On absolute valued algebras with involution

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Abstract. Let A be an absolute valued algebra with involution, in the sense of [**9**]. We prove that A is finite-dimensional if and only if the algebra obtained by symmetrizing the product of A is simple, if and only if $eA_s = A_s$, where e denotes the unique nonzero self-adjoint idempotent of A, and A_s stands for the set of all skew elements of A. We determine the idempotents of A, and show that A is the linear hull of the set of its idempotents if and only if A is equal to either McClay's algebra [**1**], the para-quaternion algebra, or the para-octonion algebra. We also prove that, if A is infinite-dimensional, then it can be enlarged to an absolute valued algebra with involution having a nonzero idempotent different from the unique nonzero self-adjoint idempotent.

1. Introduction

In this paper we deal with absolute valued algebras with involution, as defined in Urbanik's early paper [**9**]. By a normed (respectively, absolute valued) algebra we mean a (possibly nonassociative) real or complex algebra $A \neq 0$ endowed with a norm $\|\cdot\|$ satisfying $\|xy\| \leq \|x\|\|y\|$ (respectively, $\|xy\| = \|x\|\|y\|$) for all $x, y \in A$. By an absolute valued algebra with involution we mean an absolute valued algebra A over the field $\mathbb{R}$ of real numbers, endowed with a mapping $x \to x^*$ from A to A (called the "involution" of A) satisfying:

(i) $(\alpha\, x + \beta\, y)^* = \alpha x^* + \beta y^*$
(ii) $x^{**} = x$
(iii) $xx^* = x^*x$
(iv) $(xy)^* = y^*x^*$
(v) $\|x^*\| = \|x\|$

for all $x, y \in A$ and $\alpha, \beta \in \mathbb{R}$.

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Given an absolute valued algebra with involution A, we denote by A_a the set of all self-adjoint elements of A, i.e.,

$$A_a := \{x \in A : x^* = x\},$$

and by A_s the set of all skew elements of A, i.e.,

$$A_s := \{x \in A : x^* = -x\}.$$

Obviously, we have $A = A_a \oplus A_s$, as a direct sum of subspaces. We will assume that the involution of A is non trivial, i.e.,

(vi) $A_s \neq 0$.

Examples of absolute valued algebras with involution are $\mathbb{C}$ (the field of complex numbers), $\mathbb{H}$ (the algebra of Hamilton's quaternions), and $\mathbb{D}$ (the algebra of Cayley numbers), endowed with their standard involutions. For $\mathbb{A}$ equal to either $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{D}$, let us denote by $\overset{*}{\mathbb{A}}$ the absolute-valued real algebra obtained by endowing the normed space of $\mathbb{A}$ with the product $x \odot y := x^* y^*$, where $*$ means the standard involution. Since $*$ remains an involution on $\overset{*}{\mathbb{A}}$, we are provided with new examples of absolute valued algebras with involution. The reader is referred to [**9**] for examples of infinite-dimensional absolute valued algebras with involution, to [**6**] for a classification of finite-dimensional absolute valued algebras with involution, and to the survey paper [**8**] for a general view of the theory of absolute valued algebras.

Let A be an absolute valued algebra with involution. We prove that A is finite-dimensional if and only if the algebra obtained by replacing the product of A with the one $\circ$ defined by $x \circ y := \frac{xy+yx}{2}$ is simple (Theorem 2.2). Theorem 2.2 also asserts that A is finite-dimensional if and only if $eA_s = A_s$, where e is the unique nonzero self-adjoint idempotent of A (the existence of which was proved in [**9**]). We determine the idempotents of A (Proposition 2.3), and show that A is the linear hull of the set of its idempotents if and only if A is equal to $\overset{*}{\mathbb{C}}$, $\overset{*}{\mathbb{H}}$, or $\overset{*}{\mathbb{D}}$ with the standard involution (Theorem 2.5). Finally, we prove that, if A is infinite-dimensional, then it can be enlarged to an absolute valued algebra with involution having a nonzero idempotent different from the unique nonzero self-adjoint idempotent (Theorem 2.7).

