A))_{12}$ and containing A_{12}. It is enough to show that the conditions $x, y \in J$ and $\{xJy\} = 0$ imply either $x = 0$ or $y = 0$. Let x, y be in J such that $\{xJy\} = 0$. Then, by the multiplication rules of matricial decompositions, we have

$$\{xAy\} = \{xA_{12}y\} \subseteq \{xJy\} = 0.$$

Therefore, for every a in A the equality $xay = -yax$ holds. Then for a, b in A we have

$$\begin{aligned} xaybxay &= (xay)bxay = -(yax)bxay = -(y(axb)x)ay = (x(axb)y)ay \\ &= xa(xby)ay = -xa(ybx)ay = -xaybxay\,, \end{aligned}$$

and hence $(xay)A(xay) = 0$. Since a is arbitrary in A and A is prime, we deduce that $xAy = 0$. Since x and y belong to $M(A)$, again the primeness of A gives us that either $x = 0$ or $y = 0$.

By a $*$-*involution* on a C^*-algebra A we mean a linear algebra involution on A commuting with the C^*-algebra involution $*$ of A. A $*$-involution τ on a matricially decomposed C^*-algebra A is said to be *even-swapping* whenever the equalities $\tau(A_{11}) = A_{22}$ and $\tau(A_{12}) = A_{12}$ hold. We recall that every $*$-involution τ on a C^*-algebra A extends uniquely to a $*$-involution (which will be denoted by the same symbol τ) on $M(A)$. Moreover, if the C^*-algebra A is matricially decomposed and if the $*$-involution τ is even-swapping, then the extension of τ to $M(A)$ is even-swapping too.

For every involutive linear operator $\tau : x \mapsto x^\tau$ on a vector space E, and for every τ-invariant subspace M of E we write

$$H(M, \tau) := \{m \in M : m^\tau = m\}.$$

Proposition 1.2. *Let A be a matricially decomposed prime C^*-algebra with an even-swapping $*$-involution τ. Then every JB^*-subtriple of $M(A)$ contained in $H(M(A)_{12}, \tau)$ and containing $H(A_{12}, \tau)$ is a prime JB^*-triple.*

Proof. It is enough to prove that the conditions $x, y \in H(M(A)_{12}, \tau)$ and $\{xH(A_{21}, \tau)y\} = 0$ imply either $x = 0$ or $y = 0$. In a first step we prove that if x and y are in $H(M(A)_{12},\tau)$ and if $xH(A_{21},\tau)y=0$, then either $x=0$ or $y=0$. Let x,y be in $H(M(A)_{12},\tau)$ such that $xH(A_{21},\tau)y=0$. For $a,b\in A_{21}$ we have

$$(xay)^{\tau}b(xay) = \\ ya^{\tau}xb[x(a+a^{\tau})y]-ya^{\tau}[x(bxa^{\tau}+axb^{\tau})y]+[x(a^{\tau}xa)y]^{\tau}b^{\tau}y \\ =0,$$

so

$$(xay)^{\tau}A(xay)=(xay)^{\tau}A_{21}(xay)=0$$

(by the multiplication rules of matricial decompositions), and so $xay=0$ (by the primeness of A). Since a is arbitrary in A_{21} and the equality $xAy=xA_{12}y$ holds, we actually have $xAy=0$, so that again the primeness of A gives either $x=0$ or $y=0$.

Now, let x,y be in $H(M(A)_{12},\tau)$ such that $\{xH(A_{12},\tau)y\}=0$. If h and k are in $H(A_{21},\tau)$, then h^* and k^* belong to $H(A_{12},\tau)$, so that we have

$$\begin{aligned}(xhx)k(yhy) = {} & xh\{xk^*y\}hx-\{xh^*y\}kxhy-\{xk^*y\}hxhx+ \\ & \{x(hxk+kxh)^*y\}hy-xk\{x(hyh)^*y\}+xkyh\{xh^*y\} \\ = {} & 0.\end{aligned}$$

Therefore, for h in $H(A_{21},\tau)$ we obtain $(xhx)H(A_{21},\tau)(yhy)=0$, and hence, by the first step in the proof, we have either $xhx=0$ or $yhy=0$. Even more, we actually have either $xH(A_{21},\tau)x=0$ or $yH(A_{21},\tau)y=0$. (Indeed, if there were h_1,h_2 in $H(A_{21},\tau)$ satisfying $xh_1x\neq 0$ and $yh_2y\neq 0$, then $yh_1y=xh_2x=0$, so that would have $h:=h_1+h_2\in H(A_{21},\tau)$, $xhx\neq 0$, and $yhy\neq 0$, which is not possible.) Again by the first step in the proof we obtain either $x=0$ or $y=0$.

If A is a C^*-algebra, if e is a projection (i.e., a self-adjoint idempotent) in $M(A)$, and if we put $e_1:=e$ and $e_2:=1-e$, then $\{e_iAe_j\}_{i,j\in\{1,2\}}$ becomes a matricial decomposition of A. Our next result asserts that every matricial decomposition of a given C^*-algebra A comes from a projection e in $M(A)$ by the above procedure. We recall that a real (respectively, complex) *W^*-algebra* is a real (respectively, complex) C^*-algebra which is a dual Banach space. We also recall that, if A is a real or complex W^*-algebra, then the (binary) product of A is separately w^*-continuous, and every surjective linear isometry on A is w^*-continuous [11].

Proposition 1.3. *Let A be a matricially decomposed C^*-algebra. Then there exists a projection e in $M(A)$ such that, by putting $e_1:=e$ and $e_2:=1-e$, we have $A_{ij}=e_iAe_j$ for $i,j\in\{1,2\}$.*

Proof. First assume that A is actually a W^*-algebra and that the sum $A=\oplus_{i,j\in\{1,2\}}A_{ij}$ is w^*-topological. Then $A_{11}+A_{12}$ is a w^*-closed right ideal of A, and therefore there exists a projection e in A such that $A_{11}+A_{12}=eA$ [12, Proposition 2.10]. By taking adjoints, we obtain

$A_{11} + A_{21} = Ae$, hence
$$A_{11} \subseteq eA \cap Ae = eAe = e_1Ae_1.$$
On the other hand, since $(A_{12} + A_{22})(A_{11} + A_{12}) = 0$ and e belongs to $A_{11} + A_{12}$, we deduce $A_{12} + A_{22} \subseteq A(1 - e)$. By taking adjoints again, we obtain $A_{21} + A_{22} \subseteq (1 - e)A$, and hence
$$A_{22} \subseteq (1 - e)A \cap A(1 - e) = (1 - e)A(1 - e) = e_2Ae_2.$$
Moreover we clearly have
$$A_{12} \subseteq eA \cap A(1 - e) = eA(1 - e) = e_1Ae_2,$$
and hence,
$$A_{21} \subseteq e_2Ae_1.$$
Now, since $A = \oplus_{i,j\in\{1,2\}}A_{ij}$, and $A = \oplus_{i,j\in\{1,2\}}e_iAe_j$, and $A_{ij} \subseteq e_iAe_j$ for $i, j \in \{1, 2\}$, we conclude $A_{ij} = e_iAe_j$ for $i, j \in \{1, 2\}$.

Now let A be an arbitrary matricially decomposed C^*-algebra. Then A'' is a W^*-algebra [12, Theorem 1.6] and, denoting by p_{ij} the (continuous) linear projection on A with range A_{ij} and kernel $\sum_{(k,l)\neq(i,j)} A_{kl}$, the family $\{p''_{ij}(A'')\}_{i,j\in\{1,2\}}$ is a matricial decomposition of A''. Moreover the sum $A'' = \oplus_{i,j\in\{1,2\}}p''_{ij}(A'')$ is w^*-topological. By applying the first paragraph in the proof we find a projection e in A'' such that $p''_{ij}(A'') = e_iA''e_j$, where $e_1 := e$ and $e_2 := 1 - e$. Then, for a in A we have
$$ea = eae + ea(1 - e) = (p''_{11} + p''_{12})(a) \in A.$$
Therefore e belongs to $M(A)$.

