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NONASSOCIATIVE ULTRAPRIME NORMED ALGEBRAS

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Recently M. Mathieu [9] has proved that any associative ultraprime normed complex algebra is centrally closed. The aim of this note is to announce the general nonassociative extension of Mathieu's result obtained by the authors [2].

THEOREM 1. *Every ultraprime complex normed (nonassociative) algebra is centrally closed.*

We recall the concepts involved in the statement of Theorem 1. A prime (nonassociative) algebra A over a field K is called *centrally closed* if for each non cero ideal U of A and for each linear mapping $f:U \longrightarrow A$ such that $f(ax)=af(x)$ and $f(xa)=f(x)a$ for all x in U and a in A, there exists α in K such that $f(x)=\alpha x$ for all x in U. The reader is referred to [4, 6, 7, 8] for the impact of this concept in the study of associative and nonassociative prime algebras. Since, even in the associative commutative semisimple complete complex case, normed prime algebras need not be centrally closed, results about reasonable additional conditions on a prime normed algebra implying central closeability are welcome (see [1, 4, 9] and the survey in Section 2 of [10]). An ultrafilter $\mathcal{U}$ on a set I is said to be *countably incomplete* if it is not closed under countable intersections of their elements. For a normed algebra A and a countably incomplete ultrafilter $\mathcal{U}$ on a set I we define the *ultrapower of A (with respect to the ultrafilter $\mathcal{U}$)* as the quotient $A_{\mathcal{U}}:=l^{\infty}(I,A)/N_{\mathcal{U}}$, where $l^{\infty}(I,A)$ is the normed algebra of all bounded functions from I into A with the sup-norm, and $N_{\mathcal{U}}$ is the closed ideal defined as:

$$N_{\mathcal{U}}:=\{ (a_i)_{i\in I}\in l^{\infty}(I,A) \ : \ lim_{\mathcal{U}}\|a_i\|=0\}.$$

Following [9] we say that a normed (nonassociative) algebra A is *ultraprime* if some ultrapower $A_{\mathcal{U}}$ (with respect to a countably incomplete ultrafilter $\mathcal{U}$) is a prime algebra.

The proof of the associative precedent of Theorem 1 relies on the following peculiar characterization of ultraprimness for associative normed algebras which makes no reference any ultrapowers [9; Lemma 3.1]: *a normed associative algebra is ultraprime if and only if there exists a positive number* K *such that* $\|M_{a,b}\| \geq K\|a\|\|b\|$ *for all* a,b *in* A, *where* $M_{a,b}$ *denotes the operator on* A *given by* $x \longrightarrow axb$. This characterization is based on the well-known fact that the eventual primness of an associative algebra can be decided without involving ideals, indeed: *an associative algebra* A *is prime if and only if, for* a,b *in* A, $aAb=0$ *implies either* $a=0$ *or* $b=0$. Since a similar criterion to settle the primness of a nonassociative algebra is unknown, any intent to prove a general nonassociative extension of Mathieu theorem must rely on essentially different techniques. The search of such techniques allows us to introduce a particular type of prime normed algebras which we will define in what follows.

If A is a (nonassociative) algebra, we define the *unital multiplication algebra* $M(A)$ as the subalgebra of $L(A)$ (the algebra of all linear operators on A) generated by the identity operator I on A and all left and right multiplication operators on A. For each a,b in an algebra A, we denote by $N_{a,b}$ the bilinear mapping from $M(A)\times M(A)$ into A defined by $N_{a,b}(F,G)=F(a)G(b)$ for all F,G in $M(A)$. From the obvious fact that the ideal generated by any element a in A agrees with $M(A)(a)$, we obtain that A is a prime algebra if and only if, for a,b in A, $N_{a,b}=0$ implies either $a=0$ or $b=0$. Therefore, for a normed algebra A, a reasonable strengthening of primness is to require the existence of a positive number K such that: $\|N_{a,b}\| \geq K\|a\|\|b\|$ for all a,b in A, where of course for the computation of $\|N_{a,b}\|$ we consider $M(A)$ as a normed space under the operator norm. Normed algebras satisfying the above requirement will be called *totally prime algebras*. The requirement on a normed algebra of being totally prime also makes no reference to ultrapowers and is rather similar to the above cited characterization of ultraprimness for associative normed algebras, so that by refining some ideas in [9] we are able to prove

Theorem 2. *Every totally prime normed complex algebra is centrally closed.*

Theorem 1 follows from the above result once the next crucial fact is proved.

Theorem 3. *Every ultraprime normed algebra is totally prime.*

We have contrasted the new concept of total primness on the class of (nonassociative) H*-algebras. We recall that an *H*-algebra* is a complex algebra A with an algebra involution * and whose vector space is a Hilbert space satisfying $(ab|c)=(a|cb^*)=(b|a^*c)$ for all a,b,c in A. The product of any H*-algebra is continuous for the topology of the Hilbert norm $a \longrightarrow \|a\|:=\sqrt{(a|a)}$ [5; Proposition 2.(i)], so (by multiplying the inner product by a suitable positive number, if necessary) every H*-algebra becomes a (complete) normed algebra in the usual sense of the word. We refer to [3] and their references for a general view of the development of the theory of nonassociative H*-algebras. We have proved the following theorem.

THEOREM 4. *Every prime H*-algebra is totally prime. More concretely, if A is a prime H*-algebra, then* $\|N_{a,b}\|=M\|a\|\|b\|$ for all a,b in A, *where M denotes the norm of the product of A.*

This result shows that the class of totally prime algebras becomes a large class of normed algebras. Moreover, since associative prime H*-algebras are ultraprime only in the trivial finite-dimensional case, Theorem 4 also shows that (even in the associative case) the requirement of ultraprimness is strictly stronger than the one of total primness, and therefore Theorem 2 is an associative improvement of Mathieu's result.

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