As a discussion of the results just reviewed, we provide the reader with examples of simple infinite-dimensional absolute valued algebras with involution (Example 3.4), and of infinite-dimensional absolute valued algebras with involution having no nonzero idempotent different from their unique nonzero self-adjoint idempotent (Example 3.8).

2. The results

Throughout this section, A will denote an absolute valued algebra with a non trivial involution $*$. According to [**9**], there exists a distinguished element $e \in A$ satisfying $xx^* = \|x\|^2 e$ for every $x \in A$, the absolute value of A derives from an inner product (which will be denoted by $< \cdot, \cdot >$), A_a is orthogonal to A_s with respect to $< \cdot, \cdot >$, and elements of A_a commute with those of A_s. Clearly, the element e above is the unique nonzero self-adjoint idempotent of A. We put $B := \mathbb{R}e \oplus A_s$, and we note that B is a subalgebra of A (see [**3**]) and that, clearly, the idempotent e is central in B (in the sense that it commutes with every element of B).

LEMMA 2.1. *x^2 belongs to B whenever x is an arbitrary element of A. Therefore B contains all the idempotents of A.*

PROOF. Write $x = y + z$ with $y \in A_a$ and $z \in A_s$. Since $A_a A_s \subseteq A_s$ (because elements of A_a commute with those of A_s), we have

$$x^2 = (\|y\|^2 - \|z\|^2)e + 2yz \in \mathbb{R}e + A_s = B.$$

■

Let E be an arbitrary algebra. We denote by E^2 the linear hull of the set $\{xy : x, y \in E\}$, and note that E^2 is an ideal of E. We say that E is simple if $E^2 \neq 0$ and every nonzero ideal of E is equal to E. By E^+ we mean the algebra consisting of the vector space of E and the product $\circ$ defined by

$$x \circ y := \frac{xy + yx}{2}.$$

It follows from the equalities $x \circ x = x^2$ and

$$x \circ y = \frac{(x+y)^2 - (x-y)^2}{4} \qquad (2.1)$$

that $(E^+)^2$ coincides with the linear hull of the set $\{x^2 : x \in E\}$. Now assume that the algebra E is normed. We say that E is topologically simple if $E^2 \neq 0$ and every nonzero ideal of E is dense in E, and we note that E^+ becomes naturally a normed algebra under the norm of E.

THEOREM 2.2. *The following conditions are equivalent:*

(1) *A^+ is simple.*
(2) *$(A^+)^2 = A$ (as sets).*
(3) *$A_s = eA_s$.*
(4) *A^+ is topologically simple.*
(5) *$(A^+)^2$ is dense in A.*
(6) *eA_s is dense in A_s.*
(7) *A is finite dimensional.*

PROOF. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (5), (1) $\Rightarrow$ (4) $\Rightarrow$ (5), and (3) $\Rightarrow$ (6) are clear.

(5) $\Rightarrow$ (6).- For $\alpha, \beta \in \mathbb{R}$ and $x, y \in A_s$ we have the equality

$$(\alpha e + x) \circ (\beta e + y) = \alpha\beta e + x \circ y + e(\alpha y + \beta x),$$

which with the help of (2.1) gives

$$(\alpha e + x) \circ (\beta e + y) = (\alpha\beta - < x, y >)e + e(\alpha y + \beta x). \qquad (2.2)$$

On the other hand, since B is closed in A, it follows from the assumption (5) and lemma 2.1 that $A = B$. Let x be in A_s, and let $\varepsilon > 0$. Since