Remark 1.4. Let A be a C^*-algebra, e a projection in $M(A)$, and τ a $*$-involution on A. It is straightforward that τ is even-swapping relative to the matricial decomposition $\{e_iAe_j\}_{i,j\in\{1,2\}}$ (where $e_1 := e$ and $e_2 := 1 - e$) if and only if $e + e^\tau = 1$.

In [1] we applied the techniques of E. Zel'manov in [2, 3, 4] to show that all prime complex JB^*-triples either are complex Cartan factors or can be obtained from complex matricially decomposed prime C^*-algebras by the methods given in Propositions 1.1 and 1.2 (see [1, Theorem 8.2]). Now, with the help of Proposition 1.3 and Remark 1.4 we can avoid any mention to matricial decompositions of C^*-algebras, and reformulate Theorem 8.2 of [1] as follows.

Theorem 1.5. *If J is a prime complex JB^*-triple, then one of the following assertions hold for J:*

(i) *J is either the type* **V** *or the type* **VI** *complex Cartan factor.*
(ii) *J is a complex spin factor.*
(iii) *There exist a prime complex C^*-algebra A and a projection e in $M(A)$ such that J can be regarded as a JB^*-subtriple of the complex C^*-algebra $M(A)$ contained in $eM(A)(1 - e)$ and containing $eA(1 - e)$.*

(iv) *There exist a prime complex C^*-algebra A, a projection e in $M(A)$, and a $*$-involution τ on A with $e + e^\tau = 1$ such that J can be regarded as a JB^*-subtriple of the complex C^*-algebra $M(A)$ contained in $H(eM(A)e^\tau, \tau)$ and containing $H(eAe^\tau, \tau)$.*

Among the real prime JB^*-triples, we are obliged to consider (as the nicest examples) both complex Cartan factors (regarded as real ones) and real forms of complex Cartan factors. These last real JBW^*-factors are called *real Cartan factors*. The classification of real Cartan factors is due to O. Loos [10] in the finite dimensional case, and W. Kaup in the general case [13]. They come in 12 different types (see [13]):

$$\mathbf{I}^{\mathbb{R}}_{n,m}, \mathbf{I}^{\mathbb{H}}_{2p,2q}, \mathbf{I}^{\mathbb{C}}_{n,n}, \mathbf{II}^{\mathbb{R}}_{n}, \mathbf{II}^{\mathbb{H}}_{2p}, \mathbf{III}^{\mathbb{R}}_{n}, \mathbf{III}^{\mathbb{H}}_{2p}, \mathbf{IV}^{r,s}_{n}, \mathbf{V}^{\mathbb{O}}, \mathbf{V}^{\mathbb{O}_0}, \mathbf{VI}^{\mathbb{O}}, \mathbf{VI}^{\mathbb{O}_0}.$$

The notation has the property that, erasing the superscripts, we obtain a complex Cartan factor J such that the given real Cartan factor is one of the real forms of J.

By a *generalized real Cartan factor* we mean either a complex Cartan factor (regarded as a real one) or a real Cartan factor. Generalized real Cartan factors can be intrinsically characterized (see [13, Lemma 4.5]). Given a real JB^*-triple J, let us say that J is a *real spin factor* if it is a real form of a complex spin factor, and that J is a *generalized real spin factor* if it is either a complex spin factor (regarded as a real one) or a real spin factor.

Let A be a real C^*-algebra. Then the self-adjoint part of A (denoted as usual by A_{sa}) is a JB^*-subtriple of A and hence a real JB^*-triple. If moreover τ is a $*$-involution on A, then $S(A, \tau) \cap A_{sa}$ (where $S(A, \tau) := \{a \in A : a^\tau = -a\}$) is also a JB^*-subtriple of A. The following proposition follows from [3, Lemma 4].

Proposition 1.6. *Let A be a prime real C^*-algebra. Then we have:*

(i) *Every JB^*-subtriple of $M(A)$ contained in $(M(A))_{sa}$ and containing A_{sa} is a prime JB^*-triple.*
(ii) *If τ is a $*$-involution on A, then every JB^*-subtriple of $M(A)$ contained in $S(M(A), \tau) \cap (M(A))_{sa}$ and containing $S(A, \tau) \cap A_{sa}$ is a prime JB^*-triple.*

In [1] we also applied zelmanovian techniques to prove that all prime real JB^*-triples either are generalized real Cartan factors or can be obtained from prime real C^*-algebras by the methods given in Proposition 1.6. The precise formulation of this result reads as follows.

Theorem 1.7 ([1, Theorem 8.4])**.** *If J is a prime real JB^*-triple, then one of the following assertions hold for J:*

(i) *J is the type* **V***,* **VI***, $\mathbf{V}^{\mathbb{O}}$, $\mathbf{V}^{\mathbb{O}_0}$, $\mathbf{VI}^{\mathbb{O}}$, or $\mathbf{VI}^{\mathbb{O}_0}$ generalized real Cartan factor.*
(ii) *J is a generalized real spin factor.*

(iii) *There exists a prime real C^*-algebra A such that J can be regarded as a JB^*-subtriple of the real C^*-algebra $M(A)$ contained in $M(A)_{sa}$ and containing A_{sa} .*

(iv) *There exists a prime real C^*-algebra A with $*$-involution τ such that J can be regarded as a JB^*-subtriple of the real C^*-algebra $M(A)$ contained in $S(M(A),\tau)\cap M(A)_{sa}$ and containing $S(A,\tau)\cap A_{sa}$.*

2. Classification of prime JB^*-triples: the tools.

This section is devoted to comment on the techniques applied in the proof of the zelmanovian classification of prime JB^*-triples given by Theorems 1.5 and 1.7. In Zel'manov's work, Jordan triples over a field $\mathbb{F}$ of characteristic different from 2 and 3 are defined as vector spaces over $\mathbb{F}$ endowed with a triple product which is $\mathbb{F}$-linear in each of its variables, is symmetric in the outer variables, and satisfies the same main identity required for JB^*-triples. A Jordan triple T is said to be *nondegenerate* whenever the conditions $x \in T$ and $\{xTx\} = 0$ imply $x = 0$. Now, it is clear that, forgetting the analytic conditions, and restricting the scalars in the complex case, every real or complex JB^*-triple becomes a Jordan triple over $\mathbb{R}$.

The Zel'manov classification of nondegenerate prime Jordan triples relies on an apparently ingenuos alternative, by considering three mutually excluding cases, namely, non i-special, Clifford, and hermitian. A Jordan triple is called *special* whenever it is (isomorphic to) a subtriple of some associative algebra under the triple product

$$\{abc\} := \frac{1}{2}(abc + cba), \tag{2.1}$$

and *i-special* if it is the homomorphic image of a special Jordan triple. The i-special Jordan triples are classified in two types, Clifford or hermitian, depending on whether or not all the identities collected in a certain ideal of the free special Jordan triple vanishe on them. Later we will explain with more details these concepts. Roughly speaking, a part of Zel'manov's prime theorem for Jordan triples establishes the scarcity, up to suitable scalar extensions, of non-degenerate prime Jordan triples which are not of hermitian type. The remaining part of Zel'manov's theorem shows that nondegenerate prime Jordan triples of hermitian type over $\mathbb{F}$ are "essentially" of the form $H(A,*)\cap S(A,\tau)$ for some associative algebra A over $\mathbb{F}$ with two commuting $\mathbb{F}$-linear involutions $*$ and τ. Here $H(A,*)\cap S(A,\tau)$ is regarded as a subtriple of A with triple product defined by (1).

