Numerical ranges of uniformly continuous functions on the unit sphere of a Banach space

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Dedicated to Professor John Horváth on the occasion of his 80th birthday

1. Introduction

Let X be a Banach space over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). We denote by S_X, B_X, and X^* the unit sphere, the closed unit ball, and the dual space of X, respectively. For u in S_X, we denote by $D(X, u)$ the set of all states of X relative to u, namely

$$D(X, u) := \{\phi \in S_{X^*} : \phi(u) = 1\},$$

and then, for x in X, we define the *numerical range* $V(X, u, x)$ of x relative to u as the nonempty, convex, and compact subset of $\mathbb{K}$ given by the equality

$$V(X, u, x) := \{\phi(x) : \phi \in D(X, u)\}.$$

Given a mapping f from S_X into X, we can consider the so-called *spatial numerical range* $W(f)$ of f, namely

$$W(f) := \bigcup \{V(X, x, f(x)) : x \in S_X\}$$

or, equivalently,

$$W(f) := \{\phi(f(x)) : (x, \phi) \in \Pi(X)\},$$

where $\Pi(X)$ stands for the set of those couples $(x, \phi) \in S_X \times S_{X^*}$ satisfying $\phi(x) = 1$. If the mapping f above is bounded, then it also has an *intrinsic numerical range* $V(f)$, given by the equality

$$V(f) := V(\ell_\infty(S_X, X), \mathbf{1}, f).$$

(Here, for any set E, $\ell_\infty(E, X)$ denotes the Banach space of all bounded functions from E to X, and $\mathbf{1}$ stands for the natural embedding $S_X \hookrightarrow X$.)

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We note that, if f belongs to some subspace Y of $\ell_\infty(S_X, X)$ with $\mathbf{1} \in Y$, then we have

$$V(f) = V(Y, \mathbf{1}, f).$$

We also note that, for every $f \in \ell_\infty(S_X, X)$, the inclusion

$$\overline{co}W(f) \subseteq V(f)$$

holds, where $\overline{co}$ means closed convex hull. (Indeed, for $(x, \phi) \in D(X, x)$, the mapping $g \to \phi(g(x))$ from $\ell_\infty(S_X, X)$ to $\mathbb{K}$ is an element of $D(\ell_\infty(S_X, X), \mathbf{1})$.) The inclusion above is known to be an equality whenever f is (the restriction to S_X of) a continuous linear operator on X [**6**], or $\mathbb{K} = \mathbb{C}$ and f is (the restriction to S_X of) a uniformly continuous function on B_X which is holomorphic on the interior of B_X [**4**]. More generally, the equality $\overline{co}W(f) = V(f)$ is true if the bounded function $f : S_X \to X$ is uniformly continuous [**5**]. On the other hand, the equality $\overline{co}W(f) = V(f)$ for arbitrary $f \in \ell_\infty(S_X, X)$ cannot be expected in general. Indeed, such an equality holds for every $f \in \ell_\infty(S_X, X)$ if and only if X is uniformly smooth [**8**].

If f is a continuous linear operator on X, or $\mathbb{K} = \mathbb{C}$ and f is a uniformly continuous function from B_X to X which is holomorphic on the interior of B_X, then we have in fact

$$V(f) = \overline{co}\{\phi(f(x)) : (x, \phi) \in \Gamma\}, \tag{1.1}$$

where Γ is any subset of $\Pi(X)$ such that the natural projection $\pi_X(\Gamma)$ is equal to S_X (see again [**6**] and [**4**]). Even, in the case of a linear operator, the requirement that $\pi_X(\Gamma) = S_X$ can be relaxed to the one that $\pi_X(\Gamma)$ is dense in S_X [**2**, Theorem 9.3]. The aim of this note is to prove that the equality (1.1) actually holds for every bounded and uniformly continuous function $f : S_X \to X$, and every subset Γ of $\Pi(X)$ such that $\pi_X(\Gamma)$ is dense in S_X. This refines the result of [**5**] quoted above.

2. The results

Throughout this section, X will denote a Banach space over $\mathbb{K}$, and Γ will stand for a subset of $\Pi(X)$ such that $\pi_X(\Gamma)$ is dense in S_X. The following lemma is a reformulation of [**7**, Lemma 5.1].