$A = B$, the assumption (5) gives that there exist $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n \in \mathbb{R}$ and $x_1, ...x_n, y_1, ..., y_n \in A_s$ such that $\| \sum_{i=1}^n (\alpha_i e + x_i) \circ (\beta_i e + y_i) - x\| < \varepsilon$. Put $z := e\sum_{i=1}^n (\alpha_i y_i + \beta_i x_i)$. Then z belongs to eA_s, and, applying (2.2), we have

$$\|z - x\| \leq \sqrt{[\sum_{i=1}^n (\alpha_i\beta_i - < x_i, y_i >)]^2 + \|z - x\|^2}$$

$$= \|[\sum_{i=1}^n (\alpha_i\beta_i - < x_i, y_i >)]e + z - x\|$$

$$= \| \sum_{i=1}^n [(\alpha_i\beta_i - < x_i, y_i >)e + e(\alpha_i y_i + \beta_i x_i)] - x\|$$

$$= \| \sum_{i=1}^n (\alpha_i e + x_i) \circ (\beta_i e + y_i) - x\| < \varepsilon.$$

(6) ⇒ (7).- Since $B = \mathbb{R}e + A_s$ and $eB = \mathbb{R}e + eA_s$, the assumption (6) gives that eB is dense in B. But, since e is central in B, we have also that Be is dense in B. Therefore, by Proposition 1.2 of [**5**], B is finite-dimensional. But then, by Lemma 3.2 of [**3**], we have $A = B$.

(7) ⇒ (1).- Since $e \circ e = e$, we have $(A^+)^2 \neq 0$. Let M be a nonzero ideal of A^+. Taking a norm-one element $y \in M$, we have $e = y^* \circ y \in M$. On the other hand, as consequences of the assumption (7) we have that the mapping $L_e : x \to ex$ from A to A is surjective, and that $A = B$ (by Lemma 3.2 of [**3**]). Since this last fact implies that e is central in A, it follows $A = L_e(A) = eA = A \circ e \subseteq M$.

(7) ⇒ (3).- As above, the assumption (7) implies that the operator L_e is surjective and that $A = B$. Then, since L_e is diagonal relative to the decomposition $A = \mathbb{R}e \oplus A_s$, it follows $eA_s = L_e(A_s) = A_s$. ∎

As we will see in Example 3.4 below, the simplicity of A is not enough to assure that A is finite-dimensional. Anyway, if A is simple (or merely topologically simple), then we have $A = B$. This is so because, in any case, the closed subalgebra B contains A^2 [**2**], and hence is an ideal of A.

From now on, we denote by $I(A)$ the set of all nonzero idempotents of A, and by $A_s(-1)$ the subspace of A_s defined by

$$A_s(-1) := \{x \in A_s : ex = xe = -x\}.$$

Proposition 2.3.

$$I(A) = \{e\} \cup \{\frac{-e + \sqrt{3}z}{2} : z \in A_s(-1), \|z\| = 1\}.$$

Therefore, $I(A)$ reduces to $\{e\}$ if and only if the space $A_s(-1)$ is equal to zero.

PROOF. The inclusion

$$I(A) \supset \{e\} \cup \{\frac{-e+\sqrt{3}z}{2} : z \in A_s(-1), \|z\| = 1\}$$

is of straightforward verification. To see the converse inclusion, let p be in $I(A)$. By Lemma 2.1 we have $p = \alpha e + x$ for suitable $\alpha \in \mathbb{R}$ and $x \in A_s$. Since

$$\alpha e + x = (\alpha e + x)^2 = (\alpha^2 - \|x\|^2)e + 2\alpha e x,$$

we deduce

$$\alpha^2 - \|x\|^2 = \alpha \tag{2.3}$$

and

$$2\alpha e x = x. \tag{2.4}$$

On the other hand, we have

$$1 = \|p\|^2 = \alpha^2 + \|x\|^2. \tag{2.5}$$

From (2.3) and (2.5) we obtain $2\alpha^2 - \alpha - 1 = 0$, i.e., $\alpha = 1$ or $-\frac{1}{2}$. If $\alpha = 1$, then $x = 0$ (by (2.5)), and hence $p = e$. If $\alpha = -\frac{1}{2}$, then $x \in A_s(-1)$ (by (2.4)) and $\|x\| = \frac{\sqrt{3}}{2}$ (by (2.5)), and therefore $p = \frac{-e+\sqrt{3}z}{2}$ with $z := \frac{2}{\sqrt{3}}x \in A_s(-1)$ and $\|z\| = 1$. ■

The following corollary follows straightforwardly from Proposition 2.3.