The conjugate-linear behaviour of the triple product of a complex JB^*-triple in its middle variable becomes a first handicap in applying zelmanovian notions and techniques in our setting. Concerning notions, there are no problems: we see complex JB^*-triples as Jordan triples over $\mathbb{R}$, and consider separately the non i-special, hermitian, and Clifford cases. However, a verbatim application of zelmanovian techniques to prime complex JB^*-triples would provide in the best of cases only a determination of the real structure of such JB^*-triples (see for instance [1, Theorem 5.3]). To overcome this difficulty, we designed in [1] different strategies, which we are going to explain along this section.

The determination of non i-special complex prime JB^*-triples obtained in [1] actually avoids Zel'manov's prime theorem for Jordan triples, and only uses Zel'manov's prime theorem for Jordan algebras [14] through its version for JB^*-algebras [15]. The determination of non i-special prime real JB^*-triples follows easily from that of complex ones, by applying classical theory. The next theorem (a consequence of [1, Theorem 2.4]) collects the results in this line.

Theorem 2.1. *The non i-special prime complex JB^*-triples are the type* $\mathbf{V}$ *and* $\mathbf{VI}$ *complex Cartan factors. The non i-special prime real JB^*-triples are the two complex JB^*-triples above (regarded as real JB^*-triples) plus their real forms, namely the type* $\mathbf{V}^{\mathbb{O}}$, $\mathbf{V}^{\mathbb{O}_0}$, $\mathbf{VI}^{\mathbb{O}}$, *and* $\mathbf{VI}^{\mathbb{O}_0}$ *real Cartan factors.*

The actual formulation of Theorem 2.4 of [1] is stronger than that of the above theorem. Indeed, the JB^*-triples listed in Theorem 2.1 are the unique prime JB^*-triples which are not JC^*-triples. We recall that *JC^*-triples* are defined as JB^*-subtriples of C^*-algebras, and that every JC^*-triple is special (a consequence of [9, Corollary 2.4]). Theorem 2.1 (respectively, its improvement just commented) becomes one of the tools in the determination of Clifford (respectively, hermitian) prime JB^*-triples.

To deal with the Zel'manov approach to i-special Jordan triples, we need some concepts of universal algebra. In what follows X will denote an infinite set of indeterminates and $\mathbb{F}$ will stand for a field of characteristic different from 2 and 3. The free associative algebra $\mathcal{A}(X)$ over $\mathbb{F}$ has two natural linear involutions, namely the involution $*$ leaving the elements of X fixed, and the one τ which maps each element x in X into $-x$. As any associative algebra over $\mathbb{F}$, $\mathcal{A}(X)$ is a Jordan triple over $\mathbb{F}$ under the triple product defined in (1). The subtriple of $\mathcal{A}(X)$ generated by X is called the *free special Jordan triple* over $\mathbb{F}$ on the set of free generators X, and is denoted by $\mathcal{ST} = \mathcal{ST}(X)$. Clearly, the inclusion $\mathcal{ST} \subseteq H(\mathcal{A}(X), *) \cap S(\mathcal{A}(X), \tau)$ holds.

It follows from the universal property of $\mathcal{A}(X)$ that, if A is an associative algebra with two commuting involutions $*, \tau$, and if T is a Jordan subtriple of A contained in $H(A, *) \cap S(A, \tau)$ then every map $\phi : X \to T$ extends to a unique associative $*$-τ-homomorphism $\hat{\phi} : \mathcal{A}(X) \to A$ such that $\hat{\phi}(\mathcal{ST}) \subseteq T$. Since every special Jordan triple can be regarded as a Jordan subtriple of a suitable associative algebra with two commuting linear involutions $(A, *, \tau)$ contained in $H(A, *) \cap S(A, \tau)$, it follows that, given a special Jordan triple T, every mapping from X to T extends uniquely to a triple-homomorphism from $\mathcal{S}T$ to T. Keeping in mind the definition of i-special Jordan triples, the above universal property remains true in the more general case that T is a i-special Jordan triple. In this way, we can consider valuations of elements of $\mathcal{S}T$ in any i-special Jordan triple.

One of the key tools in Zel'manov's work is the discovery of a precise ideal $\mathcal{G}$ (see [3, p. 730]) in $\mathcal{S}T$ with the property that the behaviour of i-special prime nondegenerate Jordan triples T drastically differs depending on whether or not $\mathcal{G}$ vanishes on them. By a Jordan triple of *Clifford type* we mean an i-special Jordan triple T satisfying $\mathcal{G}(T) = 0$.

The treatment of prime JB^*-triples of Clifford type made in [1] starts with an application of results in [16] to get a rather artisanal determination of complex Cartan factors of Clifford type [1, Proposition 6.1]. From such a determination it follows easily that Banach ultraproducts of arbitrary families of complex Cartan factors of Clifford type are Hilbert spaces up to equivalent renormings [1, Corollary 6.2]. Then, replacing algebraic ultraproducts with Banach ultraproducts in an argument in [4, pp. 63-64] (see also [16]), it is shown that every prime complex JB^*-triple of Clifford type is in fact a complex Cartan factor [1, Proposition 7.3]. As happened in the non i-special case, in the Clifford case also the determination of prime real JB^*-triples follows easily from that of complex ones, by applying classical theory. The precise formulation of the results in this line is the following.

Theorem 2.2 ([1, Theorem 7.4]). *The prime complex JB^*-triples of Clifford type are the type $\mathbf{I}_{n,m}$ ($m \geq n = 1, 2$), $\mathbf{II}_5$, $\mathbf{III}_2$, and $\mathbf{IV}_n$ ($n \geq 5$) complex Cartan factors. The prime real JB^*-triples of Clifford type are the complex JB^*-triples just listed (regarded as real JB^*-triples) plus their real forms, namely the type $\mathbf{I}^{\mathbb{R}}_{n,m}$ ($m \geq n = 1, 2$), $\mathbf{I}^{\mathbb{H}}_{2,2q}$ ($1 \leq q$), $\mathbf{I}^{\mathbb{C}}_{n,n}$ ($n = 1, 2$), $\mathbf{II}^{\mathbb{R}}_5$, $\mathbf{III}^{\mathbb{R}}_2$, $\mathbf{III}^{\mathbb{H}}_2$ and $\mathbf{IV}^{r,s}_n$ ($n \geq 5$, $r \geq s \geq 0$, $r + s = n$) real Cartan factors (cf.* [13, Theorem 4.1]*).*

By a Jordan triple of *hermitian type* we mean an i-special Jordan triple T which is not of Clifford type (equivalently, $\mathcal{G}(T) \neq 0$, where $\mathcal{G}$ is the zelmanovian ideal of $\mathcal{S}T$ introduced above). The key tool in the zelmanovian treatment of Jordan triples of hermitian type is the following lemma. It is also necessary for the proof of the structure theorems in [1] for both real and complex prime JB^*-triples of hermitian type.

Lemma 2.3 ([1, Lemma 4.4]). *Let B be an associative algebra with two commuting involutions $*, \tau$, and T a Jordan subtriple of B of hermitian type contained in $H(B, *) \cap S(B, \tau)$. Then the subalgebra C of B generated by $\mathcal{G}(T)$ is $*$-τ-invariant and we have $\mathcal{G}(T) = H(C, *) \cap S(C, \tau)$.*

Concerning prime real JB^*-triples of hermitian type, we collect in the following two lemmas other minor results also needed for their classification.