Lemma 2.1. *Let u be in S_X, Then, for every y in X we have*

$$V(X, u, y) = \bigcap_{\delta > 0} \overline{co}\{\phi(y) : (x, \phi) \in \Gamma, \|x - u\| < \delta\}.$$

Lemma 2.2. *For every g in $\ell_\infty(\pi_X(\Gamma), X)$ we have*

$$V(\ell_\infty(\pi_X(\Gamma), X), \mathbf{1}_\Gamma, g) = \bigcap_{\delta > 0} \overline{co}\{\phi(g(y)) : (x, \phi) \in \Gamma, y \in \pi_X(\Gamma), \|x - y\| < \delta\},$$

where $\mathbf{1}_\Gamma$ stands for the natural embedding $\pi_X(\Gamma) \hookrightarrow X$.

Proof. Put $Y := \ell_\infty(\pi_X(\Gamma), X)$ and, for $(y, \phi) \in \pi_X(\Gamma) \times X^*$, denote by $y \otimes \phi$ the element of Y^* defined by $(y \otimes \phi)(h) := \phi(h(y))$ for every $h \in Y$. Now, consider the set $\widetilde{\Gamma}$ of all couples $(h, \psi) \in S_Y \times Y^*$ such that there exists $(y, \phi) \in \pi_X(\Gamma) \times X^*$ satisfying that $(h(y), \phi)$ belongs to Γ and that $\psi = y \otimes \phi$. Clearly $\widetilde{\Gamma}$ is a subset of $\Pi(Y)$. We claim that $\pi_Y(\widetilde{\Gamma})$ is dense in S_Y. Let f be in S_Y and $0 < \varepsilon < 1$. There exists $y \in \pi_X(\Gamma)$ with $\|f(y)\| > 1 - \varepsilon$, and, by the density of $\pi_X(\Gamma)$ in S_X, there exists $(x, \phi) \in \Gamma$ with $\|x - \frac{f(y)}{\|f(y)\|}\| < \varepsilon$. Consider the element h of Y defined by $h(y) = x$ and $h(z) = f(z)$ whenever z belongs to $\pi_X(\Gamma) \setminus \{y\}$. Then $(h, y \otimes \phi)$ belongs to $\widetilde{\Gamma}$ and $\|h - f\| = \|x - f(y)\| < \frac{\varepsilon(2-\varepsilon)}{1-\varepsilon}$. Now that the claim has been proved, we fix $g \in Y$, and apply Lemma 2.1 to obtain $V(Y, \mathbf{1}_\Gamma, g) = \bigcap_{\delta>0} \overline{co}(A_\delta)$, where

$$A_\delta := \{\psi(g) : (h, \psi) \in \widetilde{\Gamma}, \|h - \mathbf{1}_\Gamma\| < \delta\}.$$

Thus, to conclude the proof it is enough to show that

$$A_\delta = B_\delta := \{\phi(g(y)) : (x, \phi) \in \Gamma, y \in \pi_X(\Gamma), \|x - y\| < \delta\}.$$

Let λ be in A_δ. Then there exists $(h, \psi) \in \widetilde{\Gamma}$ such that $\|h - \mathbf{1}_\Gamma\| < \delta$ and $\psi(g) = \lambda$. By the definition of $\widetilde{\Gamma}$, there exists $(y, \phi) \in \pi_X(\Gamma) \times X^*$ such that $(h(y), \phi) \in \Gamma$ and $\psi = y \otimes \phi$. It follows that, putting $x := h(y)$, we have $\lambda = \phi(g(y))$ with $(x, \phi) \in \Gamma$, $y \in \pi_X(\Gamma)$, and $\|x - y\| < \delta$, i.e., λ belongs to B_δ. Conversely, assume that λ is in B_δ. Then there exist $(x, \phi) \in \Gamma$ and $y \in \pi_X(\Gamma)$ such that $\|x - y\| < \delta$ and $\phi(g(y)) = \lambda$. Considering the element h of Y defined by $h(y) = x$ and $h(z) = z$ whenever z belongs to $\pi_X(\Gamma) \setminus \{y\}$, and putting $\psi := y \otimes \phi$, we have $\lambda = \psi(g)$ with $(h, \psi) \in \widetilde{\Gamma}$ and $\|h - \mathbf{1}_\Gamma\| < \delta$, i.e., λ belongs to A_δ. ■

The choice $\Gamma = \Pi(X)$ in Lemma 2.2 yields the following.