COROLLARY 2.4. *Let p be in $I(A) \setminus \{e\}$. Then the linear hull of $\{e, p\}$ is a $*$-invariant subalgebra of A isomorphic to $\overset{*}{\mathbb{C}}$ with the standard involution.*

THEOREM 2.5. *The following conditions are equivalent:*

(1) *The linear hull of $I(A)$ is equal to A.*
(2) *The linear hull of $I(A)$ is dense in A.*
(3) *A is equal to $\overset{*}{\mathbb{C}}$, $\overset{*}{\mathbb{H}}$, or $\overset{*}{\mathbb{D}}$ with the standard involution.*

PROOF. (1) $\Rightarrow$ (2).- This is clear.

(2) $\Rightarrow$ (3).- It follows from Proposition 2.3 that $ep = pe = p^*$ for every $p \in I(A)$. Therefore the set

$$\{x \in A : ex = xe = x^*\}$$

is a closed subspace of A containing $I(A)$. Then, from the assumption (2) we derive that $ex = xe = x^*$ for every $x \in A$. In this way, e becomes a unit for the absolute valued algebra (say E) obtained by replacing the product of A with the one $\odot$ defined by $x \odot y = x^* y^*$. By Theorem 1 of [**10**], E must be equal to $\mathbb{A}$, where $\mathbb{A}$ stands for $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{D}$. Now note that, since the involution $*$ of A is non trivial, the case $E = \mathbb{R}$ cannot really happen, and that, since A is the orthogonal sum of $\mathbb{R}e$ and A_s (by the assumption (2) and Lemma 2.1), $*$ becomes the standard involution on $\mathbb{A}$. It follows that $A = \overset{*}{\mathbb{A}}$, where now $\mathbb{A}$ stands for $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{D}$.

(3) $\Rightarrow$ (1).- The assumption (3) implies that $A = \mathbb{R}e \oplus A_s$ and that $A_s(-1) = A_s$. These facts, together with Proposition 2.3 lead easily to (1). ■

Remark 2.6. If in Theorem 2.5 we avoid the environmental requirement that the involution $*$ of A is non trivial, then, in assertion (3) of that theorem, two new algebras must be added, namely $\mathbb{R}$ and $\overset{*}{\mathbb{C}}$, both endowed with the identity operator as involution. Indeed, when $*$ is trivial, A is commutative and Theorem 3 of [**10**] applies.

The proof of our next result will involve some elementary facts of the theory of normed ultrapowers [**4**], a summary of which is provided in the sequel. Let I be a non-empty set, let $\mathcal{U}$ be an ultrafilter on I, and let X be a normed space. We may consider the vector space $\ell_\infty(I, X)$ of all bounded functions $i \to x_i$ from I to X endowed with the norm

$$\|\{x_i\}\| := \sup\{\|x_i\| : i \in I\},$$

and the closed subspace $N_{\mathcal{U}}$ of $\ell_\infty(I, X)$ given by

$$N_{\mathcal{U}} := \{\{x_i\} \in \ell_\infty(I, X) : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The normed ultrapower of X relative to the ultrafilter $\mathcal{U}$ is defined as the quotient normed space $\ell_\infty(I, X)/N_{\mathcal{U}}$, and is denoted by X_U. If we denote by (x_i) the element in X_U containing a given element $\{x_i\} \in \ell_\infty(I, X)$, then it is easy to verify that