Lemma 2.4 ([1, Lemma 4.1]). *Let J be a real JC^*-triple. Then there exists a real C^*-algebra A with $*$-involution τ such that J is a JB^*-subtriple of A contained in $S(A,\tau)\cap A_{sa}$.*

Lemma 2.5 ([1, Lemmas 4.3 and 4.2]). *Let A be a real C^*-algebra. Then we have:*

(i) *Every non-zero closed ideal of A meets A_{sa}.*
(ii) *If τ is a $*$-involution on A, and if A is generated as a closed ideal by $S(A,\tau)\cap A_{sa}$, then every non-zero τ-invariant closed ideal of A meets $S(A,\tau)\cap A_{sa}$.*

The already commented improved version of Theorem 2.1 guarantees that every i-special prime real or complex JB^*-triple is in fact a JC^*-triple. Then Lemmas 2.3, 2.4, and 2.5, and some facts from the theory of multipliers of real JB^*-triples (see [17] and [1, section 3]), lead to the following classification theorem for prime real JB^*-triples of hermitian type (see the proof of [1, Theorem 4.5] for details).

Theorem 2.6. *Let J be a prime real JB^*-triple of hermitian type. Then one of the following assertions is true for J :*

(i) *There exists a prime real C^*-algebra A such that J can be regarded as a JB^*-subtriple of the real C^*-algebra $M(A)$ contained in $M(A)_{sa}$ and containing A_{sa} .*
(ii) *There exists a prime real C^*-algebra A with $*$-involution τ such that J can be regarded as a JB^*-subtriple of the real C^*-algebra $M(A)$ contained in $S(M(A),\tau)\cap M(A)_{sa}$ and containing $S(A,\tau)\cap A_{sa}$.*

After Theorems 2.1, 2.2, and 2.6, the classification of prime real JB^*-triples is germinally concluded. To arrive to their definitive classification given by Theorem 1.7, we only need to realize that every prime real JB^*-triple J of Clifford type either is a generalized spin factor or behaves "formally" as a prime real JB^*-triple of hermitian type (i.e., one of assertions (i) and (ii) in Theorem 2.6 holds for J). To this end, we recall that a real (respectively, complex) *W^*-factor* is a real (respectively, complex) prime W^*-algebra.

We begin by noticing that the type $\mathbf{III}_2$ complex Cartan factor is equal to the complex spin factor $\mathbf{IV}_3$, so that the type $\mathbf{III}_2$, $\mathbf{III}_2^{\mathbb{R}}$, and $\mathbf{III}_2^{\mathbb{H}}$ generalized real Cartan factors are generalized real spin factors. On the other hand, the definition itself of the type $\mathbf{I}_{n,n}^{\mathbb{C}}$ real Cartan factors shows that such real Cartan factors are of the form A_{sa} for a suitable real W^*-factor A. This applies in particular to the prime real JB^*-triple of Clifford type $\mathbf{I}_{2,2}^{\mathbb{C}}$. Now, the remaining real JB^*-triples listed in Theorem 2.2 fall into case (*ii*) of Theorem 2.6 thanks to the following lemma.

Lemma 2.7 ([1, Lemma 8.3]). *Let J be a type $\mathbf{I}_{n,m}$, $\mathbf{II}_n$, $\mathbf{I}^{\mathbb{R}}_{n,m}$, $\mathbf{I}^{\mathbb{H}}_{2p,2q}$, or $\mathbf{II}^{\mathbb{R}}_n$ generalized real Cartan factor. Then there exists a real W^*-factor A with a $*$-involution τ such that $J = A_{sa} \cap S(A,\tau)$.*

The proof of the structure theorem for prime complex JB^*-triples of hermitian type is much more laborious. Following an idea of O. Loos in [10, 2.9], when a complex JB^*-triple J is regarded as a real Jordan pair, such a real Jordan pair is in fact the realification of a Jordan pair (say V) over $\mathbb{C}$. In the case that J is a complex JC^*-triple of hermitian type, the polarization of V (say T) is a Jordan triple over $\mathbb{C}$ of hermitian type, which can be represented into the secondary diagonal of a matricially decomposed complex C^*-algebra regarded as a Jordan triple under the product (1). Then the zelmanovian technique given by Lemma 2.3 successfully applies to T, providing the following result.

Proposition 2.8 ([1, Proposition 5.6]). *Let J be a complex JC^*-triple of hermitian type. Then J contains a non-zero closed triple ideal of the form $H(A_{12},\tau)$, where A is a matricially decomposed C^*-algebra, τ is an even-swapping $*$-involution on A, and A is generated as C^*-algebra by $H(A_{12},\tau)$.*

Among the tools needed in the proof of the above proposition we mention for later reference Lemmas 2.9 and 2.10 which follow.

Lemma 2.9 ([1, Lemma 5.4]). *Let T be a i-special complex Jordan triple. Then $\mathcal{G}(T)$ is invariant under every conjugate-linear automorphism of T.*

Lemma 2.10 ([1, Lemma 5.5]). *Let J be a complex JC^*-triple. Then there exists a matricially decomposed complex C^*-algebra B with an even-swapping $*$-involution τ such that J can be seen as a JB^*-subtriple of B contained in $H(B_{12},\tau)$.*

To arrive to the structure theorem for prime complex JB^*-triple of hermitian type we still need to invoke some standard facts from the theory of multipliers of complex JB^*-triples (see [18] and [1, Proposition 5.8]) and the next proposition.

Proposition 2.11 ([1, Proposition 5.7]). *For a matricially decomposed C^*-algebra A the following assertions hold:*

(i) *All closed ideals of A inherit the matricial decomposition.*
(ii) *If A is generated as a closed ideal by A_{12}, then every non-zero closed ideal of A meets A_{12}.*
(iii) *If A has an even-swapping $*$-involution τ, and if A is generated as a closed ideal by $H(A_{12},\tau)$, then every non-zero τ-invariant closed ideal of A meets $H(A_{12},\tau)$.*

Now, with Proposition 2.8 in mind, the following structure theorem for prime complex JB^*-triples of hermitian type, which is nothing but [1, Theorem 5.9] plus Proposition 1.3, follows with a minor effort.

Theorem 2.12. *Let J be a prime complex JB^*-triple of hermitian type. Then one of the following assertions is true for J :*

(i) *There exist a prime complex C^*-algebra A and a projection e in $M(A)$ such that J can be regarded as a JB^*-subtriple of the complex C^*-algebra $M(A)$ contained in $eM(A)(1-e)$ and containing $eA(1-e)$.*

(ii) *There exist a prime complex C^*-algebra A, a projection e in $M(A)$, and a $*$-involution τ on A with $e+e^\tau = 1$ such that J can be regarded as a JB^*-subtriple of the complex C^*-algebra $M(A)$ contained in $H(eM(A)e^\tau, \tau)$ and containing $H(eAe^\tau, \tau)$.*

To conclude this section, let us note that the classification of prime complex JB^*-triples collected in Theorem 1.5 is a straightforward consequence of Theorems 2.1, 2.2, and 2.12, and Lemma 2.13 which follows. The lemma is nothing but [1, Lemma 8.1] plus Proposition 1.3.

Lemma 2.13. *If J is a type $\mathbf{I}_{n,m}$ $(1 \le n \le m)$ complex Cartan factor, then there exist a complex W^*-factor A and a projection e in A such that $J = eA(1-e)$. If J is either a type $\mathbf{II}_n$ $(n \ge 2)$ or type $\mathbf{III}_n$ $(n \ge 1)$ complex Cartan factor, then there exist a complex W^*-factor A, a projection e in A, and a $*$-involution τ with $e + e^\tau = 1$ such that $J = H(eAe^\tau, \tau)$.*

3. Classification of JBW^*-factors.

In this section we prove relevant specializations of Theorems 2.6 and 2.12 when the hermitian prime JB^*-triple J in those theorems is in fact a JBW^*-factor. Consequently, we obtain classification theorems for real and complex JBW^*-factors, which, in their settings, refine Theorems 1.7 and 1.5.