Corollary 2.3. *Let f be a bounded function from S_X to X. Then we have*

$$V(f) = \bigcap_{\delta>0} \overline{co}\{\phi(f(y)) : (x, \phi) \in \Pi(X), y \in S_X, \|x - y\| < \delta\}.$$

Corollary 2.4. *Let f be a bounded continuous function from S_X to X. Then we have*

$$V(f) = \bigcap_{\delta>0} \overline{co}\{\phi(f(y)) : (x, \phi) \in \Gamma, y \in \pi_X(\Gamma), \|x - y\| < \delta\}$$
$$= \bigcap_{\delta>0} \overline{co}\{\phi(f(y)) : (x, \phi) \in \Gamma, y \in S_X, \|x - y\| < \delta\}.$$

Proof. Denoting by Z the subspace of $\ell_\infty(S_X, X)$ consisting of all bounded continuous functions from S_X to X, and putting $Y := \ell_\infty(\pi_X(\Gamma), X)$, the mapping $f \to f_{|\pi_X(\Gamma)}$ from Z to Y becomes a linear isometry sending $\mathbf{1}$ to $\mathbf{1}_\Gamma$. Therefore, for every $f \in Z$ we have

$$V(f) = V(Z, \mathbf{1}, f) = V(Y, \mathbf{1}_\Gamma, f_{|\pi_X(\Gamma)}).$$

Now, the first equality in the present corollary follows from Lemma 2.2. The second equality is a consequence of the former, obvious inclusions, and Corollary 2.3. ∎

Now, we are ready to prove the main result in this note.

THEOREM 2.5. *For every bounded and uniformly continuous function $f : S_X \to X$, we have*

$$V(f) = \overline{co}\{\phi(f(x)) : (x, \phi) \in \Gamma\}.$$

PROOF. Let $\varepsilon > 0$. Take $\delta > 0$ such that $\|f(x)-f(y)\| < \varepsilon$ whenever x, y are in S_X and $\|x - y\| < \delta$. Then we have $\Re e(\phi(f(y))) < \Re e(\phi(f(x)))) + \varepsilon$ whenever (x, ϕ) is in Γ, y is in S_X, and $\|x - y\| < \delta$. Since

$$V(f) \subseteq \overline{co}\{\phi(f(y)) : (x, \phi) \in \Gamma, y \in S_X, \|x - y\| < \delta\}$$

(by Corollary 2.4), we deduce

$$\max \Re e(V(f)) \leq \max \Re e(C) + \varepsilon,$$

where

$$C := \overline{co}\{\phi(f(x)) : (x, \phi) \in \Gamma\}.$$

Now, the arbitraryness of ε yields

$$\max \Re e(V(f)) \leq \max \Re e(C).$$

Finally, for $\mu \in S_{\mathbb{K}}$, the last inequality, applied to μf instead f, gives

$$\max \Re e(\mu V(f)) \leq \max \Re e(\mu C),$$

which implies that $V(f) \subseteq C$ because C is a compact and convex subset of $\mathbb{K}$. ∎

Keeping in mind the Bishop-Phelps theorem, Corollaries 2.6, 2.7, and 2.8 immediately below are straightforward consequences of Lemma 2.1, Corollary 2.4, and Theorem 2.5, respectively.

COROLLARY 2.6. *Let χ be in S_{X^*}, Then, for every ψ in X^* we have*

$$V(X^*, \chi, \psi) = \bigcap_{\delta>0} \overline{co}\{\psi(x) : (x, \phi) \in \Pi(X), \|\phi - \chi\| < \delta\}.$$

COROLLARY 2.7. *Let f be a bounded continuous function from S_{X^*} to X^*. Then we have*

$$V(f) = \bigcap_{\delta>0} \overline{co}\{f(\varphi)(x) : (x, \phi) \in \Pi(X), \varphi \in S_{X^*}, \|\phi - \varphi\| < \delta\}.$$

COROLLARY 2.8. *For every bounded and uniformly continuous function $f : S_{X^*} \to X^*$, we have*

$$V(f) = \overline{co}\{f(\phi)(x) : (x, \phi) \in \Pi(X)\}.$$

Invoking the Bishop-Phelps-Bollobás theorem [**3**, Theorem 16.1], it is proved in [**1**, Lemma 2.7] that, for every bounded continuous function $f : S_{X^*} \to X^*$, $W(f)$ is contained in the closure in $\mathbb{K}$ of the set

$$\{f(\phi)(x) : (x, \phi) \in \Pi(X)\}.$$

Thus, Corollary 2.8 can be also derived from the result in [**1**] just quoted and the one in [**5**], already commented in the introduction, that $V(f) = \overline{co}W(f)$ whenever the bounded function f is actually uniformly continuous.

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