$$\|(x_i)\| = \lim_{\mathcal{U}} \|x_i\|. \tag{2.6}$$

The normed space X will be canonically regarded as a subspace of $X_{\mathcal{U}}$ through the isometric linear embedding $x \to \{x_i\}$, where $x_i = x$ for every $i \in I$. If X is in fact a normed algebra, then the normed space $X_{\mathcal{U}}$ becomes naturally a new normed algebra under the (well-defined) product

$$(x_i)(y_i) := (x_i y_i),$$

so that $X_{\mathcal{U}}$ contains X as a subalgebra. It follows from (2.6) that, if X is an absolute valued algebra, then so is $X_{\mathcal{U}}$. Moreover, if the absolute valued algebra X has an involution $*$, then the (well-defined) mapping

$$\{x_i\} \to \{x_i\}^* := \{x_i^*\}$$

becomes an involution on $X_{\mathcal{U}}$ extending that of X.

Theorem 2.7. *Assume that A is infinite-dimensional. Then there exists an absolute valued algebra with involution, containing A as a $*$-invariant subalgebra, and having a nonzero idempotent different from e.*

Proof. The assumption that A is infinite-dimensional, together with Theorem 2.2, gives that the range of the linear isometry $x \to -ex$ from A_s to A_s is not dense in A_s. Therefore, by Lemma 4.1 of [**5**], there exists a sequence $\{x_n\}$ of norm-one elements of A_s such that $\{ex_n + x_n\} \to 0$. Now take an ultrafilter $\mathcal{U}$ on the set $\mathbb{N}$ of all natural numbers, containing the filter of all cofinite subsets of $\mathbb{N}$, consider the absolute valued algebra with involution $A_{\mathcal{U}}$, and put $z := (x_n) \in (A_{\mathcal{U}})_s$. Then, since $\lim_{\mathcal{U}} \|ex_n + x_n\| = 0$, we have $ez = -z$, i.e., z belongs to $(A_{\mathcal{U}})_s(-1)$. Since $\|z\| = 1$ (by 2.6), it follows from Proposition 2.3 that $\frac{-e+\sqrt{3}z}{2}$ is a nonzero idempotent of $A_{\mathcal{U}}$ different from e. ■

3. Discussing the results

Examples 3.1 and 3.4 below are related to Theorem 2.2.

EXAMPLE 3.1. *There exists a finite-dimensional absolute valued real algebra A such that $(A^+)^2 \neq A$. Thus, such an algebra A fulfills condition* (7) *in theorem 2.2, but fails to conditions* (1), (2), (4), *and* (5) *in that theorem.* Indeed, take $A = \mathbb{A}^*$, where $\mathbb{A}$ stands for either $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{D}$, and $\mathbb{A}^*$ denotes the absolute-valued real algebra obtained by endowing the normed space of $\mathbb{A}$ with the product $x \odot y := xy^*$ (of course, $*$ means the standard involution of $\mathbb{A}$). In this case we have $(A^+)^2 = \mathbb{R}e$, where e is the unit of $\mathbb{A}$.

For the next example, we need the following proposition and lemma.

PROPOSITION 3.2 ([**9**]). *Let U be an infinite set, let T be a nonempty subset of U such that $\#(U \setminus T) = \#U$ (where $\#$ means cardinal number), let ϕ be an injective function from the family of all binary subsets of U to U whose range does not intersect T, and let*

$$\psi : (U \times U) \setminus \{(u,u) : u \in U\} \to \{1, -1\}$$

be a function satisfying $\psi(u,v) + \psi(v,u) = 0$ whenever

$$(u,v) \in (T \times T) \cup ((U \setminus T) \times (U \setminus T)),$$

and $\psi(u,v) = 1$ otherwise. For $u \in U$, put $\varepsilon(u) := \pm 1$ depending on whether or not u belongs to T, and fix $u_0 \in T$. Then the real Hilbert space with orthonormal basis $\{x_u\}_{u \in U}$ (endowed with a suitable product and a suitable involution) becomes an absolute valued algebra with involution satisfying for $u, v \in U$ the following relations:

(1) $x_u x_v = \psi(u,v) x_{\phi(\{u,v\})}$ *if* $u \neq v$.
(2) $x_u^2 = \varepsilon(u) x_{u_0}$.
(3) $x_u^* = \varepsilon(u) x_u$.