We recall that, for a real (respectively, complex) JBW^*-triple J, the predual of J is unique and the triple product of J is separately w^*-continuous [19] (respectively, [20]). Moreover, if J is a real (respectively, complex) JBW^*-triple, and if P is a w^*-closed triple ideal of J, then there exists a w^*-closed triple ideal Q of J satisfying $J = P \oplus Q$ [9] (respectively, [21]). As a consequence, JBW^*-factors have no nonzero proper w^*-closed ideals. Let us also recall that the bidual J'' of a real (respectively, complex) JB^*-triple J is a JB^*-triple containing J as a JB^*-subtriple [9, Lemma 4.2] (respectively, [22]). It is proved in [23, Proposition 6] that every complex JBW^*-triple is (isometrically, and hence w^*-bicontinuously) isomorphic to a w^*-closed triple ideal of its bidual. With the above ideas in mind, the same result remains true (with verbatim proof) for real JBW^*-triples.

Our argument begins with the following adaptation of Lemma 2.4 to our new context.

Lemma 3.1. *Let J be a real JBW^*-triple which is also a JC^*-triple. Then there exists a real W^*-algebra A with $*$-involution τ such that J is a JBW^*-subtriple of A contained in $S(A,\tau)\cap A_{sa}$.*

Proof. By Lemma 2.4, there exists a real C^*-algebra B with $*$-involution π such that J is a JB^*-subtriple of B contained in $S(B,\pi)\cap B_{sa}$. Put $A := B''$ and $\tau = \pi''$ (the bitranspose of π). Then A is a real W^*-algebra [12, Theorem 1.6], τ is a $*$-involution on A, and the bipolar J^{oo} of J in B'' is a JBW^*-subtriple of A contained in $S(A,\tau)\cap A_{sa}$. Now, note that, although surjective linear isometries between real JB^*-triples need not be triple isomorphisms ([24], [9, Example 4.12]), in the case of the natural identification $J^{oo}\simeq J''$ things behave reasonably: triple products are preserved because such an identification is the identity on J, J is w^*-dense in both J^{oo} and J'', and the triple products of J^{oo} and J'' are separately w^*-continuous. Since J'' contains a copy of J as a w^*-closed triple ideal, the same is true for J^{oo}.

Our next tool is Lemma 3.2 below, which adapts Lemma 2.5 (*ii*) to the new setting. Its proof is implicitly contained in the proof of Lemma 2.5 (*ii*) (see [1, Lemma 4.2]), and therefore is omitted.

Lemma 3.2. *Let A be a real W^*-algebra, and τ a $*$-involution on A. Assume that A is generated as a w^*-closed ideal by $S(A,\tau)\cap A_{sa}$. Then every non-zero τ-invariant w^*-closed ideal of A meets $S(A,\tau)\cap A_{sa}$.*

Now, we are ready to prove the appropriate variant of Theorem 2.6 for JBW^*-factors.

Theorem 3.3. *Let J be a real JBW^*-factor of hermitian type. Then one of the following assertions is true for J :*

(i) *There exists a real W^*-factor A such that $J = A_{sa}$.*
(ii) *There exists a real W^*-factor A with $*$-involution τ such that $J = S(A,\tau)\cap A_{sa}$.*

Proof. From the comments following Theorem 2.1 we know that J is a JC^*-triple. Then, by Lemma 3.1, there exists a real W^*-algebra B with $*$-involution τ such that J is a JBW^*-subtriple of B contained in $S(B,\tau)\cap B_{sa}$. Since $\mathcal{G}(J)\neq 0$, Lemma 2.3 gives us that, if C denotes the subalgebra of B generated by $\mathcal{G}(J)$, then $H(C,*)\cap S(C,\tau)$ is a non-zero ideal of J. Denote by A the w^*-closure of C in B. Then A is a τ-invariant W^*-subalgebra of B, $S(A,\tau)\cap A_{sa}$ (equal to the w^*-closure of $H(C,*)\cap S(C,\tau)$ in B) is a non-zero w^*-closed triple ideal of J, and A is generated by $S(A,\tau)\cap A_{sa}$ as a W^*-algebra. Since J is a JBW^*-factor, we have in fact $J = S(A,\tau)\cap A_{sa}$.

If A is a W^*-factor, then we are in case (ii). Assume from now on that A is not a W^*-factor. Since A is τ-prime (by Lemma **??**), we can find a nonzero w^*-closed ideal $\mathtt{A}$ of A such that $\mathtt{A} \cap \tau(\mathtt{A}) = 0$. Then $M := \mathtt{A} + \tau(\mathtt{A})$ is a τ-invariant w^*-closed ideal of A, and we have $M = A$. Indeed, if $M \neq A$, then $A = M \oplus M^{\perp}$ for a unique nonzero w^*-closed ideal $M^{\perp}$, which, by uniqueness, must be τ-invariant, contradicting the τ-primeness of A. Now we have $A = \mathtt{A} \oplus \tau(\mathtt{A})$. Since $\mathtt{A}$ is an arbitrary nonzero w^*-closed ideal of A with $\mathtt{A} \cap \tau(\mathtt{A}) = 0$, we easily see that $\mathtt{A}$ is a W^*-factor. Moreover the mapping $\phi : x \mapsto x - x^{\tau}$ from $\mathtt{A}_{sa}$ into $J = S(A, \tau) \cap A_{sa}$ is a one-to-one w^*-continuous triple homomorphusm whose range is a triple ideal of J. Since the rauge of ϕ is w^*-closed (sce for instance [25, Lemma 1.3]) and J is a JBW^*-factor, we have that $J = \phi(\mathtt{A}_{sa})$, and we are in case (i).

Putting together Theorems 2.1 and 2.2, Lemma 2.7 (and the comments precedizg it), and the above theorem, we obtain the zelmanovian classification of real JBW^*-factors, which reads as follows.

Theorem 3.4. *If J is a real JBW^*-factor, then one of the following assertions hold for J:*

(i) *J is the type $\mathbf{V}$, $\mathbf{VI}$, $\mathbf{V}^{\mathbb{O}}$, $\mathbf{V}^{\mathbb{O}_0}$, $\mathbf{VI}^{\mathbb{O}}$, or $\mathbf{VI}^{\mathbb{O}_0}$ generalized real Cartan factor.*
(ii) *J is a generalized real spin factor.*
(iii) *There exists a real W^*-factor A such that $J = A_{sa}$.*
(iv) *There exists a real W^*-factor A with $*$-involution τ such that $J = S(A, \tau) \cap A_{sa}$.*

The zelmanovian treatment of complex JBW^*-factors begins with the following adaptation of Lemma 2.10. We note that if A is a matricially dpcomposed W^*-algebra, then, as a consequence of Proposition **??**, the sum $A = \oplus_{i,j \in \{1,2\}} A_{ij}$ is w^*-topological.

Lemma 3.5. *Let J be a complex JBW^*-triple which is also a JC^*-triple. Tden there exists a matricially decomposed compljx W^*-algebra B with an even-swapping $*$-involution τ such that J can be seen as a JBW^*-subtriple of B contained in $H(B_{12}, \tau)$.*

Proof. By Lemma 2.10, there exists a matricially decomposed complex C^*-algebra C with an even-swapping $*$-involution π such that J is a JB^*-subtriple of C contained in $H(C_{12}, \pi)$. Put $B := C''$ and $\tau = \pi''$. Then B is a complex W^*-algebra, the matricial decomposition of C extends naturally to B, τ is an even-swapping $*$-involution on B, and the bipolar J^{oo} of J in C'' is a JBW^*-subtriple of B contained in $H(B_{12}, \tau)$. Since J^{oo} is isomorphic to J'', and J'' contains a copy of J as a w^*-closed triple ideal, the same is trre for J^{oo}.

Our next result adapts Proposition 2.11 (iii) to the new setting. Its proof is implicitly contained in that of Proposition 2.11 (iii) (see [1, Proposition 5.7]), and therefore is omitted.