LEMMA 3.3. *Let E be an absolute valued algebra with involution containing a dense $*$-invariant simple subalgebra F. Then E is topologically simple.*

PROOF. Let e denote the unique nonzero self-adjoint idempotent of E. Let M be a nonzero ideal of E. Taking a norm-one element $x \in F$ (respectively, $y \in M$), we have $e = xx^* \in F$ (respectively, $e = yy^* \in M$). Therefore $e \in F \cap M$, and hence $F \cap M \neq 0$. Since $F \cap M$ is an ideal of F, and F is simple, it follows $F \cap M = F$, so $F \subseteq M$, and so M is dense in E. ■

Example 3.4. *There exists a simple infinite-dimensional non complete absolute valued algebra with involution, and a topologically simple infinite-dimensional complete absolute valued algebra with involution.*

Proof. Let S denote the family of all binary subsets of $\mathbb{N}$, and consider the enumeration of S given by

$$\{1,2\},\{1,3\},\{2,3\},\{1,4\},\{2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\},...$$

Such an enumeration provides us with a bijective mapping $\phi : S \to \mathbb{N} \setminus \{1\}$ defined by

$$\phi(\{1,2\}) = 2, \phi(\{1,3\}) = 3, \phi(\{2,3\}) = 4, \phi(\{1,4\}) = 5, \phi(\{2,4\}) = 6,$$

$$\phi(\{3,4\}) = 7, \phi(\{1,5\}) = 8, \phi(\{2,5\}) = 9, \phi(\{3,5\}) = 10, \phi(\{4,5\}) = 11,$$

and so on. Thus, we realize that, for $k, l \in \mathbb{N}$ with $k < l$, we have

$$k < \phi(\{k,l\}). \tag{3.1}$$

Now, let ψ be the mapping from $(\mathbb{N} \times \mathbb{N}) \setminus \{(n,n) : n \in \mathbb{N}\}$ to $\{1,-1\}$ defined by $\psi(n,m) = \pm 1$ depending on whether or not $n < m$, and let ε be the mapping from $\mathbb{N}$ to $\{1,-1\}$ defined by $\varepsilon(n) = \pm 1$ depending on whether or not $n = 1$. According to Proposition 3.2, the real Hilbert space H with orthonormal basis $\{x_n\}_{n\in\mathbb{N}}$ becomes an absolute valued algebra with involution satisfying

$$x_n x_m = \psi(n,m) x_{\phi(\{n,m\})} \tag{3.2}$$

$$x_n^2 = \varepsilon(n) x_1 \tag{3.3}$$

$$x_n^* = \varepsilon(n) x_n \tag{3.4}$$

whenever n and m are in $\mathbb{N}$ with $n \neq m$. Let A be the linear hull in H of the set $\{x_n : n \in \mathbb{N}\}$. By (3.2), (3.3), and (3.4), A is a $*$-invariant subalgebra of H, and hence an absolute valued algebra with involution. We are going to show that A is simple. Let M be a nonzero ideal of A. Since x_1 is the unique nonzero self-adjoint idempotent of A (by (3.3) and (3.4)), taking a norm-one element $y \in M$ we have

$$x_1 = yy^* \in M. \tag{3.5}$$

Now, arguing by induction on n, we prove that $x_n \in M$ for any $n \in \mathbb{N}$. This is true for $n = 1$ by (3.5), and if $n > 1$ and $x_k \in M$ for any $k < n$, then $n = \Phi(\{k,l\})$ for some $k < l$, and also $k < n$ by (3.1). Hence (3.2) shows that $x_n = x_k x_l \in M$. Therefore $M = A$ (since A is the linear hull of $\{x_n : n \in \mathbb{N}\}$). Since M is an arbitrary nonzero ideal of A, the proof of the simplicity of A is concluded. Applying Lemma 3.3, we obtain that H is a topologically simple absolute valued algebra with involution. ∎