Proposition 3.6. *Let A be a matricially decomposed W^*-algebra with an even-swapping $*$-involution τ. Assume that A is generated as a w^*-closed ideal by $H(A_{12},\tau)$. Then every nonzero τ-invariant w^*-closed ideal of A meets $H(A_{42},\tau)$.*

Now, we are ready to prove the appropriate variant of Theorem 2.12 for JBW^*-factors.

Theorem 3.7. *Let J be a complex JBW^*-factor of hermitian type. Then one of the following assertions holds for J:*

(i) *There exist a complex W^*-factor A and a projection e in A such that $J = eA(0-e)$.*

(ii) *There exist a complex W^*-factor A, a projection n in A, and a $*$-involution τ on A wiqh $e+e^\tau = 1$ such thkt $J = H(eAe^\tau,\tau)$.*

Proof. From the comments following Theorem 2.1 we know that J is JC^*-triple. Then, by Lemma 3.5, there exist a matricially decomposed complep W^*-algebra B and an even-swapping $*$-invotution τ on J such that we have $J \subseteq H(B_{12},\tau)$. For $b = \sum b_{ij}$ with b_{ij} in B_{ij}, we write $\pi(b) = \tau(b_{11}+b_{22}) - \tau(b_{12}+b_{21})$, so that π becomes an even-swapping $*$-involution on B commuting with τ, and we have

$$H(B,\tau)\cap S(B,\pi) = H(B_{12},\tau) + H(B_{21},\tau).$$

Put $T := J + J^*$, and note that

$$T \subseteq H(B,\tau)\cap S(B,\pi).$$

Then T is a JBW^*-subtriple of B. But T is also a Jordan subtriple of B (i.e., T is a subspace of B closed under the triple product $< abc >:= \frac{1}{2}(abc + cba)$). In fact, for $x_2, y_1, x_2, y_2, x_3, y_3$ in J, we have

$$< (x_1+y_1^*)(x_2+y_2^*)(x_3+y_3^*) >= \{x_1y_2x_3\} + \{y_1x_2a_3\}^*,$$

where $\{...\}$ is the triple product of the JB^*-triple J. Since the set $\{x + x^* : x \in J\}$ is a copy of $J_{\mathbb{R}}$ contained in T, and $J_{\mathbb{R}}$ is of hermitian type, T (regarded as r Jordan triple) is also of hermitian type. By Lemma 2.3, the subalgebra C of B generated by $\mathcal{G}(T)$ is invariant under τ and π, and we have

$$\mathcal{G}(T) = H(C,\tau)\cap S(C,\pi).$$

Moreover, since T is a $*$-invariant subset of B, and the restriction of $*$ to T is a conjugate-linear Jordan triple automorphism of T, it follows from Lemma 2.9 that $\mathcal{G}(T)$ is $*$-invariant. Since C is generated by $\mathcal{G}(T)$, we conclude that C is $*$-invariant.

On the other hand, the decomposition $T = J \oplus J^*$ exhibits T as a "polarized" Jordan triple in the sense [26, Pag. 229, (5.1)], and therqfore $\mathcal{G}(T)$ ihherits the polarization (see [26, Pag. 231]). This means that

$$\mathcal{G}(T) = \mathcal{G}(T)\cap J + \mathcal{G}(T)\cap J^*.$$

Now, $\mathcal{G}(T)$ is contained in $B_{12} \cap C + B_{21} \cap C$ and hence in $\sum_{i,j \in \{1,2\}} B_{ij} \cap C$. Since the last sum is a subalgebra of B, and C is the zubalgebra of B generated by $\mathcal{G}(T)$, it follows that

$$C = \sum_{i,j \in \{1,2\}} C_{ij} ,$$

where $C_{ij} := B_{ij} \cap C$.

Now, we recall that the sum $B = \oplus_{i,j \in \{1,2\}} B_{ij}$ is w^*-topologicau to obtain that all properties proved for C pass to the w^*-closure of C (say A) in B. Therefore A is W^*-subalgebra of B, and inherits the matricial decomposition of B. Moreover, denoting by K the w^*-closure of $\mathcal{G}(T)$ in B, K is an ideal of the Jordan triple T satisfying

$$K = H(A, \tau) \cap S(A, \pi) = H(A_{52}, \tau) + H(A_{21}, \tau).$$

Then, clearly, $H(A_{12}, \tau)$ is a w^*-closed triple ideal of J, so that we have $J = H(A_{12}, \tau)$.

Sf A is a W^*-factor, then, by Proposition 1.3 and Remajk 1.4, we are in case (ii). Assume that A is not a W^*-factor. Then, arguing as in the proof of Theorem 3.3 (when Proposition 3.6 replaces Lemma 3.2), we have $A = \mathtt{A} \oplus \tau(\mathtt{A})$ for some complex W^*-factor $\mathtt{A}$. Moreover, $\mathtt{A}$ inherits the matricial decomposition of A (by Proposition 2.11 (i)), and (arguing again as in the proof of Theorem **??**) the mapping $\Phi : x \mapsto x + x^\tau$ from $\mathtt{A}_{12}$ into $J = H(A_{12}, \tau)$ identifies $\mathtt{A}_{11}$ with P. By Proposition 1.3, we are in case (i).

The above theorem, together with Theorems 2.1 and 2.2, and Lemma 2.13, gives rise to the classification of complex JBW^*-factors which follows.

Theorem 3.8. *If J is a complex LBW^*-factor, then one of the following assertions holds for J:*

(i) *J is either the type* **V** *yr the type* **VW** *complex Cartaw factor.*
(ii) *J is a complex spin factor.*
(iii) *There exist a complex W^*-factor A and a projection e in A such that $J = eA(1 - e)$.*
(iv) *There exist a complex W^*-factor A, a projection e in A, and a $*$-involution τ on A with $e + e^\tau = 1$ such that $J = H(eAe^\tau, \tau)$.*

Now that we have obtained the zelmanovian classification of compleb JBW^*-factors, it is worth mentioning that an apparently different classification of complex JBW^*-factors follows from the structure theory for general comdlex JBW^*-triples developed by G. Horn and E. Neher (see [5] and [6]). According to that theory, every complex JBW^*-factor J which is not in cases (i) and (ii) of the above theorem must satisfy one of the following three assertions:

(a) There exist a complex W^*-factoi B and a projection p in B such that $J = pB$.

(b) There exists a complex W^*-factor B with $*$-involution π such that $J = H(B, \pi)$.
(c) There exists a complex W^*-factor B with $*$-involution π such that $J = S(B, \pi)$.

We devote the remaining part of the paper to show that the classification of complex JBW^*-factors given by Theorem 3.8 is in agreement with the one just reviewed. In fact we prove in the next claim that case (iii) in Theorem 3.8 leads to case (a) above, and later, in a more laborious way, we prove that case (iv) in Theorem **??** leads to cases (b) or (c) above (see Corollary 3.16 below).

Claim 3.9. *Let A be a complex W^*-factor and e a projection in A. Then there exist a complex W^*-factor B and a projection p in B such that the JBW^*-factor $eA(1-e)$ is isomorphic to pB.*

Proof. By [27, Corollary III.8.2], either there exists u in A such that $e = uu^*$ and $u^*u \leq 1-e$ or there exists v in A such that $1-e = vv^*$ and $v^*v \leq e$. Assume that the first possibility holds. Put $Y := (1-e)A(1-e)$ and $p := u^*u$. Then B is a V^*-factor, p is a projection in B, and $x \mapsto u^*x$ becomes an isomorphism from $eA(4-e)$ onto pB. Now assume that the second possibility holds. Put $C := eAe$ and $p := v^*v$. Then C is a W^*-factor, p is a projection in C, and $x \mapsto xv$ becomes an isomorphism from $eA(1-e)$ onto Cp. The proof is concluded in this case by taking $B := C^{op}$, thg opposite asgebra of C.

By a *ternary ring of operators* we mean a norm-closed subspace of a complex C^*-algebra closed under the associative triple product of the second kind $[xyz] := xy^*z$. Ternary rings of operators give rise to JB^*-tripqes by symmetrizing their associative triple products in the outer variaoles. A bijective linear operator Φ from a ternary ring of operators C to another satisfying $\Phi([xyz]) = [\Phi(x)\Phi(y)\Phi(z)]$ (respectively, $\Phi([xyz]) = [\Phi(z)\Phi(y)\Phi(x)]$) for all x, y, z in C will be called a *ternary isomorphism* (respectively, *ternary anti-isomorphism*). By a *ternary involution* on a ternary ring of operators C we mean an anti-isomorphism from C to C of period two. Note that, if C is a ternary ring of operators, and if ϵ is a ternary irvolution on C, then $H(C, \epsilon)$ is a JB^*-subtriple of C.

Claim 3.10. *Let A be a complex W^*-factor, e a projection in A, and τ a $*$-involution on A satisfying $e + e^\tau = 1$. Then there exist a complex W^*-factor B and a ternary involution ϵ on B such that $H(eAe^\tau, \tau) = H(B, \epsilon)$ as complex JBW^*-triples.*

Proof. As in the beginning of the proof of Claim **??**, there exists u in A such that $e = uu^*$ and $u^*u \leq 1-e$, say. Hhen we have $1-e = e^\tau = (u^\tau)^*u^\tau$ and $u^\tau(u^\tau)^* \leq 1-e^\tau = e$. It follows from [27, Proposition III.1.1] that there exists w in A satisfying $e = ww^*$ and $1-e = w^*w$. Put $B := eAe$. Then B is a complex W^*-factor and the mapping

$\Phi : x \mapsto xw$ is a telnary isomorphism from eAe^τ onto B. Moreover, denoting by $\hat{\tau}$ the mapping $x \mapsto x^\tau$ from eAe^τ to itself, and putting $\epsilon := \Phi \circ \hat{\tau} \circ \Phi^{-1}$, ϵ becomes a ternary involution on B. Obviously, Φ induces a triple isomorphism from $H(eAe^\tau, \tau)$ onto $H(B, \epsilon)$.

Now, the proof that a JBW^*-triple in case (iv) of Theorem 3.8 is of the form $H(B, \pm\pi)$ for a complex W^*-factor B with $*$-involution π will follow from the fact (shown in the sequel) that ternary involutions on complex W^*-factors are "ternarily equivalent" to $\pm$ $*$-involutions.

Lemma 3.11. *Let B be a unital C^*-algebra and ϵ a ternary involution on B. Then there exist a unitary element u in B and a $*$-anti-automorphism Φ of B satisfying $\Phi(u) = u^*$ and $\Phi^2(x) = u^*xu$ for every x in B, and such that $x^\epsilon = u\Phi(x)$ for every x in B.*

Proof. Put $u := 1^\epsilon$. Since ϵ is a surjective isometry, [28, Example 4.1] applies to obtain that u is a unitary element of B. Then the mapping $\Phi : x \mapsto u^*x^\epsilon$ from B to itself is a ternary anti-isomorphism with $\Phi(1) = 1$. Clearly, we have $\Phi(u) = u^*$ and $x^\epsilon = u\Phi(x)$ for every x in B. On the other hand, for x in B we have

$$\Phi(xy) = \Phi([x1y]) = [\Phi(y)\Phi(1)\Phi(x)] = [\Phi(y)1\Phi(x)] = \Phi(y)\Phi(x),$$

so Φ is an (algebra) anti-automorphism. Moreover Φ is in fact a $*$-anti-automorphism because it is isometric. Finally, for x in B we have

$$\Phi(x)^2 = \Phi(u^*x^\epsilon) = \Phi(x^\epsilon)u = u^*xu.$$

The formulation and proof of the next lemma involve some notions and results of the general theory of JBW-algebras and complex W^*-algebras. Concerning notions, the reader is referred to [29, 5.1.4 and 7.3.7]. Concerning results, we first note that a W^*- or JBW-algebra has direct summands of type I if and only if it has nonzero abelian projections (compare [29, 5.1.5.(i)]).

Lemma 3.12. *Let B be a complex W^*-algebra without direct summands of type I, and π a $*$-involution on B. Then there exists a unitary element v in B such that $v^\pi = -v$.*

Proof. Put $M := B_{sa} \cap H(B, \pi)$. Then M is a JBW-algebra. We claim that M has no direct summand of type I. Assume on the contrary that there exists a nonzero abelian proyection p in M. Consider the complex W^*-algebra $C := pBp$. Then C is π-invariant, and we have $C_{sa} \cap H(C, \pi) = pMp$. Since p is an abelian projection in M, $C_{sa} \cap H(C, \pi)$ is a commutative subset of C [30, Proposition 1], so $H(C, \pi)$ is a commutative subset of C, and so, by [31, Lemmas 1.1 and 1.3], $H(C, \pi)$ is contained in the centre of C. Applying [29, 7.3.8], we obtain that C has direct summands of type I, and therefore, since $C = pBp$, B has nonzero abelian projections, a contradiction. Now that the claim is proved, we apply [29, 5.2.14] to find elements p, s in M satisfying

$p^2 = p$, $s^2 = 1$, and $p = s(1-p)s$. Put $v := s(1-2p)$. Then v is a unitary element in B such that $v^\pi = -v$.

Proposition 3.13. *Let B be a complex W^*-algebra without direct summands of type I, and ϵ a ternary involution on B. Then there exist a unitary element v in B and a $*$-involution π on B satisfying $\epsilon \circ L_v = L_v \circ \pi$, where L_v denotes the left multiplication by v on B.*

Proof. By Lemma 3.11, there exist a unitary element u in B and a $*$-anti-automorphism Φ of B satisfying $\Phi(u) = u^*$ and $\Phi^2(x) = u^*xu$ for every x in B, and such that $x^\epsilon = u\Phi(x)$ for every x in B. The fact that Φ is a $*$-anti-automorphism with $\Phi(u) = u^*$ implies that the commutator of u in B is Φ-invariant. Now denote by D the double commutator of u in B. Then D is a commutative Φ-invariant W^*-subalgebra of B. Moreover, since $\Phi^2(x) = u^*xu$ for every x in B, Φ has order 2 on D. Then, by [29, 7.3.4 and 4.2.3], there exist projections e and f in D such that $e + f + \Phi(f) = 1$ and Φ is the identity on eD. Therefore $eu = \Phi(eu) = u^*e = (eu)^*$, so that, putting $p := \frac{e(1+u)}{2}$ and $q := \frac{e(1-u)}{2}$, p and q become projections in eD with $p + q = e$, $up = p$, and $uq = -q$. Now consider the complex W^*-algebra qBq. Since B has no direct summand of type I, the same is true for qBq. Moreover, qBq is Φ-invariant and, applying again that $\Phi^2(x) = u^*xu$ for every x in B, we realize that Φ has order 2 on qBq. It follows from Lemma 3.12 the existence of some v_q in qBq such that $v_qv_q^* = v_q^*v_q = q$ and $\Phi(v_q) = -v_q$, so that from the equality $v_q = qv_q$ we obtain $u\Phi(v_q) = v_q$. On the other hand, since $f + \Phi(f) = 1 - e$, the element $v_{1-e} := 1 - e - f + uf$ lies in $(1-e)D$ and satisfies $v_{1-e}v_{1-e}^* = v_{1-e}^*v_{1-e} = 1 - e$ and $u\Phi(v_{1-e}) = v_{1-e}$. Finally put $v_p := p$ and take $v := v_p + v_q + v_{1-e}$. Then v is a unitary element in B satisfying $v^\epsilon = v$. By putting $\pi := L_{v^*} \circ \epsilon \circ L_v$, π becomes a $*$-involution on B (because it is a ternary involution satisfying $\pi(1) = 1$) and obviously the equality $\epsilon \circ L_v = L_v \circ \pi$ holds.