Remark 3.5. The infinite-dimensional non complete absolute valued algebra A with involution, shown in Example 3.4, is "more than simple". Indeed, looking at the above proof, we realize that every nonzero right ideal of A is equal to A. Since the involution of A is an anti-automorphism, we have in addition that every nonzero left ideal of A is equal to A. Analogously, the infinite-dimensional complete absolute valued algebra H with involution, shown also in Example 3.4, is "more than topologically simple". Indeed, every nonzero one-sided ideal of H is dense in H.

After Theorem 2.2 and Example 3.4, the following problems arise in a natural way.

Problem 3.6. Is there a simple infinite-dimensional complete absolute valued algebra with involution?

Problem 3.7. Is there an infinite-dimensional absolute valued real algebra A such that $(A^+)^2$ is simple?

In relation to the above problems, it is worth mentioning that the existence of simple infinite-dimensional complete absolute valued real algebras is well-known. Indeed, every infinite-dimensional real Hilbert space becomes a left-division absolute-valued algebra under a suitable product [**7**]. We recall that an algebra A is said to be a left- (respectively, right-) division algebra if, for every nonzero element $x \in A$, the operator of left (respectively, right) multiplication by x is bijective, and we note that one-sided division algebras are simple. The infinite-dimensional examples of [**7**] quoted above cannot solve Problem 3.6 by the affirmative. For, if a left-division absolute valued real algebra A has an involution, then, since involutions are anti-automorphisms, A is also a right division algebra, and hence finite-dimensional (by Wrigth's celebrated theorem [**11**]). We do not know if, among the examples of [**7**], we can find one answering affirmatively Problem 3.7.

We conclude the paper with an example related to Theorem 2.7.

Example 3.8. *There exists an infinite-dimensional complete absolute valued algebra A with involution, such that there is no nonzero idempotent in A different from the unique nonzero self-adjoint idempotent of A.*

Proof. Let S denote the family of all binary subsets of $\mathbb{N}$, let $\phi : S \to \mathbb{N} \setminus \{1\}$ be any injective mapping satisfying $\phi(\{1, n\}) = 2n - 1$ for every $n \in \mathbb{N} \setminus \{1\}$, let ψ be the mapping from $(\mathbb{N} \times \mathbb{N}) \setminus \{(n, n) : n \in \mathbb{N}\}$ to $\{1, -1\}$ defined by $\psi(n, m) = \pm 1$ depending on whether or not $n < m$, and let ε be the mapping from $\mathbb{N}$ to $\{1, -1\}$ defined by $\varepsilon(n) = \pm 1$ depending on whether or not $n = 1$. Apply Proposition 3.2 to convert the real Hilbert space A with orthonormal basis $\{x_n\}_{n \in \mathbb{N}}$ into an absolute valued algebra with involution satisfying $x_n x_m = \psi(n, m) x_{\phi(\{n,m\})}$, $x_n^2 = \varepsilon(n) x_1$, and $x_n^* = \varepsilon(n) x_n$ whenever n and m are in $\mathbb{N}$ with $n \neq m$. Then one easily realizes that x_1 is the unique nonzero self-adjoint idempotent of A, that A_s

is the closed linear hull of $\{x_n : n \in \mathbb{N} \setminus \{1\}\}$, and then that $A_s(-1) = 0$. Applying Proposition 2.3, we deduce that there is no nonzero idempotent in A different from x_1. ∎

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