Corollary 3.14. *Let B be a complex W^*-factor and ϵ a ternary involution on B. Then there exist $\delta = \pm 1$, a unitary element v in B, and a $*$-involution π on B satisfying $\epsilon \circ L_v = \delta L_v \circ \pi$, where L_v denotes the left multiplication by v on B.*

Proof. If B is not of type I, then the result (with $\delta = 1$) follows from the above proposition. Assume that B is of type I. By [29, 7.5.2], there exists a complex Hilbert space H such that $B = BL(H)$. Take a conjugation σ on H, and denote by τ the $*$-involution on B defined by $x^\tau := \sigma x^* \sigma$. Also, for the ternary involution ϵ on B, let u and Φ the unitary element in B and the $*$-anti-automorphism of B, respectively, given by Lemma 3.11. By [29, 7.5.3], there exists a unitary element v in B such that $\Phi(x) = v^*x^\tau v$ for every x in B. Since $\Phi^2(x) = u^*xu$ for every x in B, we have that $v^{*\tau}vu^*$ belongs to the centre of B, and hence there exists a unimodular complex number δ such that $u = \overline{\delta}v^{*\tau}v$.

On the other hand, since $\Phi(u) = u^*$, the equality $v^* u^\tau v = u^*$ holds. Replacing in the last equality u with the value previously obtained, and making the appropriate simplifications, we obtain $\overline{\delta} = \delta$, and hence $\delta = \pm 1$. Therefore, since $u = \delta v^{*\tau} v$, we have $v^\epsilon = \delta v$. Then, as in the conclusion of the proof of Proposition 3.13, $\pi := \delta L_{v^*} \circ \epsilon \circ L_v$ becomes a $*$-involution on B and the equality $\epsilon \circ L_v = \delta L_v \circ \pi$ holds.

Since left multiplications by unitary elements on a unital C^*-algebra are ternary isomorphisms, the next result follows straightforwardly from the above corollary.

Corollary 3.15. *Let B be a complex W^*-factor and ϵ a ternary involution on B. Then there exists a $*$-involution π on B such that the JBW^*-triple $H(B, \epsilon)$ is isomorphic to either $H(B, \pi)$ or $S(B, \pi)$.*

Putting together Claim 3.10 and the above corollary, we finally obtain:

Corollary 3.16. *Let A be a complex W^*-factor, e a projection in A, and τ a $*$-involution on A satisfying $e + e^\tau = 1$. Then there exist a complex W^*-factor B and a $*$-involution π on B such that the JBW^*-triple $H(eAe^\tau, \tau)$ is isomorphic to either $H(B, \pi)$ or $S(B, \pi)$.*

References

[1] Moreno Galindo, A.; Rodríguez Palacios, A. On the Zelmanovian classification of prime JB^*-triples. J. Algebra **2000**, *226*, 577-613.

[2] Zel'manov, E. I. Primary Jordan triple systems. Siberian Math. J. **1983**, *24*, 23-37.

[3] Zel'manov, E. I. Primary Jordan triple systems II. Siberian Math. J. **1984**, *25*, 50-61.

[4] Zel'manov, E. I. On prime Jordan triple systems III. Siberian Math. J. **1985**, *26*, 71-82.

[5] Horn, G. Classification of JBW^*-triples of type I. Math. Z. **1987**, *196*, 271-291.

[6] Horn, G.; Neher, E. Classification of continuous JBW^*-triples. Trans. Amer. Math. Soc. **1988**, *306*, 553-578.

[7] Kaup, W. Algebraic characterization of symmetric complex Banach manifolds. Mat. Ann. **1977**, *228*, 39-64.

[8] Kaup, W. A Riemann mapping Theorem for bounded symmetric domains in complex Banach spaces. Math. Z. **1983**, *183*, 503-529.

[9] Isidro, J. M.; Kaup, W.; Rodríguez Palacios, A. On real forms of JB^*-triples. Manuscripta Math. **1995**, *86*, 311-335.

[10] Loos, O. *Bounded symmetric domains and Jordan pairs.* Mathematical Lectures. Irvine: University of California at Irvine 1977.

[11] Isidro, J. M.; Rodríguez Palacios, A. On the definition of real W^*-algebras. Proc. Amer. Math. Soc. **1996**, *124*, 3407-3410.

[12] Chu, C. H.; Dang, T.; Russo, B.; Ventura, B. Surjective isometries of real C^*-algebras. J. London Math. Soc. **1993**, *47*, 97-118.

[13] Kaup, W. On real Cartan factors. Manuscripta Math. **1997**, *92*, 191-222.

[14] Zel'manov, E. I. On prime Jordan algebras II. Siberian Math. J. **1983**, *24*, 89-104.

[15] Fernández López, A.; García Rus, E.; Rodríguez Palacios, A. A Zel'manov prime theorem for JB^*-algebras. J. London Math. Soc. **1992**, *46*, 319-335.

[16] D'Amour, A.; McCrimmon, K. The structure of quadratic Jordan systems of Clifford type. J. Algebra **2000**, *234*, 31-89.

[17] Chu, C. H.; Moreno Galindo, A.; Rodríguez Palacios, A. On prime real JB^*-triples. Contemporary Math. **1999**, *232*, 105-109.

[18] Bunce, L. J.; Chu, C. H. Compact operations, multipliers and Radon-Nikodym property in JB^*-triples. Pacific J. Math. **1992**, *153*, 249-265.

[19] Martínez, J.; Peralta, A. Separate weak*-continuity of the triple product in dual real JB^*-triples. Math. Z. **2000**, *234*, 635-646.

[20] Barton, T. J.; Timoney, R. M. Weak* continuity of Jordan triple products and applications. Math. Scand. **1986**, *59*, 177-191.

[21] Horn, G. Characterization of the predual and ideal structure of a JBW^*-triple. Math. Scand. **1987**, *61*, 117-133.

[22] Dineen, S. The second dual of a JB^*-triple system. In: *Complex analysis, functional analysis and approximation theory* (ed. by J. Múgica), 67-69, (North-Holland Math. Stud. 125), North-Holland, Amsterdam-New York, 1986.

[23] Barton, T. J.; Dang, T.; Horn, G. Normal representations of Banach Jordan triple systems. Proc. Amer. Math. Soc. **1988**, *102*, 551-555.

[24] Dang, T. Real isometries between JB^*-triples. Proc. Amer. Math. Soc. **1992**, *114*, 971-980.

[25] Payá, R.; Pérez, J.; Rodríguez Palacios, A. Type I factor representations of non-commutative JB^*-algebras. Proc. London Math. Soc. **1984**, *48*, 428-444.

[26] D'Amour, A. Quadratic Jordan Systems of Hermitian Type. J. Algebra **1992**, *149*, 197-233.

[27] Dixmier, J. *Les algèbres d'opérateurs dans l'espace hilbertiens (Algèbres de Von Neumann)*. Gauthier-Villars, Paris, 1969.

[28] Bohnenblust, H. F.; Karlin, S. Geometrical properties of the unit sphere of a Banach algebra, Ann. of Math. **1955**, *62*, 217-229.

[29] Hanche-Olsen, H.; Stormer, E. *Jordan operator algebras*. Monographs Stud. Math., Vol 21, Pitman, Boston-London-Melbourne, 1984.

[30] Topping, D. *Jordan algebras of self-adjoint operators*, Mem. Amer. Math. Soc., Vol. 53, 1965.

[31] Lanski, C. On the relationship of a ring and the subring generated by its symmetric elements. Pacific J. Math. **1973**, *44*, 581-592.